Positive solutions for a class of fractional boundary value problems

Jiafa Xu\textsuperscript{a}, Zhongli Wei\textsuperscript{b,c}

\textsuperscript{a}School of Mathematical Sciences, Chongqing Normal University, Chongqing 401331, China
xujiafa292@sina.com

\textsuperscript{b}School of Mathematics, Shandong University, Jinan 250100, Shandong, China

\textsuperscript{c}Department of Mathematics, Shandong Jianzhu University, Jinan 250101, Shandong, China
jnwzl32@163.com

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Abstract. In this work, by virtue of the Krasnoselskii–Zabreiko fixed point theorem, we investigate the existence of positive solutions for a class of fractional boundary value problems under some appropriate conditions concerning the first eigenvalue of the relevant linear operator. Moreover, we utilize the method of lower and upper solutions to discuss the unique positive solution when the nonlinear term grows sublinearly.

Keywords: fractional boundary value problem, Krasnoselskii–Zabreiko fixed point theorem, positive solution, uniqueness.

1 Introduction

In this paper we consider the existence of positive solutions for the boundary value problem of fractional order involving Riemann–Liouville’s derivative

\[
\begin{align*}
D_0^\alpha D_0^\alpha u &= f(t, u, u', -D_0^\alpha u), \quad t \in [0, 1], \\
u(0) = u'(0) = u'(1) = D_0^\alpha u(0) = D_0^{\alpha+1} u(0) = D_0^{\alpha+1} u(1) = 0,
\end{align*}
\]

(1)

where \( \alpha \in (2, 3] \) is a real number, \( D_0^\alpha \) is the standard Riemann–Liouville fractional derivative of order \( \alpha \) and \( f \in C([0, 1] \times \mathbb{R}^3_+, \mathbb{R}_+) \) (\( \mathbb{R}_+ := [0, +\infty) \)).

Recently, the fractional differential calculus and fractional differential equation have drawn more and more attention due to the applications of such constructions in various
sciences such as physics, mechanics, chemistry, engineering, etc. Many books on fractional calculus, fractional differential equations have appeared, for instance, see [7,10,11]. This may explain the reason that the last two decades have witnessed an overgrowing interest in the research of such problems, with many papers in this direction published. We refer the interested reader to [1,2,4,5,6,12,13,14,15] and the references therein.

In [4,6], by using the fixed point index theory and Krein–Rutman theorem, Jiang et al. obtained the existence of positive solutions for the multi-point boundary value problems of fractional differential equations

\[ D^\alpha_0 u(t) + f(t,u(t)) = 0, \quad 0 < t < 1, \quad 1 < \alpha \leq 2, \]
\[ u(0) = 0, \quad D^\beta_0 u(1) = \sum_{i=1}^{n-2} a_i D^\beta_0 u(\xi_i), \quad (2) \]

and

\[ D^\alpha u - Mu = \lambda f(t,u(t)), \quad t \in [0,1], \quad 0 < \alpha < 1, \]
\[ u(0) = \sum_{i=1}^{n} \beta_i u(\xi_i). \quad (3) \]

In this paper, we first construct an integral operator for the corresponding linear boundary value problem and find out its first eigenvalue and eigenfunction. Then we establish a special cone associated with the Green’s function of (1). Finally, by employing the Krasnoselskii–Zabreiko fixed point theorem, combined with a priori estimates of positive solutions, we obtain the existence of positive solutions for (1). Note that our nonlinear term \( f \) involves the fractional derivatives of the dependent variable–this is seldom studied in the literature and most research articles on boundary value problems consider nonlinear terms that involve the unknown function \( u \) only, for example, [1,2,4,5,6,12,13,15]. Moreover, we adopt the method of lower and upper solutions to discuss the uniqueness of positive solutions for (1), and prove that the unique positive solution can be uniformly approximated by an iterative sequence beginning with any function which is continuous, nonnegative and not identically vanishing on \([0,1]\). This, together with the fact that our nonlinearity may be of distinct growth, means that our methodology and main results here are entirely different from those in the above papers.

2 Preliminaries

For convenience, we give some background materials from fractional calculus theory to facilitate analysis of problem (1). These materials can be found in the recent books, see [7,10,11].

**Definition 1.** (See [7,10], [11, pp. 36–37].) The Riemann–Liouville fractional derivative of order \( \alpha > 0 \) of a continuous function \( f : (0, +\infty) \rightarrow (-\infty, +\infty) \) is given by

\[ D^\alpha_0 f(t) = \frac{1}{\Gamma(n-\alpha)} \left( \frac{d}{dt} \right)^n \int_0^t (t-s)^{n-\alpha-1} f(s) \, ds, \]
where \( n = \lceil\alpha\rceil + 1 \), \( \lceil\cdot\rceil \) denotes the integer part of number \( \alpha \), provided that the right side is pointwise defined on \((0, +\infty)\).

**Definition 2.** (See [11, Def. 2.1].) The Riemann–Liouville fractional integral of order \( \alpha > 0 \) of a function \( f : (0, +\infty) \to (-\infty, +\infty) \) is given by

\[
I_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_{0}^{t} (t-s)^{\alpha-1} f(s) \, ds,
\]

provided that the right side is pointwise defined on \((0, +\infty)\).

From the definition of the Riemann–Liouville derivative, we can obtain the following statement.

**Lemma 1.** (See [1].) Let \( \alpha > 0 \). If we assume \( u \in C(0,1) \cap L(0,1) \), then the fractional differential equation \( D_{0+}^{\alpha} u(t) = 0 \) has a unique solution

\[
u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_N t^{\alpha-N}, \quad c_i \in \mathbb{R}, \; i = 1, 2, \ldots, N,
\]

where \( N \) is the smallest integer greater than or equal to \( \alpha \).

**Lemma 2.** (See [1].) Assume that \( u \in C(0,1) \cap L(0,1) \) with a fractional derivative of order \( \alpha > 0 \) that belongs to \( C(0,1) \cap L(0,1) \). Then

\[
I_{0+}^{\alpha} D_{0+}^{\alpha} u(t) = u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_N t^{\alpha-N}, \quad c_i \in \mathbb{R}, \; i = 1, 2, \ldots, N,
\]

where \( N \) is the smallest integer greater than or equal to \( \alpha \).

In what follows, we shall discuss some properties of the Green’s function for fractional boundary value problem (1). Let

\[
G_1(t, s) := \frac{1}{\Gamma(\alpha)} \begin{cases} 
(1-s)^{\alpha-2} - (t-s)^{\alpha-1}, & 0 \leq s \leq t \leq 1, \\
(1-s)^{\alpha-2}, & 0 \leq t \leq s \leq 1.
\end{cases}
\]

Then we can easily obtain that

\[
G_2(t, s) := \frac{\partial}{\partial t} G_1(t, s)
= \frac{\alpha - 1}{\Gamma(\alpha)} \begin{cases} 
(1-s)^{\alpha-2} - (t-s)^{\alpha-2}, & 0 \leq s \leq t \leq 1, \\
(1-s)^{\alpha-2}, & 0 \leq t \leq s \leq 1.
\end{cases}
\]

**Lemma 3.** (See [2, Lemma 2.7].) Let \( f \) be as in (1) and \( -D_{0+}^{\alpha} u := v \). Then (1) is equivalent to

\[
v(t) = \int_{0}^{1} G_1(t, s) f(s, 1) \, ds + \int_{0}^{1} \int_{0}^{1} G_2(s, \tau) v(\tau) \, d\tau + \int_{0}^{1} G_2(s, 1) v(s) \, ds.
\]

Lemma 4. (See [2, Lemma 2.8] and [5, Thms. 1.1, 1.2].) The functions $G_i(t, s) \in C([0, 1] \times [0, 1], \mathbb{R}_+)$ $(i = 1, 2)$, moreover, the following two inequalities hold:

$$t^\alpha s(1-s)^{\alpha-2} \leq \Gamma(\alpha) G_1(t, s) \leq s(1-s)^{\alpha-2} \quad \forall t, s \in [0, 1].$$ (7)

$$(\alpha - 1)(\alpha - 2)t^{\alpha-2}(1-t)s(1-s)^{\alpha-2} \leq \Gamma(\alpha) G_2(t, s) \leq (\alpha - 1)t^{\alpha-3}s(1-s)^{\alpha-2} \quad \forall t, s \in [0, 1].$$ (8)

In what follows, we shall define two extra functions by $G_1, G_2$. Let

$$G_3(t, s) := \int_0^1 G_1(t, \tau)G_1(\tau, s) \, d\tau \quad \forall t, s \in [0, 1],$$

$$G_4(t, s) := \int_0^1 G_1(t, \tau)G_2(\tau, s) \, d\tau \quad \forall t, s \in [0, 1].$$ (9)

Then $G_i(t, s) \in C([0, 1] \times [0, 1], \mathbb{R}_+)$ $(i = 3, 4)$. Moreover, by Lemma 4, we easily have

$$\frac{\alpha}{(\alpha - 1)\Gamma(2\alpha)} t^{\alpha-1}s(1-s)^{\alpha-2}$$

$$= \int_0^1 \frac{t^{\alpha-1}\tau(1-\tau)^{\alpha-2}}{\Gamma(\alpha)} \cdot \frac{\tau^{\alpha-1}s(1-s)^{\alpha-2}}{\Gamma(\alpha)} \, d\tau \leq G_3(t, s)$$

$$\leq \int_0^1 \frac{s(1-s)(1-\tau)^{\alpha-2}}{\Gamma^2(\alpha)} \, d\tau = \frac{s(1-s)^{\alpha-2}}{\alpha(\alpha - 1)\Gamma^2(\alpha)}. (10)$$

Similarly,

$$\frac{(\alpha - 1)(\alpha - 2)}{\Gamma(2\alpha)} t^{\alpha-1}s(1-s)^{\alpha-2}$$

$$= \int_0^1 \frac{t^{\alpha-1}\tau(1-\tau)^{\alpha-2}}{\Gamma(\alpha)} \cdot \frac{(\alpha - 1)(\alpha - 2)s(1-s)^{\alpha-2}}{\Gamma(\alpha)} \, d\tau$$

$$\leq G_4(t, s) \leq \int_0^1 \frac{(\alpha - 1)s^{\alpha-3}(1-s)(1-\tau)^{\alpha-2}}{\Gamma^2(\alpha)} \, d\tau$$

$$= \frac{s(1-s)^{\alpha-2}}{(\alpha - 1)\Gamma(2\alpha - 2)}. (11)$$

Let

$$E := C[0, 1], \quad \|v\| := \max_{t \in [0, 1]} |v(t)|, \quad P := \{ v \in E: v(t) \geq 0 \quad \forall t \in [0, 1] \}.$$
Then \((E, \| \cdot \|)\) becomes a real Banach space and \(P\) is a cone on \(E\). Define \(B_\rho := \{ v \in E : \| v \| < \rho \}\) for \(\rho > 0\) in the sequel.

Let

\[
(Av)(t) := \int_0^1 G_1(t, s) f \left( s, \int_0^1 G_1(s, \tau) v(\tau) \, d\tau, \int_0^1 G_2(s, \tau) v(\tau) \, d\tau, v(s) \right) \, ds
\]  

(12)

for all \(v \in E\). The continuity of \(G_1, G_2\) and \(f\) implies that \(A : E \to E\) is a completely continuous nonlinear operator. As mentioned in Lemma 3, \(-D_{0+}^{\alpha} u = v\), together with the boundary conditions \(u(0) = u'(0) = u'(1) = 0\), we have

\[
u(t) = \int_0^1 G_1(t, s) v(s) \, ds.
\]  

(13)

where \(G_1\) is determined by (4). Therefore, we find the existence of solutions of (1) is equivalent to that of fixed points of \(A\).

For \(a, b, c \geq 0\) with \(a^2 + b^2 + c^2 \neq 0\), let

\[
G_{a,b,c}(t, s) := aG_3(t, s) + bG_4(t, s) + cG_1(t, s) \quad \forall t, s \in [0, 1],
\]

and define a linear operator \(L_{a,b,c}\) as follows:

\[
(L_{a,b,c}v)(t) := \int_0^1 G_{a,b,c}(t, s) v(s) \, ds \quad \forall v \in E.
\]  

(14)

Obviously, \(L_{a,b,c}\) is positive, i.e., \(L_{a,b,c}(P) \subset P\). The continuity of \(G_1, G_3, G_4\) implies that \(L_{a,b,c}\) is a completely continuous operator. From now on, we utilize \(r(L_{a,b,c})\) to denote the spectral radius of \(L_{a,b,c}\). Furthermore, Gelfand’s theorem enables us to obtain the following result.

**Lemma 5.** Let

\[
\xi_{a,b,c} := \frac{a\alpha}{(\alpha - 1)\Gamma(2\alpha)} + \frac{b(\alpha - 1)(\alpha - 2)}{\Gamma(2\alpha)} + \frac{c}{\Gamma(\alpha)},
\]

\[
\eta_{a,b,c} := \frac{a}{\alpha(\alpha - 1)\Gamma^2(\alpha)} + \frac{b}{(\alpha - 1)\Gamma(2\alpha - 2)} + \frac{c}{\Gamma(\alpha)}.
\]

Then

\[
\frac{\xi_{a,b,c} \Gamma(\alpha + 1) \Gamma(\alpha - 1)}{\Gamma(2\alpha)} \leq r(L_{a,b,c}) \leq \frac{\eta_{a,b,c}}{\alpha(\alpha - 1)}.
\]

**Proof.** By (7), (10), and (11), we obtain

\[
\| L_{a,b,c} \| = \max_{t \in [0,1]} \int_0^1 G_{a,b,c}(t, s) \, ds \leq \eta_{a,b,c} \int_0^1 (1 - s)^{\alpha - 2} \, ds = \frac{\eta_{a,b,c}}{\alpha(\alpha - 1)}.
\]

Similarly, we find, for all $n \in \mathbb{N}_+$,

$$
\|L^n_{a,b,c}\| = \max_{t \in [0,1]} \int_0^1 \cdots \int_0^1 G_{a,b,c}(t, s_{n-2}) \cdots G_{a,b,c}(s_2, s_1) G_{a,b,c}(s_1, s) \cdots G_{a,b,c}(s, \tau) \, ds_{n-2} \cdots ds_1 \, d\tau \\
\leq \left[ \frac{\eta_{a,b,c}}{\alpha(\alpha - 1)} \right]^n.
$$

Gelfand’s theorem implies that

$$
r(L_{a,b,c}) = \lim_{n \to \infty} \sqrt[n]{\|L^n_{a,b,c}\|} \leq \frac{\eta_{a,b,c}}{\alpha(\alpha - 1)}.
$$

On the other hand,

$$
\|L_{a,b,c}\| = \max_{t \in [0,1]} \int_0^1 G_{a,b,c}(t, s) \, ds \geq \max_{t \in [0,1]} \int_0^1 \xi_{a,b,c} t^{\alpha-1} s(1-s)^{\alpha-2} \, ds \\
= \frac{\xi_{a,b,c}}{\alpha(\alpha - 1)}.
$$

Similarly, we also obtain

$$
\|L^2_{a,b,c}\| = \max_{t \in [0,1]} \int_0^1 \int_0^1 G_{a,b,c}(t, s)G_{a,b,c}(s, \tau) \, d\tau \, ds \\
\geq \max_{t \in [0,1]} \int_0^1 \int_0^1 \xi^2_{a,b,c} t^{\alpha-1} s(1-s)^{\alpha-2} \tau(1-\tau)^{\alpha-2} \, d\tau \, ds \\
= \xi^2_{a,b,c} \int_0^1 s^{\alpha}(1-s)^{\alpha-2} \, ds \int_0^1 \tau(1-\tau)^{\alpha-2} \, d\tau \\
\text{and}
$$

$$
\|L^3_{a,b,c}\| \geq \xi^3_{a,b,c} \left( \int_0^1 s^{\alpha}(1-s)^{\alpha-2} \, ds \right)^2 \int_0^1 \tau(1-\tau)^{\alpha-2} \, d\tau.
$$

Therefore, for all $n \in \mathbb{N}_+$,

$$
\|L^n_{a,b,c}\| \geq \xi^n_{a,b,c} \left( \int_0^1 s^{\alpha}(1-s)^{\alpha-2} \, ds \right)^{n-1} \int_0^1 \tau(1-\tau)^{\alpha-2} \, d\tau.
$$
By Gelfand’s theorem, we see
\[
r(L_{a,b,c}) = \lim_{n \to \infty} \sqrt[n]{\|L_{a,b,c}^n\|} \geq \xi_{a,b,c} \int_0^1 s^\alpha (1-s)^{\alpha-2} \, ds
\]
\[
= \frac{\xi_{a,b,c} \Gamma(\alpha + 1) \Gamma(\alpha - 1)}{\Gamma(2\alpha)}.
\]
This completes the proof. □

By Lemma 5, we see \( r(L_{a,b,c}) > 0 \), and thus the Krein–Rutman theorem [9] asserts that there are \( \varphi_{a,b,c} \in P \setminus \{0\} \) and \( \psi_{a,b,c} \in P \setminus \{0\} \) such that
\[
\int_0^1 G_{a,b,c}(t,s)\varphi_{a,b,c}(s) \, ds = r(L_{a,b,c})\varphi_{a,b,c}(t),
\]
\[
\int_0^1 G_{a,b,c}(t,s)\psi_{a,b,c}(t) \, dt = r(L_{a,b,c})\psi_{a,b,c}(s).
\]
(15)

Note that we can normalize \( \psi_{a,b,c} \) such that
\[
\int_0^1 \psi_{a,b,c}(t) \, dt = 1.
\]
(16)

Let \( \omega_{a,b,c} = \xi_{a,b,c} \eta_{a,b,c}^{-1} \int_0^1 t^{\alpha-1} \psi_{a,b,c}(t) \, dt \) and define
\[
P_0 := \left\{ v \in P : \int_0^1 v(t)\psi_{a,b,c}(t) \, dt \geq \omega_{a,b,c}\|v\| \right\}.
\]
Clearly, \( P_0 \) is also a cone of \( E \).

**Lemma 6.** \( L_{a,b,c}(P) \subset P_0 \).

**Proof.** We easily have the following inequality:
\[
G_{a,b,c}(t,s) \geq \xi_{a,b,c} \eta_{a,b,c}^{-1} t^{\alpha-1} G_{a,b,c}(\tau,s) \quad \forall t, s, \tau \in [0,1].
\]
For \( v(t) \geq 0, \ t \in [0,1] \), we have
\[
\int_0^1 (L_{a,b,c}v)(t)\psi_{a,b,c}(t) \, dt = \int_0^1 \int_0^1 G_{a,b,c}(t,s)v(s)\psi_{a,b,c}(t) \, ds \, dt
\]

\[ \int_0^1 \int_0^1 \xi_{a,b,c}^{-1} \eta_{a,b,c}^{-1} t^{\alpha - 1} G_{a,b,c}(\tau, s) v(s) \psi_{a,b,c}(t) \, ds \, dt \]

\[ = \xi_{a,b,c}^{-1} \eta_{a,b,c}^{-1} \int_0^1 t^{\alpha - 1} \psi_{a,b,c}(t) \, dt \int_0^1 G_{a,b,c}(\tau, s) v(s) \, ds \quad \forall \tau \in [0, 1]. \]

Consequently, we see

\[ \int_0^1 (L_{a,b,c} v)(t) \psi_{a,b,c}(t) \, dt \geq \omega_{a,b,c} \| L_{a,b,c} v \|. \]

This completes the proof. \( \square \)

**Lemma 7.** (See [8].) Let \( E \) be a real Banach space and \( W \) a cone of \( E \). Suppose that \( A : (\overline{B}_R \setminus B_r) \cap W \to W \) is a completely continuous operator with \( 0 < r < R \). If either

(i) \( Au \leq u \) for each \( \partial B_r \cap W \) and \( Au \not\leq u \) for each \( \partial B_R \cap W \) or

(ii) \( Au \not\leq u \) for each \( \partial B_r \cap W \) and \( Au \leq u \) for each \( \partial B_R \cap W \),

then \( A \) has at least one fixed point on \( (\overline{B}_R \setminus B_r) \cap W \).

**Lemma 8.** (See [3].) Let \( E \) be a partial order Banach space, and \( x_0, y_0 \in E \) with \( x_0 \leq y_0, D = [x_0, y_0] \). Suppose that \( A : D \to E \) satisfies the following conditions:

(i) \( A \) is an increasing operator;

(ii) \( x_0 \leq Ax_0, y_0 \geq Ay_0 \), i.e., \( x_0 \) and \( y_0 \) is a subsolution and a supersolution of \( A \);

(iii) \( A \) is a completely continuous operator.

Then \( A \) has the smallest fixed point \( x^* \) and the largest fixed point \( y^* \) in \([x_0, y_0]\), respectively. Moreover, \( x^* = \lim_{n \to \infty} A^n x_0 \) and \( y^* = \lim_{n \to \infty} A^n y_0 \).

## 3 Main results

We first offer twelve fixed numbers \( \alpha_i, \beta_i, \gamma_i \geq 0 \) which are not all zero and let \( r^{-1}(L_{\alpha_1, \beta_1, \gamma_1}) = \lambda_{\alpha_1, \beta_1, \gamma_1} \) for \( i = 1, 2, 3, 4 \). Now, we list our assumptions on \( f \):

(H1) \( f \in C([0, 1] \times \mathbb{R}_+^3, \mathbb{R}_+) \); \hspace{1cm} (H1)’ \( f \in C([0, 1] \times \mathbb{R}_+^3, (0, +\infty)) \).

(H2) \[ \liminf_{\alpha_1 x_1 + \beta_1 x_2 + \gamma_1 x_3 \to +\infty} \frac{f(t, x_1, x_2, x_3)}{\alpha_1 x_1 + \beta_1 x_2 + \gamma_1 x_3} > \lambda_{\alpha_1, \beta_1, \gamma_1} \quad (17) \]

uniformly for \( t \in [0, 1] \).

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Proof. (H2) implies that there are
\[ \text{(H5)} \lim_{\alpha x_1 + \beta_1 x_2 + \gamma_2 x_3 \to 0^+} \frac{f(t, x_1, x_2, x_3)}{\alpha_2 x_1 + \beta_2 x_2 + \gamma_2 x_3} < \lambda_{\alpha_2, \beta_2, \gamma_2} \] (18)
unifies for \( t \in [0, 1] \).

(H4) \[ \liminf_{\alpha x_1 + \beta_1 x_2 + \gamma_2 x_3 \to 0^+} \frac{f(t, x_1, x_2, x_3)}{\alpha_3 x_1 + \beta_3 x_2 + \gamma_3 x_3} > \lambda_{\alpha_3, \beta_3, \gamma_3} \] (19)
unifies for \( t \in [0, 1] \).

(H5) \[ \limsup_{\alpha x_1 + \beta_1 x_2 + \gamma_2 x_3 \to +\infty} \frac{f(t, x_1, x_2, x_3)}{\alpha_4 x_1 + \beta_4 x_2 + \gamma_4 x_3} < \lambda_{\alpha_4, \beta_4, \gamma_4} \] (20)
unifies for \( t \in [0, 1] \).

(H6) There exists a positive constant \( \mu < 1 \) such that
\[ \kappa^\mu f(t, x_1, x_2, x_3) \leq f(t, \kappa x_1, \kappa x_2, \kappa x_3) \quad \forall \kappa \in (0, 1). \]

(H7) \( f(t, x_1, x_2, x_3) \) is increasing in \( x_1, x_2, x_3 \), that is, the inequality
\[ f(t, x_1, x_2, x_3) \leq f(t, x_1', x_2', x_3') \]
holds for \( x_1 \leq x_1', x_2 \leq x_2', x_3 \leq x_3' \).

3.1 Existence of positive solutions

Theorem 1. Assume that (H1)–(H3) hold. Then (1) has at least one positive solution.

Proof. (H2) implies that there are \( \varepsilon > 0 \) and \( c_1 > 0 \) such that
\[ f(t, x_1, x_2, x_3) \geq (\lambda_{\alpha_1, \beta_1, \gamma_1} + \varepsilon)(\alpha x_1 + \beta_1 x_2 + \gamma_1 x_3) - c_1 \quad \forall x_i \in \mathbb{R}^+, t \in [0, 1]. \] (21)

Let \( \mathcal{M}_1 := \{ v \in P : v \geq A v \} \). We claim that \( \mathcal{M}_1 \) is bounded in \( P \). Indeed, if \( v \in \mathcal{M}_1 \), by (12) and (21), we can obtain
\[
v(t) \geq \frac{1}{0} G_1(t, s) f \left( \frac{1}{0} G_1(s, \tau) v(\tau) d\tau, \frac{1}{0} G_2(s, \tau) v(\tau) d\tau, v(s) \right) ds \geq (\lambda_{\alpha_1, \beta_1, \gamma_1} + \varepsilon) \left[ \int_0^1 \alpha_1 G_3(t, \tau) v(\tau) d\tau + \int_0^1 \beta_1 G_4(t, \tau) v(\tau) d\tau + \int_0^1 \gamma_1 G_1(t, s) v(s) ds \right] - \frac{c_1}{\alpha(\alpha - 1)\Gamma(\alpha)} \]
\[ = (\lambda_{\alpha_1, \beta_1, \gamma_1} + \varepsilon) \int_0^1 G_{\alpha_1, \beta_1, \gamma_1}(t, s) v(s) ds - \frac{c_1}{\alpha(\alpha - 1)\Gamma(\alpha)}. \] (22)
Multiply (22) by \( \psi_{\alpha_1, \beta_1, \gamma_1}(t) \) on both sides and integrate over \([0, 1]\) and use (15), (16) to obtain
\[
\int_0^1 v(t) \psi_{\alpha_1, \beta_1, \gamma_1}(t) \, dt \geq \frac{\lambda_{\alpha_1, \beta_1, \gamma_1}}{\lambda_{\alpha_1, \beta_1, \gamma_1}} + \varepsilon \int_0^1 v(t) \psi_{\alpha_1, \beta_1, \gamma_1}(t) \, dt - \frac{c_1}{\alpha(\alpha - 1)\Gamma(\alpha)}.
\tag{23}
\]
Therefore, we have
\[
\int_0^1 v(t) \psi_{\alpha_1, \beta_1, \gamma_1}(t) \, dt \leq \varepsilon^{-1}\lambda_{\alpha_1, \beta_1, \gamma_1} c_1 \tag{24}
\]
Consequently, Lemma 6 implies that
\[
\omega_{\alpha_1, \beta_1, \gamma_1} \|v\| \leq \varepsilon^{-1}\lambda_{\alpha_1, \beta_1, \gamma_1} c_1, \tag{25}
\]
and hence,
\[
\|v\| \leq \frac{\varepsilon^{-1}\omega_{\alpha_1, \beta_1, \gamma_1} \lambda_{\alpha_1, \beta_1, \gamma_1} c_1}{\alpha(\alpha - 1)\Gamma(\alpha)}, \tag{26}
\]
for all \( v \in \mathcal{M}_1 \). Taking \( R > \sup\{\|v\|: v \in \mathcal{M}_1\} \), we obtain
\[
v \not\geq Av \quad \forall v \in \partial B_R \cap P. \tag{27}
\]
On the other hand, by (H3), there exist \( r \in (0, R) \) and \( \varepsilon \in (0, \lambda_{\alpha_2, \beta_2, \gamma_2}) \) such that
\[
f(t, x_1, x_2, x_3) \leq (\lambda_{\alpha_2, \beta_2, \gamma_2} - \varepsilon) (\alpha_2 x_1 + \beta_2 x_2 + \gamma_2 x_3) \tag{28}
\]
for all \( x_i \in [0, r] \) and \( t \in [0, 1] \). This implies that
\[
(Av)(t) \leq (\lambda_{\alpha_2, \beta_2, \gamma_2} - \varepsilon) \int_0^1 G_1(t, s) \, ds \\
\times \left( \alpha_2 \int_0^1 G_1(s, \tau) v(\tau) \, d\tau + \beta_2 \int_0^1 G_2(s, \tau) v(\tau) \, d\tau + \gamma_2 v(s) \right) \, ds \\
= (\lambda_{\alpha_2, \beta_2, \gamma_2} - \varepsilon) \int_0^1 G_{\alpha_2, \beta_2, \gamma_2}(t, s) v(s) \, ds \tag{29}
\]
for all \( v \in \overline{B}_r \cap P \). Let \( \mathcal{M}_2 := \{v \in \overline{B}_r \cap P: v \leq Av\} \). Now, we claim \( \mathcal{M}_2 = \{0\} \). Indeed, if \( v \in \mathcal{M}_2 \), by (29), we have
\[
v(t) \leq (\lambda_{\alpha_2, \beta_2, \gamma_2} - \varepsilon) \int_0^1 G_{\alpha_2, \beta_2, \gamma_2}(t, s) v(s) \, ds.
\]
Multiply (22) by $\psi_{\alpha_2, \beta_2, \gamma_2}(t)$ on both sides and integrate over $[0, 1]$ and use (15), (16) to obtain
\[
\int_0^1 v(t)\psi_{\alpha_2, \beta_2, \gamma_2}(t) \, dt \leq (\lambda_{\alpha_2, \beta_2, \gamma_2} - \varepsilon)\lambda_{\alpha_2, \beta_2, \gamma_2}^{-1} \int_0^1 v(t)\psi_{\alpha_2, \beta_2, \gamma_2}(t) \, dt
\]
and thus $\int_0^1 v(t)\psi_{\alpha_2, \beta_2, \gamma_2}(t) \, dt = 0$. Consequently, we have $v(t) \equiv 0$, i.e., $\mathcal{M}_2 = \{0\}$. Therefore,
\[
v \not\in Av \quad \forall v \in \partial B_r \cap P.
\] (30)
Now Lemma 7 indicates that the operator $A$ has at least one fixed point on $(B_r \setminus \overline{B}_r) \cap P$. That is, (1) has at least one positive solution. This completes the proof.

**Theorem 2.** Assume that (H1), (H4) and (H5) hold. Then (1) has at least one positive solution.

**Proof.** By (H4), there exist $r > 0$ and $\varepsilon > 0$ such that
\[
f(t, x_1, x_2, x_3) \geq (\lambda_{\alpha_3, \beta_3, \gamma_3} + \varepsilon)(\alpha_3 x_1 + \beta_3 x_2 + \gamma_3 x_3) \quad \forall x_i \in [0, r], \ t \in [0, 1].
\] (31)
This implies
\[
(Av)(t) \geq (\lambda_{\alpha_3, \beta_3, \gamma_3} + \varepsilon)\int_0^1 G_{\alpha_3, \beta_3, \gamma_3}(t, s)v(s) \, ds
\] (32)
for all $v \in \overline{B}_r \cap P$. Let $\mathcal{M}_3 := \{v \in \overline{B}_r \cap P: v \geq Av\}$. We claim that $\mathcal{M}_3 = \{0\}$. Indeed, if $v \in \mathcal{M}_3$, combining with (32), we find
\[
v(t) \geq (\lambda_{\alpha_3, \beta_3, \gamma_3} + \varepsilon)\int_0^1 G_{\alpha_3, \beta_3, \gamma_3}(t, s)v(s) \, ds.
\] (33)
Multiply (33) by $\psi_{\alpha_3, \beta_3, \gamma_3}(t)$ on both sides and integrate over $[0, 1]$ and use (15), (16) to obtain
\[
\int_0^1 v(t)\psi_{\alpha_3, \beta_3, \gamma_3}(t) \, dt \geq (\lambda_{\alpha_3, \beta_3, \gamma_3} + \varepsilon)\lambda_{\alpha_3, \beta_3, \gamma_3}^{-1} \int_0^1 v(t)\psi_{\alpha_3, \beta_3, \gamma_3}(t) \, dt
\]
and thus $\int_0^1 v(t)\psi_{\alpha_3, \beta_3, \gamma_3}(t) \, dt = 0$. Hence, we see $v(t) \equiv 0$, i.e., $\mathcal{M}_3 = \{0\}$. Consequently,
\[
v \not\in Av \quad \forall v \in \partial B_r \cap P.
\] (34)
In addition, by (H5), there exist $\varepsilon \in (0, \lambda_{\alpha_4, \beta_4, \gamma_4})$ and $c_2 > 0$ such that
\[
f(t, x_1, x_2, x_3) \leq (\lambda_{\alpha_4, \beta_4, \gamma_4} - \varepsilon)(\alpha_4 x_1 + \beta_4 x_2 + \gamma_4 x_3) + c_2 \quad \forall x_i \geq 0, \ t \in [0, 1].
\] (35)
Let $\mathcal{M}_4 := \{ v \in P : v \leq Av \}$. We shall prove that $\mathcal{M}_4$ is bounded in $P$. Indeed, if $v \in \mathcal{M}_4$, then we have

$$v(t) \leq (\lambda_{\alpha_4, \beta_4, \gamma_4} - \varepsilon) \int_0^1 G_{\alpha_4, \beta_4, \gamma_4}(t,s) v(s) \, ds + \frac{c_2}{\alpha(\alpha - 1)\Gamma(\alpha)}. \quad (36)$$

Multiply (36) by $\psi_{\alpha_4, \beta_4, \gamma_4}(t)$ on both sides and integrate over $[0, 1]$ and use (15), (16) to obtain

$$\int_0^1 v(t)\psi_{\alpha_4, \beta_4, \gamma_4}(t) \, dt \leq (\lambda_{\alpha_4, \beta_4, \gamma_4} - \varepsilon)\lambda_{\alpha_4, \beta_4, \gamma_4}^{-1} \int_0^1 v(t)\psi_{\alpha_4, \beta_4, \gamma_4}(t) \, dt + \frac{c_2}{\alpha(\alpha - 1)\Gamma(\alpha)}$$

and then

$$\int_0^1 v(t)\psi_{\alpha_4, \beta_4, \gamma_4}(t) \, dt \leq \frac{\varepsilon^{-1}\lambda_{\alpha_4, \beta_4, \gamma_4}^{-1} c_2}{\alpha(\alpha - 1)\Gamma(\alpha)}.$$ 

It follows from Lemma 6 that

$$\|v\| \leq \frac{\varepsilon^{-1}\lambda_{\alpha_4, \beta_4, \gamma_4}^{-1} c_2}{\alpha(\alpha - 1)\Gamma(\alpha)} \quad (37)$$

for all $v \in \mathcal{M}_4$. Choosing $R > \sup\{\|v\| : v \in \mathcal{M}_4\}$ and $R > r$, we have

$$v \notin Av \quad \forall v \in \partial B_R \cap P. \quad (38)$$

Now Lemma 7 implies that $A$ has at least one fixed point on $(B_R \setminus \overline{P}_r) \cap P$. Equivalently, (1) has at least one positive solution. This completes the proof. \qed

### 3.2 Uniqueness of positive solutions

In order to obtain our main results in this subsection, we first offer some lemmas. From now on, we always assume that (H1)' holds.

**Lemma 9.** If $v(t) \in C[0, 1]$ is a positive fixed point of $A$ in (12), then there exist two positive constants $a_v$ and $b_v$ such that $a_v \rho(t) \leq v(t) \leq b_v \rho(t)$, where $\rho(t) = \int_0^t G_1(t,s) \, ds$.

**Proof.** The continuity of $G_1$, $G_2$ and $v$ implies that there exists $M > 0$ such that $|v(t)| \leq M$ and $\int_0^1 G_1(t,s)v(s) \, ds \leq M$ for all $t \in [0, 1]$. Taking

$$a_v = \min_{(t,x_1,x_2,x_3) \in [0,1] \times [0,M]^3} f(t,x_1,x_2,x_3) > 0,$$

$$b_v = \max_{(t,x_1,x_2,x_3) \in [0,1] \times [0,M]^3} f(t,x_1,x_2,x_3) > 0,$$

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Then we have
\[ a_i \rho(t) \leq v(t) = (Av)(t) \]
\[ = \int_0^1 G_1(t, s)f\left(s, \int_0^1 G_1(s, \tau)v(\tau) d\tau, \int_0^1 G_2(s, \tau)v(\tau) d\tau, v(s)\right) ds \]
\[ \leq b_i \rho(t). \]

This completes the proof. \(\square\)

**Lemma 10.** Suppose that (H1)', (H4)–(H7) hold. Then the operator \(A\) has exactly one positive fixed point.

**Proof.** By Theorem 2, \(A\) has at least one positive fixed point. It then remains to prove that \(A\) has at most one positive fixed point. Indeed, if \(v_1\) and \(v_2\) are two positive fixed points of \(A\), then

\[ v_i(t) = \int_0^1 G_1(t, s)f\left(s, \int_0^1 G_1(s, \tau)v_1(\tau) d\tau, \int_0^1 G_2(s, \tau)v_1(\tau) d\tau, v_i(s)\right) ds, \]

where \(i = 1, 2\). By Lemma 9, there exist four positive numbers \(a_i, b_i\) for which \(a_i \rho(t) \leq v_i(t) \leq b_i \rho(t)\) for \(t \in [0, 1]\) and \(i = 1, 2\). Clearly, \(v_2 \geq (a_2/b_1)v_1\).

Let \(\gamma_0 := \sup\{\gamma > 0: v_2 \geq \gamma v_1\} \neq \emptyset\). Then \(\gamma_0 > 0\) and \(v_2 \geq \gamma_0 v_1\). We shall claim that \(\gamma_0 \geq 1\). Suppose the contrary. Then \(\gamma_0 < 1\) and

\[ v_2(t) \geq \int_0^1 G_1(t, s)f\left(s, \int_0^1 G_1(s, \tau)\gamma_0 v_1(\tau) d\tau, \int_0^1 G_2(s, \tau)\gamma_0 v_1(\tau) d\tau, \gamma_0 v_1(s)\right) ds \]
\[ = \int_0^1 G_1(t, s)g(s) ds + \gamma_0^2 v_1(t), \]

where

\[ g(s) = f\left(s, \int_0^1 G_1(s, \tau)\gamma_0 v_1(\tau) d\tau, \int_0^1 G_2(s, \tau)\gamma_0 v_1(\tau) d\tau, \gamma_0 v_1(s)\right) \]
\[ - \gamma_0^2 f\left(s, \int_0^1 G_1(s, \tau)v_1(\tau) d\tau, \int_0^1 G_2(s, \tau)v_1(\tau) d\tau, v_1(s)\right). \]

(H6) implies that \(g \in P \setminus \{0\}\) and there is a \(a_3 > 0\) such that \(\int_0^1 G_1(t, s)g(s) ds \geq a_3 \rho(t)\) by Lemma 9. Consequently, \(v_2(t) \geq a_3 \rho(t) + \gamma_0^2 v_1(t) \geq (a_3/b_1) v_1(t) + \gamma_0 v_1(t)\), which contradicts the definition of \(\gamma_0\). As a result, \(\gamma_0 \geq 1\) and \(v_2 \geq v_1\). Similarly, \(v_1 \geq v_2\). Hence, \(v_1 = v_2\). This completes the proof. \(\square\)
Proof. Clearly, $\rho(t) = \int_0^1 G_1(t, s) \, ds$ is a bounded function on $[0, 1]$. Then by Lemma 9, there exist $\alpha, \beta > 0$ such that

$$a_\alpha \rho(t) \leq \int_0^1 G_1(t, s) f \left( s, \int_0^1 G_1(s, \tau) \rho(\tau) \, d\tau, \int_0^1 G_2(s, \tau) \rho(\tau) \, d\tau, \rho(s) \right) \, ds$$

$$= \eta(t) \leq b_\beta \rho(t).$$

Let $\beta_1(t) = \delta \eta(t)$ with $0 < \delta \leq \min\{1/b_\beta, a_\alpha^{\mu/(1-\mu)}\}$. Then we can choose $0 < \epsilon < \min\{1/a_\beta, b_\beta^{\mu/(1-\mu)}\}$, and

$$(\lambda \epsilon \beta_1)(t)$$

$$= \int_0^1 G_1(t, s) f \left( s, \int_0^1 G_1(s, \tau) \epsilon \beta_1(\tau) \, d\tau, \int_0^1 G_2(s, \tau) \epsilon \beta_1(\tau) \, d\tau, \epsilon \beta_1(s) \right) \, ds$$

$$= \int_0^1 G_1(t, s) f \left( s, \int_0^1 \frac{G_1(s, \tau) \epsilon \beta_1(\tau)}{\rho(\tau)} \, d\tau, \int_0^1 \frac{G_2(s, \tau) \epsilon \beta_1(\tau)}{\rho(\tau)} \, d\tau, \frac{\epsilon \beta_1(s)}{\rho(s)} \, ds \right)$$

$$\geq \epsilon^{a}(\epsilon \alpha_\beta)^{\mu} \int_0^1 G_1(t, s) f \left( s, \int_0^1 G_1(s, \tau) \rho(\tau) \, d\tau, \int_0^1 G_2(s, \tau) \rho(\tau) \, d\tau, \rho(s) \right) \, ds$$

$$= \epsilon^{a}(\epsilon \alpha_\beta)^{\mu} \eta(t) \geq \epsilon^{a} \delta \eta(t) \geq \epsilon \delta \eta(t) = \epsilon \beta_1(t).$$

Thus we have $\lambda \epsilon \beta_1 \geq \epsilon \beta_1$. On the other hand, let $\beta_2(t) = \xi \eta(t)$ with $\xi > \max\{1/a_\beta, b_\beta^{\mu/(1-\mu)}\}$. Taking $\bar{\eta} > \max\{1, b_\beta\}$, we find

$$(\lambda \beta_2)(t) \geq \lambda \bar{\eta} \xi \eta(t)$$

$$= \frac{\lambda \xi}{1} \int_0^1 G_1(t, s) f \left( s, \int_0^1 G_1(s, \tau) \rho(\tau) \, d\tau, \int_0^1 G_2(s, \tau) \rho(\tau) \, d\tau, \rho(s) \right) \, ds$$

$$\geq \frac{\lambda \xi}{1} \int_0^1 G_1(t, s) f \left( s, \int_0^1 G_1(s, \tau) \rho(\tau) \bar{\beta}_2(\tau) \, d\tau, \bar{\beta}_2(\tau) \right) \, ds$$

$$\geq \frac{\lambda \xi}{1} \int_0^1 G_1(t, s) f \left( s, \int_0^1 \frac{G_1(s, \tau) \rho(\tau) \bar{\beta}_2(\tau)}{\bar{\beta}_2(\tau)} \, d\tau, \frac{\rho(s) \bar{\beta}_2(s)}{\bar{\beta}_2(s)} \right) \, ds$$

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Example 1. Let \( \alpha = 2.5, \alpha_1 = \Gamma^2(\alpha) = 9\pi/16 \approx 1.77, \beta_1 = \Gamma(2\alpha) = 24, \gamma_1 = \Gamma(\alpha) = 3\sqrt{\pi}/4 \approx 1.33. \) Then by Lemma 5, we get \( 0.23 < r(L_{1,\beta_1,\gamma_1}) < 2.47, \) and \( 0.40 < \lambda_{1,\beta_1,\gamma_1} < 4.35. \)

Let

\[
 f(t, x_1, x_2, x_3) = \frac{1}{4} |\sin(\alpha_1 x_1 + \beta_1 x_2 + \gamma_1 x_3)| + (\alpha_1 x_1 + \beta_1 x_2 + \gamma_1 x_3)^2.
\]

Then

\[
\lim\inf_{\alpha_1 x_1 + \beta_1 x_2 + \gamma_1 x_3 \to +\infty} \frac{\frac{1}{4} |\sin(\alpha_1 x_1 + \beta_1 x_2 + \gamma_1 x_3)| + (\alpha_1 x_1 + \beta_1 x_2 + \gamma_1 x_3)^2}{\alpha_1 x_1 + \beta_1 x_2 + \gamma_1 x_3} = \infty > \lambda_{1,\beta_1,\gamma_1},
\]

and

\[
\lim\sup_{\alpha_1 x_1 + \beta_1 x_2 + \gamma_1 x_3 \to 0^+} \frac{\frac{1}{4} |\sin(\alpha_1 x_1 + \beta_1 x_2 + \gamma_1 x_3)| + (\alpha_1 x_1 + \beta_1 x_2 + \gamma_1 x_3)^2}{\alpha_1 x_1 + \beta_1 x_2 + \gamma_1 x_3} = \frac{1}{4} < \lambda_{1,\beta_1,\gamma_1}.
\]
uniformly for $t \in [0, 1]$. All conditions of Theorem 1 hold. Therefore, (1) has at least one positive solution.

**Example 2.** Let $\alpha = 2.5$, $\alpha_3 = \Gamma^2(\alpha) = 9\pi/16 \approx 1.77$, $\beta_3 = \Gamma(2\alpha - 2) = 2$, $\gamma_3 = \Gamma(\alpha - 1) = \sqrt{\pi}/2 \approx 0.89$. Then from Lemma 5 we have $0.10 \leq r(L_{\alpha_3, \beta_3, \gamma_3}) \leq 0.43$, and $2.33 \leq \lambda_{\alpha_3, \beta_3, \gamma_3} \leq 10$.

Let $$f(t, x_1, x_2, x_3) = e^t + \ln\left(1 + (\alpha_3 x_1 + \beta_3 x_2 + \gamma_3 x_3)\right).$$

Then
$$\liminf_{\alpha_3 x_1 + \beta_3 x_2 + \gamma_3 x_3 \to 0^+} \frac{e^t + \ln(1 + (\alpha_3 x_1 + \beta_3 x_2 + \gamma_3 x_3))}{\alpha_3 x_1 + \beta_3 x_2 + \gamma_3 x_3} = \infty > \lambda_{\alpha_3, \beta_3, \gamma_3}$$

and
$$\limsup_{\alpha_3 x_1 + \beta_3 x_2 + \gamma_3 x_3 \to +\infty} \frac{e^t + \ln(1 + (\alpha_3 x_1 + \beta_3 x_2 + \gamma_3 x_3))}{\alpha_3 x_1 + \beta_3 x_2 + \gamma_3 x_3} = 0 < \lambda_{\alpha_3, \beta_3, \gamma_3}$$

uniformly for $t \in [0, 1]$. Hence, (H4), (H5) hold, and Theorem 2 implies that (1) has at least one positive solution.

**Example 3.** Let $\alpha = 2.5$, $\alpha_3 = \Gamma(2\alpha - 2) = 4$, $\beta_3 = \gamma_3 = \Gamma(\alpha - 2) = \sqrt{\pi} \approx 1.77$. By Lemma 5, we can obtain $\lambda_{\alpha_3, \beta_3, \gamma_3} \in [1.48, 4.90]$.

Let $$f(t, x_1, x_2, x_3) = e^t + \sqrt{\alpha_3 x_1 + \beta_3 x_2 + \gamma_3 x_3}.$$ 

Similar with Example 2, we can show (H4) and (H5) hold. On the other hand, for any $\kappa \in (0, 1)$, we have $\sqrt{\kappa} \leq 1$ and

$$\sqrt{\kappa}\left[e^t + \sqrt{\alpha_3 x_1 + \beta_3 x_2 + \gamma_3 x_3}\right]$$

$$= \sqrt{\kappa}e^t + \sqrt{\alpha_3 \kappa x_1 + \beta_3 \kappa x_2 + \gamma_3 \kappa x_3} \leq e^t + \sqrt{\alpha_3 \kappa x_1 + \beta_3 \kappa x_2 + \gamma_3 \kappa x_3}.$$

As a result, (H6) is also satisfied. In addition, (H1)' and (H7) automatically hold. Hence, from Theorem 3, (1) has a unique positive solution.

**References**


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