Fixed points of multivalued nonlinear $F$-contractions on complete metric spaces

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Abstract. We introduce a new concept for multivalued maps, also called multivalued nonlinear $F$-contraction, and give a fixed point result. Our result is a proper generalization of some recent fixed point theorems including the famous theorem of Klim and Wardowski [D. Klim, D. Wardowski, Fixed point theorems for set-valued contractions in complete metric spaces, J. Math. Anal. Appl., 334(1):132–139, 2007].

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1 Introduction and preliminaries

Let $(X, d)$ be a metric space. $P(X)$ denotes the family of all nonempty subsets of $X$, $C(X)$ denotes the family of all nonempty, closed subsets of $X$, $CB(X)$ denotes the family of all nonempty, closed, and bounded subsets of $X$, and $K(X)$ denotes the family of all nonempty compact subsets of $X$. It is clear that, $K(X) \subseteq CB(X) \subseteq C(X) \subseteq P(X)$. For $A, B \in C(X)$, let

$$H(A, B) = \max \left\{ \sup_{x \in A} d(x, B), \sup_{y \in B} d(y, A) \right\},$$

where $d(x, B) = \inf \{d(x, y) : y \in B\}$. Then $H$ is called generalized Pompeiu–Hausdorff distance on $C(X)$. It is well known that $H$ is a metric on $CB(X)$, which is called Pompeiu–Hausdorff metric induced by $d$. We can find detailed information about the Pompeiu–Hausdorff metric in [1, 5, 9]. Let $T : X \rightarrow CB(X)$ be a map, then $T$ is called multivalued contraction (see [14]) if for all $x, y \in X$, there exists $L \in [0, 1)$ such that

$$H(Tx, Ty) \leq Ld(x, y).$$
In 1969, Nadler [14] proved that every multivalued contraction on complete metric space has a fixed point.

Nadler’s fixed point theorem has been extended in many directions [4, 6, 7, 10, 13, 16, 17]. The following generalization of it is given by Feng and Liu [8].

**Theorem 1.** (See [8].) Let \((X, d)\) be a complete metric space and \(T : X \to C(X)\). Assume that the following conditions hold:

(i) the map \(x \mapsto d(x, Tx)\) is lower semi-continuous;
(ii) there exist \(b, c \in (0, 1)\) with \(b < c\) such that for any \(x \in X\), there is \(y \in I^x_b\) satisfying

\[
d(y, Ty) \leq cd(x, y),
\]

where

\[
I^x_b = \{ y \in Tx : bd(x, y) \leq d(x, Tx) \}.
\]

Then \(T\) has a fixed point.

Recently, another interesting result have been obtained by Klim and Wardowski [11]. They proved the following theorem.

**Theorem 2.** (See [11].) Let \((X, d)\) be a complete metric space and \(T : X \to C(X)\). Assume that the following conditions hold:

(i) the map \(x \mapsto d(x, Tx)\) is lower semi-continuous;
(ii) there exists \(b \in (0, 1)\) and a function \(\varphi : [0, \infty) \to [0, b)\) satisfying

\[
\limsup_{t \to s^+} \varphi(t) < b \quad \text{for} \quad s \geq 0
\]

and for any \(x \in X\), there is \(y \in I^x_b\) satisfying

\[
d(y, Ty) \leq \varphi(d(x, y))d(x, y).
\]

Then \(T\) has a fixed point.

In this paper, we introduce a new class of multivalued maps and give a fixed point result, which extend and generalize many fixed point theorems including Theorems 1 and 2. Our results are based on \(F\)-contraction which is a new approach to contraction mapping. The concept of \(F\)-contraction for single valued maps on complete metric space was introduced by Wardowski [18]. First, we recall this new concept and some related results.

Let \(F : (0, \infty) \to \mathbb{R}\) be a function. For the sake of completeness, we will consider the following conditions:

\(F1\) \(F\) is strictly increasing, i.e., for all \(\alpha, \beta \in (0, \infty)\) such that \(\alpha < \beta\), \(F(\alpha) < F(\beta)\).

\(F2\) For each sequence \(\{\alpha_n\}\) of positive numbers,

\[
\lim_{n \to \infty} \alpha_n = 0 \quad \text{if and only if} \quad \lim_{n \to \infty} F(\alpha_n) = -\infty.
\]

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Thus, every ordinary contraction is also an $F$-contraction. Thus, $x, y \in X$, it is clear that for $d(x, y) = 0$.

If we define $F_3(\alpha) = \ln \alpha$ for $\alpha \leq 1$ and $F_3(\alpha) = 2\alpha$ for $\alpha > 1$, then $F_3 \in F \setminus F_\tau$.

Remark 1. If $F$ satisfies (F1), then it satisfies (F4) if and only if it is right continuous.

Definition 1. (See [18].) Let $(X, d)$ be a metric space and $T : X \to X$ be a mapping. Then $T$ is an $F$-contraction if $F \in F$ and there exists $\tau > 0$ such that for all $x, y \in X$,

$$d(Tx, Ty) > 0 \iff \tau + F(d(Tx, Ty)) \leq F(d(x, y)). \quad (1)$$

If we take $F(\alpha) = \ln \alpha$ in Definition 1, inequality (1) turns into

$$d(Tx, Ty) \leq e^{-\tau} d(x, y) \quad \text{for all } x, y \in X, Tx \neq Ty. \quad (2)$$

It is clear that for $x, y \in X$ such that $Tx = Ty$, the inequality $d(Tx, Ty) \leq e^{-\tau} d(x, y)$ also holds. Thus, $T$ is an ordinary contraction with contractive constant $e^{-\tau}$. Therefore, every ordinary contraction is also an $F$-contraction with $F(\alpha) = \ln \alpha$, but the converse may not be true as shown in Example 2.5 of [18]. If we choose $F(\alpha) = \alpha + \ln \alpha$, inequality (1) turns into

$$d(Tx, Ty) e^{d(Tx, Ty) - d(x, y)} \leq e^{-\tau} \quad \text{for all } x, y \in X, Tx \neq Ty. \quad (3)$$

In addition, Wardowski showed that every $F$-contraction $T$ is a contractive mapping, i.e.,

$$d(Tx, Ty) < d(x, y) \quad \text{for all } x, y \in X, Tx \neq Ty.$$

Thus, every $F$-contraction is a continuous map. Also, Wardowski concluded that if $F_1, F_2 \in F$ with $F_1(\alpha) \leq F_2(\alpha)$ for all $\alpha > 0$ and $G = F_2 - F_1$ is nondecreasing, then every $F_1$-contraction $T$ is an $F_2$-contraction. He noted that for the mappings $F_1(\alpha) = \ln \alpha$ and $F_2(\alpha) = \alpha + \ln \alpha$, $F_1 < F_2$ and the mapping $F_2 - F_1$ is strictly increasing. Hence, every Banach contraction satisfies the contractive condition (3). On the other hand, Example 2.5 in [18] shows that the mapping $T$ is not $F_1$-contraction (Banach contraction), but still is an $F_2$-contraction. Thus, the following theorem is a proper generalization of Banach contraction principle.

Theorem 3. (See [18].) Let $(X, d)$ be a complete metric space and $T : X \to X$ be an $F$-contraction. Then $T$ has a unique fixed point in $X$.

By combining the ideas of Wardowski’s and Nadler’s, Altun et al. [3] introduced the concept of multivalued $F$-contractions and obtained some fixed point results for these type mappings on complete metric space.

Definition 2. (See [3].) Let \((X, d)\) be a metric space and \(T : X \to \mathcal{CB}(X)\) be a mapping. Then \(T\) is a multivalued \(F\)-contraction if \(F \in \mathcal{F}\) and there exists \(\tau > 0\) such that for all \(x, y \in X\),

\[
H(Tx, Ty) > 0 \implies \tau + F(H(Tx, Ty)) \leq F(d(x, y)).
\]

By considering \(F(\alpha) = \ln \alpha\), every multivalued contraction in the sense of Nadler is also a multivalued \(F\)-contraction.

Theorem 4. (See [3].) Let \((X, d)\) be a complete metric space and \(T : X \to \mathcal{K}(X)\) be a multivalued \(F\)-contraction, then \(T\) has a fixed point in \(X\).

At this point, one can ask if \(\mathcal{CB}(X)\) can be used instead of \(\mathcal{K}(X)\) in Theorem 4. As shown in Example 1 of [2], the answer is negative. But, by adding condition (F4) on \(F\), we can take \(\mathcal{CB}(X)\) instead of \(\mathcal{K}(X)\).

Theorem 5. (See [3].) Let \((X, d)\) be a complete metric space and \(T : X \to \mathcal{CB}(X)\) be a multivalued \(F\)-contraction. Suppose \(F \in \mathcal{F}^*\), then \(T\) has a fixed point in \(X\).

On the other hand, Olgun et al. [15] proved the following theorems. Theorem 7 is a generalization of famous Mizoguchi–Takahashi’s fixed point theorem for multivalued contraction maps. These results are nonlinear cases of Theorems 4 and 5, respectively.

Theorem 6. (See [15].) Let \((X, d)\) be a complete metric space and \(T : X \to \mathcal{K}(X)\). If there exists \(F \in \mathcal{F}\) and \(\tau : (0, \infty) \to (0, \infty)\) such that

\[
\liminf_{t \to s^+} \tau(t) > 0 \quad \text{for all } s \geq 0
\]

and for all \(x, y \in X\),

\[
H(Tx, Ty) > 0 \implies \tau(d(x, y)) + F(H(Tx, Ty)) \leq F(d(x, y)),
\]

then \(T\) has a fixed point in \(X\).

Theorem 7. (See [15].) Let \((X, d)\) be a complete metric space and \(T : X \to \mathcal{CB}(X)\). If there exists \(F \in \mathcal{F}^*\) and \(\tau : (0, \infty) \to (0, \infty)\) such that

\[
\liminf_{t \to s^+} \tau(t) > 0 \quad \text{for all } s \geq 0
\]

and for all \(x, y \in X\),

\[
H(Tx, Ty) > 0 \implies \tau(d(x, y)) + F(H(Tx, Ty)) \leq F(d(x, y)),
\]

then \(T\) has a fixed point in \(X\).

2 Main results

Let \(T : X \to \mathcal{P}(X)\) be a multivalued map, \(F \in \mathcal{F}\) and \(\sigma \geq 0\). For \(x \in X\) with \(d(x, Tx) > 0\), define a set \(F^x_\sigma \subseteq X\) as

\[
F^x_\sigma = \{ y \in Tx : F(d(x, y)) \leq F(d(x, Tx)) + \sigma \}.
\]
We need to consider the following cases.

Case 1. If $T : X \to K(X)$, then for all $\sigma > 0$ and $x \in X$ with $d(x, Tx) > 0$, we have $F^x_\sigma \neq \emptyset$. Indeed, since $Tx$ is compact, for every $x \in X$, we have $y \in Tx$ such that $d(x, y) = d(x, Tx)$. Therefore, for every $x \in X$ with $d(x, Tx) > 0$, we have $F(d(x, y)) = F(d(x, Tx))$. Thus, $y \in F^x_\sigma$ for all $\sigma > 0$.

Case 2. If $T : X \to C(X)$, then $F^x_\sigma$ may be empty for some $x \in X$ and $\sigma \geq 0$. For example, let $F(\alpha) = \ln \alpha$ for $\alpha \leq 1$ and $F(\alpha) = 2\alpha$ for $\alpha > 1$ and let $X = \{0\} \cup (1, 2)$ with the usual metric. Define $T : X \to C(X)$ by $T(0) = (1, 2)$ and $Tx = \{0\}$ for $x \in (1, 2)$. Then for $x = 0$, we have (note that $d(0, T0) = 1 > 0$)

$$F^0_\sigma = \{y \in T0 : F(d(0, y)) \leq F(d(0, T0)) + 1\} = \{y \in (1, 2) : F(y) \leq F(1) + 1\} = \{y \in (1, 2) : 2y \leq 1\} = \emptyset.$$

Case 3. If $T : X \to C(X)$ (even if $T : X \to P(X)$) and $F \in \mathcal{F}_\alpha$, then for all $\sigma > 0$ and $x \in X$ with $d(x, Tx) > 0$, we have $F^x_\sigma \neq \emptyset$. Indeed, by (F4), we have

$$F^x_\sigma = \{y \in Tx : F(d(x, y)) \leq F(d(x, Tx)) + \sigma\} = \{y \in Tx : F(d(x, y)) \leq \inf\{F(d(x, y)) : y \in Tx\} + \sigma\} \neq \emptyset.$$

Mnak et al. [12] proved the following fixed point theorems. Note that Theorem 1 is a special case of Theorem 9.

**Theorem 8.** Let $(X, d)$ be a complete metric space, $T : X \to K(X)$ and $F \in \mathcal{F}$. If there exists $\tau > 0$ such that for any $x \in X$ with $d(x, Tx) > 0$, there exists $y \in F^x_\sigma$ satisfying

$$\tau + F(d(y, Ty)) \leq F(d(x, y)),$$

where

$$F^x_\sigma = \{y \in Tx : F(d(x, y)) \leq F(d(x, Tx)) + \sigma\},$$

then $T$ has a fixed point in $X$ provided $\sigma < \tau$ and $x \to d(x, Tx)$ is lower semi-continuous.

**Theorem 9.** Let $(X, d)$ be a complete metric space, $T : X \to C(X)$ and $F \in \mathcal{F}_\alpha$. If there exists $\tau > 0$ such that for any $x \in X$ with $d(x, Tx) > 0$, there exists $y \in F^x_\sigma$ satisfying

$$\tau + F(d(y, Ty)) \leq F(d(x, y)),$$

then $T$ has a fixed point in $X$ provided $\sigma < \tau$ and $x \to d(x, Tx)$ is lower semi-continuous.

By considering the above facts, we give the following theorems, which are nonlinear form of Theorems 8 and 9. Note that Theorem 10 is a proper generalization of Theorem 2.
Theorem 10. Let \((X, d)\) be a complete metric space, \(T : X \to C(X)\) and \(F \in \mathcal{F}_*\). Assume that the following conditions hold:

(i) the map \(x \to d(x, Tx)\) is lower semi-continuous;
(ii) there exist \(\sigma > 0\) and a function \(\tau : (0, \infty) \to (\sigma, \infty)\) such that

\[
\liminf_{t \to s^+} \tau(t) > \sigma \quad \text{for all } s \geq 0
\]

and for any \(x \in X\) with \(d(x, Tx) > 0\), there exists \(y \in F^x_\sigma\) satisfying

\[
\tau(d(x, y)) + F(d(y, Ty)) \leq F(d(x, y)).
\]

Then \(T\) has a fixed point.

Proof. Suppose that \(T\) has no fixed point. Then for all \(x \in X\), we have \(d(x, Tx) > 0\). Since \(Tx \in C(X)\) for every \(x \in X\), the set \(F^x_\sigma\) is nonempty for any \(\sigma > 0\). Let \(x_0 \in X\) be any initial point, then there exists \(x_1 \in F^x_{\sigma_1}\) such that

\[
\tau(d(x_0, x_1)) + F(d(x_1, Tx_1)) \leq F(d(x_0, x_1))
\]

and for \(x_1 \in X\), there exists \(x_2 \in F^x_{\sigma_2}\) satisfying

\[
\tau(d(x_1, x_2)) + F(d(x_2, Tx_2)) \leq F(d(x_1, x_2)).
\]

Continuing this process, we get an iterative sequence \(\{x_n\}\), where \(x_{n+1} \in F^x_{\sigma_n}\) and

\[
\tau(d(x_n, x_{n+1})) + F(d(x_{n+1}, Tx_{n+1})) \leq F(d(x_n, x_{n+1})). \tag{4}
\]

We will verify that \(\{x_n\}\) is a Cauchy sequence. Since \(x_{n+1} \in F^x_{\sigma_n}\), we have

\[
F(d(x_n, x_{n+1})) \leq F(d(x_n, Tx_n)) + \sigma. \tag{5}
\]

From (4) and (5) we have

\[
F(d(x_{n+1}, Tx_{n+1})) \leq F(d(x_n, Tx_n)) + \sigma - \tau(d(x_n, x_{n+1})) \tag{6}
\]

and

\[
F(d(x_{n+1}, x_{n+2})) \leq F(d(x_n, x_{n+1})) + \sigma - \tau(d(x_n, x_{n+1})). \tag{7}
\]

Let \(a_n = d(x_n, x_{n+1})\) for \(n \in \mathbb{N}\), then \(a_n > 0\) and from (7) \(\{a_n\}\) is decreasing. Therefore, there exists \(\delta \geq 0\) such that \(\lim_{n \to \infty} a_n = \delta\). Now let \(\delta > 0\). Using (7), the following holds:

\[
F(a_{n+1}) \leq F(a_n) + \sigma - \tau(a_n)
\]

\[
\leq F(a_{n-1}) + 2\sigma - \tau(a_n) - \tau(a_{n-1})
\]

\[\vdots\]

\[
\leq F(a_0) + n\sigma - \tau(a_n) - \tau(a_{n-1}) - \cdots - \tau(a_0). \tag{8}
\]

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Let $\tau(a_{p_n}) = \min\{\tau(a_0), \tau(a_1), \ldots, \tau(a_n)\}$ for all $n \in \mathbb{N}$. From (8) we get
\[
F(a_n) \leq F(a_0) + n(\sigma - \tau(a_{p_n})).
\] (9)

In a similar way, from (6) we can obtain
\[
F(d(x_{n+1}, Tx_{n+1})) \leq F(d(x_0, Tx_0)) + n(\sigma - \tau(a_{p_n})).
\] (10)

Now consider the sequence $\{\tau(a_{p_n})\}$. We distinguish two cases.

**Case 1.** For each $n \in \mathbb{N}$, there is $m > n$ such that $\tau(a_{p_n}) > \tau(a_{p_m})$. Then we obtain a subsequence $\{a_{p_{n_k}}\}$ of $\{a_{p_n}\}$ with $\tau(a_{p_{n_k}}) > \tau(a_{p_{n_k+1}})$ for all $k$. Since $a_{p_{n_k}} \to \delta^+$, we deduce that
\[
\liminf_{k \to \infty} \tau(a_{p_{n_k}}) > \sigma.
\]

Hence, $F(a_{n_k}) \leq F(a_0) + n_k(\sigma - \tau(a_{p_{n_k}}))$ for all $k$. Consequently, $\lim_{k \to \infty} F(a_{n_k}) = -\infty$, and by (F2), $\lim_{k \to \infty} a_{p_{n_k}} = 0$, which contradicts that $\lim_{n \to \infty} a_n > 0$.

**Case 2.** There is $n_0 \in \mathbb{N}$ such that $\tau(a_{p_{n_0}}) = \tau(a_{p_{m}})$ for all $m > n_0$. Then $F(a_{m}) \leq F(a_0) + m(\sigma - \tau(a_{p_{n_0}}))$ for all $m > n_0$. Hence, $\lim_{m \to \infty} F(a_{m}) = -\infty$, so $\lim_{m \to \infty} a_{m} = 0$, which contradicts that $\lim_{n \to \infty} a_n > 0$. Thus, $\lim_{n \to \infty} a_n = 0$. From (F3) there exists $k \in (0, 1)$ such that
\[
\lim_{n \to \infty} a_{n}^k F(a_n) = 0.
\]

By (9), the following holds for all $n \in \mathbb{N}$:
\[
a_{n}^k F(a_n) - a_{n}^k F(a_0) \leq a_{n}^k n(\sigma - \tau(a_{p_n})) \leq 0.
\] (11)

Letting $n \to \infty$ in (11), we obtain that
\[
\lim_{n \to \infty} a_{n}^k = 0.
\] (12)

From (12) there exits $n_0 \in \mathbb{N}$ such that $n_0 a_{n}^k \leq 1$ for all $n \geq n_0$. So, for all $n \geq n_0$, we have
\[
a_n \leq \frac{1}{n^{1/k}}.
\] (13)

In order to show that $\{x_n\}$ is a Cauchy sequence, consider $m, n \in \mathbb{N}$ such that $m > n \geq n_1$. Using the triangular inequality for the metric and from (13) we have
\[
d(x_n, x_m) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m)
\]
\[
= \sum_{i=n}^{m-1} d(x_i, x_{i+1}) \leq \sum_{i=n}^{\infty} d(x_i, x_{i+1}) \leq \sum_{i=n}^{\infty} \frac{1}{i^{1/k}}.
\]

By the convergence of the series $\sum_{i=1}^{\infty} (i^{-1/k})$, passing to limit $n, m \to \infty$, we get $d(x_n, x_m) \to 0$. This yields that $\{x_n\}$ is a Cauchy sequence in $(X, d)$. Since $(X, d)$ is

a complete metric space, the sequence \( \{x_n\} \) converges to some point \( z \in X \), that is, 
\[
\lim_{n \to \infty} x_n = z.
\]
On the other hand, from (10) and (F2) we have 
\[
\lim_{n \to \infty} d(x_n, Tx_n) = 0.
\]
Since \( x \to d(x, Tx) \) is lower semi-continuous, then 
\[
0 \leq d(z, Tz) \leq \lim \inf_{n \to \infty} d(x_n, Tx_n) = 0.
\]
This is a contradiction. Hence, \( T \) has a fixed point.

In the following example, we show that there are some multivalued maps such that our result can be applied, but Theorem 2 can not.

Example 1. Let \( X = \{x_n = n(n + 1)/2, n \in \mathbb{N}\} \) and \( d(x, y) = |x - y| \). Then \( (X, d) \) is a complete metric space. Define a mapping \( T : X \to C(X) \) as 
\[
Tx = \begin{cases} \{x_1\}, & x = x_1, \\ \{x_1, x_{n-1}\}, & x = x_n. \end{cases}
\]
Then, since \( \tau_d \) is discrete topology, the map \( x \to d(x, Tx) \) is continuous. Now we claim that condition (ii) of Theorem 2.1 of [11] is not satisfied. Indeed, let \( x = x_n \) for \( n > 1 \), then \( Tx = \{x_1, x_{n-1}\} \). In this case, for all \( b \in (0, 1) \), there exists \( n_0(b) \in \mathbb{N} \) such that for all \( n \geq n_0(b) \), \( T_{n_0} = \{x_{n-1}\} \). Thus, for \( n \geq n_0(b) \), we have 
\[
d(y, Ty) = n - 1, \quad d(x, y) = n.
\]
Therefore, since \( d(y, Ty)/d(x, y) = (n - 1)/n \), we can not find a function \( \varphi : [0, \infty) \to [0, b) \) satisfying 
\[
d(y, Ty) \leq \varphi(d(x, y))d(x, y).
\]
Now we show that condition (ii) of Theorem 10 is satisfied with \( F(\alpha) = \alpha + \ln \alpha \), \( \sigma = 1/2 \) and \( \tau(t) = 1/t + 1/2 \). Note that if \( d(x, Tx) > 0 \), then \( x = x_n \) for \( n > 1 \). In this case, \( d(x_n, Tx_n) = n \). Therefore, for \( y = x_{n-1} \in Tx_n \), we have \( y \in F_{1/2}^{x_n} \) and 
\[
\tau(d(x, y)) + F(d(y, Ty)) = \tau(n) + F(n - 1) \\
= \frac{n}{n} + \frac{1}{2} + n - 1 + \ln(n - 1) \\
\leq n + \ln n = F(n) = F(d(x, Tx)).
\]

Remark 2. If we take \( K(X) \) instead of \( C(X) \) in Theorem 10, we can remove condition (F4) on \( F \). Further, by taking into account Case 1, we can take \( \sigma \geq 0 \). Therefore, the proof of the following theorem is obvious.

Theorem 11. Let \((X, d)\) be a complete metric space and \( T : X \to K(X) \). Assume that the following conditions hold:
(i) the map $x \mapsto d(x, Tx)$ is lower semi-continuous;
(ii) there exists $\sigma \geq 0$, $F \in \mathcal{F}$ and a function $\tau : (0, \infty) \to (\sigma, \infty)$ such that
\[
\lim \inf_{t \to s^+} \tau(t) > \sigma \quad \text{for all } s \geq 0
\]
and for any $x \in X$ with $d(x, Tx) > 0$, there exists $y \in F^x_\sigma$ satisfying
\[
\tau(d(x, y)) + F(d(y, Ty)) \leq F(d(x, y)).
\]
Then $T$ has a fixed point.

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