Random convolution of $\mathcal{O}$-exponential distributions

Svetlana Danilenko, Jonas Šiaulys

$^a$Faculty of Fundamental Sciences, Vilnius Gediminas Technical University
Saulėtekio ave. 11, LT-10223 Vilnius, Lithuania
svetlana.danilenko@vgtu.lt

$^b$Faculty of Mathematics and Informatics, Vilnius University
Naugarduko str. 24, LT-03225 Vilnius, Lithuania
jonas.siaulys@mif.vu.lt

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Abstract. Assume that $\xi_1, \xi_2, \ldots$ are independent and identically distributed non-negative random variables having the $\mathcal{O}$-exponential distribution. Suppose that $\eta$ is a nonnegative non-degenerate at zero integer-valued random variable independent of $\xi_1, \xi_2, \ldots$. In this paper, we consider the conditions for $\eta$ under which the distribution of random sum $\xi_1 + \xi_2 + \cdots + \xi_\eta$ remains in the class of $\mathcal{O}$-exponential distributions.

Keywords: long tail, random sum, closure property, $\mathcal{O}$-exponential distribution.

1 Introduction

Let $\xi_1, \xi_2, \ldots$ be independent copies of a random variable (r.v.) $\xi$ with distribution function (d.f.) $F_\xi$. Let $\eta$ be a nonnegative non-degenerate at zero integer-valued r.v. independent of $\{\xi_1, \xi_2, \ldots\}$. We suppose that $F_\xi$ is $\mathcal{O}$-exponential and we find minimal conditions under which the d.f.

$$F_{S_\eta}(x) := \mathbb{P}(\xi_1 + \xi_2 + \cdots + \xi_\eta \leq x)$$

$$= \sum_{n=0}^{\infty} \mathbb{P}(\eta = n) \mathbb{P}(\xi_1 + \xi_2 + \cdots + \xi_n \leq x)$$

$$= \sum_{n=0}^{\infty} \mathbb{P}(\eta = n) F_\xi^{*n}(x)$$

belongs to the class of $\mathcal{O}$-exponential distributions as well. Here and elsewhere in this paper, $F_\xi^{*n}$ denotes the n-fold convolution of d.f. $F$. Theorem 1 below is the main result of this paper. Before the exact formulation of this theorem, we recall the definition of $\mathcal{O}$-exponential and some related d.f.’s classes. In all definitions below, we assume that $F(x) = 1 - F(x) > 0$ for all $x \in \mathbb{R}$.

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Definition 1. For $\gamma > 0$, by $L(\gamma)$ we denote the class of exponential d.f.s, i.e. $F \in L(\gamma)$ if for any fixed real $y$,
\[
\lim_{x \to \infty} \frac{F(x + y)}{F(x)} = e^{-\gamma y}.
\]
In the case $\gamma = 0$, class $L(0)$ is called the long-tailed distribution class and is denoted by $L$.

Definition 2. A d.f. $F$ belongs to the dominated varying-tailed class ($F \in D$) if for any fixed $y \in (0, 1)$,
\[
\limsup_{x \to \infty} \frac{F(xy)}{F(x)} < \infty.
\]

Definition 3. A d.f. $F$ is $O$-exponential ($F \in OL$) if for any fixed $y \in \mathbb{R}$,
\[
0 < \liminf_{x \to \infty} \frac{F(x + y)}{F(x)} \leq \limsup_{x \to \infty} \frac{F(x + y)}{F(x)} < \infty.
\]

It is easy to see that the following inclusions hold:
\[
\mathcal{D} \subset OL, \quad L \subset OL, \quad \bigcup_{\gamma \geq 0} L(\gamma) \subset OL.
\]

In [2, 3], Cline claimed that d.f. $F_{\xi_1 + \xi_2 + \cdots + \xi_\eta}$ remains in the class $L(\gamma)$ if $F_\xi \in L(\gamma)$ and $\eta$ is any nonnegative non-degenerate at zero integer-valued r.v. Albin [1] observed that Cline’s result is false in general. He obtained that d.f. $F_{\xi_1 + \xi_2 + \cdots + \xi_\eta}$ remains in the class $L(\gamma)$ and $Ee^{\delta \eta} < \infty$ for each $\delta > 0$. In order to prove this claim, author used the upper estimate
\[
\frac{F_{\xi_1 + \xi_2 + \cdots + \xi_\eta}(x - t)}{F_{\xi_1 + \xi_2 + \cdots + \xi_\eta}(x)} \leq (1 + \varepsilon)e^{\delta t}, \quad (1)
\]
provided that $\varepsilon > 0$, $t \in \mathbb{R}$, $F \in L(\gamma)$, $x \geq n(c_1 - t) + t$ and $c_1 = c_1(\varepsilon, t)$ is sufficiently large such that
\[
\frac{F(x - t)}{F(x)} \leq (1 + \varepsilon)e^{\delta t}
\]
for $x \geq c_1$ (see [1, Lemma 1]). Unfortunately, the obtained estimate holds for positive $t$ only. If $t$ is negative, then the above estimate is incorrect in general. This fact was shown by Watanabe and Yamamuro (see [8, Remark 6.1]). Thus, the Cline proposition that $P(\xi_1 + \xi_2 + \cdots + \xi_\eta \leq x)$ belongs to the class $L(\gamma)$ remains not proved.

In this paper, we investigate a wider class, $OL$, instead of the class $L(\gamma)$. We show that the d.f. of the sum $\xi_1 + \xi_2 + \cdots + \xi_\eta$ remains in the class $OL$, if r.v. $\eta$ satisfies the conditions similar to that in [1]. The following theorem is the main statement in this paper.

**Theorem 1.** Let $\xi_1, \xi_2, \ldots$ be independent copies of a nonnegative r.v. $\xi$ with d.f. $F_\xi$. Let $\eta$ be a nonnegative, non-degenerate at zero, integer-valued and independent of $\{\xi_1, \xi_2, \ldots\}$ r.v. with d.f. $F_\eta$. If $F_\xi$ belongs to the class $OL$ and $\overline{F_\eta}(\delta x) = O(\sqrt{x}F_\xi(x))$ for each $\delta \in (0, 1)$, then $F_{\sigma_1 + \sigma_2 + \cdots + \sigma_\eta} \in OL$. 

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A detailed proof of Theorem 1 is presented in Section 3. Note that the proof is similar to that of Theorem 6 in [5].

The following assertion actually shows that Albin’s conditions for the counting r.v. \( \eta \) are sufficient for d.f. \( F_{S_n} \) to remain in the class \( \mathcal{OL} \). The proof of the following corollary is also presented in Section 3.

**Corollary 1.** Let \( \xi_1, \xi_2, \ldots \) be a sequence of independent nonnegative r.v.s with common d.f. \( F_\xi \in \mathcal{OL} \).

(i) D.f. \( P(\xi_1 + \cdots + \xi_n \leq x) \) belongs to the class \( \mathcal{OL} \) for each fixed \( n \in \mathbb{N} \).

(ii) Let \( \eta \) be a r.v. which is nonnegative, non-degenerate at zero, integer-valued and independent of \( \{\xi_1, \xi_2, \ldots\} \). If \( \mathbb{E}e^{\varepsilon \eta} < \infty \) for each \( \varepsilon > 0 \), then \( F_{S_n} \in \mathcal{OL} \).

**2 Auxiliary lemmas**

Before proving our main results, we give three auxiliary lemmas. The first lemma is well known classical estimate for the concentration function of a sum of independent and identically distributed r.v.s. The proof of Lemma 1 can be found in [6] (see Theorem 2.22), for instance.

**Lemma 1.** Let \( X_1, X_2, \ldots, \) be a sequence of independent r.v.s with a common non-degenerate d.f. Then there exists a constant \( c_2 \), independent of \( \lambda \) and \( n \), such that

\[
\sup_{x \in \mathbb{R}} P(x \leq X_1 + X_2 + \cdots + X_n \leq x + \lambda) \leq c_2(\lambda + 1)n^{-1/2}
\]

for all \( \lambda \geq 0 \) and all \( n \in \mathbb{N} \).

The second auxiliary lemma is due to Shimura and Watanabe (see [7, Prop. 2.2]). The lemma describes an important property of a d.f. from the class \( \mathcal{OL} \).

**Lemma 2.** Let \( F \) be a d.f. from the class \( \mathcal{OL} \). Then there exists positive \( \Delta \) such that

\[
\lim_{x \to \infty} e^{\Delta x} F(x) = \infty.
\]

The last auxiliary lemma is crucial in the proof of Theorem 1. The elements of the statement below can be found in [4] (see the proof of Theorem 3(b)). Inequality (1), which is a particular case of the statement below, is proved in [1] (see Lemma 2.1). Leipus and Šiaulys [5] generalized Albin’s inequality (1) for an arbitrary d.f. with unbounded support. The analytical proof of Lemma 3 is given in [5] (see proof of Lemma 4). In this paper, we present another, completely probabilistic proof of the lemma below having in mind the importance of the statement.

**Lemma 3.** Let d.f. \( F \) be such that \( F(x) > 0 \) for all \( x \in \mathbb{R} \). Suppose that

\[
\sup_{x \geq d_2} \frac{F(x - t)}{F(x)} \leq d_1
\]
for some positive constants \( t, d_1 \) and \( d_2 > t \). Then, for all \( n = 1, 2, \ldots \), we have:

\[
\sup_{x \geq n(d_2 - t) + t} \frac{F^{n+1}(x - t)}{F^n(x)} \leq d_1.
\]

Proof of Lemma 3. Let \( X \) be a r.v. with d.f. \( F \). Then the condition of Lemma 3 says that

\[
\sup_{x \geq d_2} \frac{P(X > x - t)}{P(X > x)} \leq d_1
\]

for some positive \( t, d_1, d_2 > t \), and we need to prove that

\[
\sup_{x \geq (nd_2 - t) + t} \frac{P(S_n^X > x - t)}{P(S_n^X > x)} \leq d_1
\]

for all \( n \in \mathbb{N} \), where \( S_n^X = X_1 + \cdots + X_n \), and \( X_1, X_2, \ldots \) are independent copies of \( X \).

The proof is proceeded by induction on \( n \). According to condition (2), inequality (3) holds for \( n = 1 \). Suppose now that \( N \geq 1 \). For arbitrary real \( x, z \) and \( t > 0 \), we obtain

\[
P(S_{N+1}^X > x) = P(S_N^X + X_{N+1} > x, X_{N+1} \leq x - z) + P(S_N^X + X_{N+1} > x, S_N^X \leq z) + P(X_{N+1} > x - z) P(S_N^X > z) \
\geq P(S_N^X > x - X_{N+1}, x - X_{N+1} \geq z) + P(X_{N+1} > x - S_N^X, x - S_N^X \geq x - z + t) + P(X_{N+1} > x - z) P(S_N^X > z).
\]

If we replace \( x \) by \( x - t \) and \( z \) by \( z - t \) then we get

\[
P(S_{N+1}^X > x - t) = P(S_N^X + X_{N+1} > x - t, X_{N+1} \leq x - z) + P(S_N^X + X_{N+1} > x - t, S_N^X \leq z - t) + P(X_{N+1} > x - z) P(S_N^X > z - t) \
= P(S_N^X > x - X_{N+1} - t, x - X_{N+1} \geq z) + P(X_{N+1} > x - S_N^X - t, x - S_N^X \geq x - z + t) + P(X_{N+1} > x - z) P(S_N^X > z - t).
\]

R.v.s \( X_1, X_2, \ldots \) are independent. Therefore,

\[
P(S_N^X > x - X_{N+1} - t, x - X_{N+1} \geq z) \
= E(E(1_{\{S_N^X > x - X_{N+1} - t\}} 1_{\{x - X_{N+1} = y\}} | x - X_{N+1} = y)) \
= E(1_{\{y \geq z\}} E(1_{\{S_N^X > y - t\}} | x - X_{N+1} = y)) \
= E(1_{\{y \geq z\}} P(S_N^X > y - t))
\]

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\[ \leq \sup_{y \geq z} \frac{\mathbb{P}(S_N^X > y - t)}{\mathbb{P}(S_N^X > y)} \mathbb{E}(1_{(y \geq z)} \mathbb{P}(S_N^X > y)) \]

\[ = \sup_{y \geq z} \frac{\mathbb{P}(S_N^X > y - t)}{\mathbb{P}(S_N^X > y)} \mathbb{P}(S_N^X > x - X_{N+1}, x - X_{N+1} \geq z), \quad (6) \]

where \(1_A\) denotes the indicator function of an event \(A\). Similarly,

\[ \mathbb{P}(X_{N+1} > x - S_N^X - t, x - S_N^X \geq x - z + t) \]

\[ \leq \sup_{y \geq x - z + t} \frac{\mathbb{P}(X_{N+1} > y - t)}{\mathbb{P}(X_{N+1} > y)} \mathbb{P}(X_{N+1} > x - S_N^X, x - S_N^X \geq x - z + t). \quad (7) \]

Using estimates (4)–(7), we obtain

\[ \frac{\mathbb{P}(S_N^X_{N+1} > x - t)}{\mathbb{P}(S_N^X_{N+1} > x)} \leq \max \left\{ \sup_{y \geq z} \frac{\mathbb{P}(S_N^X > y - t)}{\mathbb{P}(S_N^X > y)}, \sup_{y \geq x - z + t} \frac{\mathbb{P}(X > y - t)}{\mathbb{P}(X > y)} \right\} \quad (8) \]

if \(x, z \in \mathbb{R}, t > 0\) and \(N \geq 1\).

Suppose now that (3) is satisfied for \(n = N\). We will show that (3) holds for \(n = N + 1\).

Condition (2) and estimate (8) imply, taking \(z = z_N = Nx/(N+1) + t/(N+1)\) and \(w_N = x - z_N + t = x/(N+1) + Nt/(N+1),\) that

\[ \frac{\mathbb{P}(S_N^X_{N+1} > x - t)}{\mathbb{P}(S_N^X_{N+1} > x)} \leq \max \left\{ \sup_{y \geq z_N} \frac{\mathbb{P}(S_N^X > y - t)}{\mathbb{P}(S_N^X > y)}, \sup_{y \geq w_N} \frac{\mathbb{P}(X > y - t)}{\mathbb{P}(X > y)} \right\} \leq d_1 \]

if \(x \geq (N + 1)(d_2 - t) + t,\) because, in this case,

\[ z_N \geq N(d_2 - t) + t \quad \text{and} \quad w_N \geq d_2. \]

So, estimate (3) holds for \(n = N + 1\) and the validity of (3) for all \(n\) follows by induction.

\[ \square \]

3 Proofs of main results

In this section, we present detailed proofs of our main results.

**Proof of Theorem 1.** First, we show that

\[ \limsup_{x \to \infty} \frac{F_{S_a}(x - a)}{F_{S_a}(x)} = \limsup_{x \to \infty} \frac{\mathbb{P}(S_a > x - a)}{\mathbb{P}(S_a > x)} < \infty \quad (9) \]

for each \(a \in \mathbb{R}\).

If \(a \leq 0\), then \(\mathbb{P}(S_a > x - a) \leq \mathbb{P}(S_a > x)\) for all \(x \in \mathbb{R},\) and estimate (9) is obvious.

Suppose now that $a > 0$. Since $F_\xi \in OL$, we derive that
\[
\limsup_{x \to \infty} \frac{F_\xi(x-a)}{F_\xi(x)} = c_3
\]
for some finite positive quantity $c_3$ maybe depending on $a$. So, there exists some $K = K_a > a + 1$ such that
\[
\sup_{x \geq K} \frac{F_\xi(x-a)}{F_\xi(x)} \leq 2c_3.
\]
Applying Lemma 3, we obtain that
\[
\sup_{x > n(K-a)+a} \frac{P(S_n > x-a)}{P(S_n > x)} = \sup_{x > n (K-a)+a} \frac{F_{\eta n}(x-a)}{F_{\eta n}(x)} \leq 2c_3,
\]
where and below $S_n = \xi_1 + \xi_2 + \cdots + \xi_n$ if $n \in \mathbb{N}$.

For an arbitrarily chosen positive $x$, we have
\[
P(S_\eta > x) = \sum_{n=1}^{\infty} P(S_n > x)P(\eta = n) \geq \sum_{n=1}^{\infty} P(\xi_1 > x)P(\eta = n)
= F_\xi(x)P(\eta \geq 1).
\]
If $x \geq K$, then, using (12), we get:
\[
P(S_\eta > x-a) = P\left(S_\eta > x-a, \eta \leq \frac{x-a}{K-a}\right) + P\left(S_\eta > x-a, \eta > \frac{x-a}{K-a}\right)
\leq 2c_3 \sum_{n \leq (x-a)/(K-a)} P(S_n > x)P(\eta = n)
+ \sum_{n > (x-a)/(K-a)} P(S_n > x-a)P(\eta = n).
\]
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\[ \leq c_4 \sum_{n=1}^{\infty} P(S_n > x)P(\eta = n) \]

\[ + \sum_{n>(x-a)/(K-a)} P(x-a < S_n \leq x)P(\eta = n) \]  \hspace{1cm} (14)

with $c_4 = \max\{2c_3, 1\}$.

According to Lemma 1, we obtain

\[ \sup_{x \in \mathbb{R}} P(x-a < S_n \leq x) \leq c_5(a+1) \frac{1}{\sqrt{n}}, \]

where the constant $c_5$ is independent of $a$ and $n$. Thus, inequality (14) implies

\[ P(S_\eta > x-a) \leq c_4P(S_\eta > x) + c_5(a+1) \sum_{n>(x-a)/(K-a)} \frac{P(\eta = n)}{\sqrt{n}} \]

\[ \leq c_4P(S_\eta > x) + c_5\sqrt{\frac{K-a}{x-a}}(a+1)P(\eta > \frac{x-a}{K-a}) \]  \hspace{1cm} (15)

provided that $x \geq K$.

Inequalities (13) and (15) imply that, for $x \geq K$, it holds

\[ \frac{P(S_\eta > x-a)}{P(S_\eta > x)} \leq c_4 + \frac{c_5\sqrt{K-a}(a+1)}{\sqrt{x-a}P(\eta \geq 1)F_\xi(x)} \]

Consequently,

\[ \limsup_{x \to \infty} \frac{P(S_\eta > x-a)}{P(S_\eta > x)} \]

\[ \leq c_4 + c_5 \frac{(a+1)\sqrt{K-a}}{P(\eta \geq 1)} \limsup_{x \to \infty} \frac{F_\eta((x-a)/(K-a))}{\sqrt{x-a}F_\xi(x-a)} \limsup_{x \to \infty} \frac{F_\xi(x-a)}{F_\xi(x)} \]

\[ = c_4 + c_3c_5 \frac{(a+1)\sqrt{K-a}}{P(\eta \geq 1)} \limsup_{x \to \infty} \frac{F_\eta(x/(K-a))}{\sqrt{F_\xi(x)} < \infty} \]

due to equality (10) and requirement $F_\eta(\delta x) = O(\sqrt{x}F_\xi(x))$ which holds for arbitrary $\delta \in (0, 1)$. Therefore, relation (9) is satisfied for all $a \in \mathbb{R}$.

It remains to prove that

\[ \liminf_{x \to \infty} \frac{F_{S_\eta}(x-a)}{F_{S_\eta}(x)} = \liminf_{x \to \infty} \frac{P(S_\eta > x-a)}{P(S_\eta > x)} > 0, \]

where $a$ is an arbitrarily chosen real number. But this relation follows from the proved estimate (9), because

\[ P(S_\eta > x) \geq F_\xi(x)P(\eta \geq 1) > 0 \]

for each positive number $x$, and so
\[
\liminf_{x \to \infty} \frac{P(S_n > x - a)}{P(S_n > x)} = \left( \limsup_{x \to \infty} \frac{P(S_n > x + a)}{P(S_n > x)} \right)^{-1} > 0.
\]
The last inequality, together with estimate (9), implies that d.f. $F_{S_n}$ belongs to the class $\mathcal{OL}$. Theorem 1 is proved.

Proof of Corollary 1. Part (i) of Corollary 1 is evident. So we only prove part (ii). Let $\delta \in (0, 1)$. According to the Markov inequality, we have
\[
\mathcal{F}_\eta(\delta x) = P(\eta > \delta x) = P(e^{y\eta} > e^{y\delta x}) \leq e^{-\delta y x} E e^{y\eta}
\]
for each $y > 0$. The d.f. $F_\xi$ belongs to the class $\mathcal{OL}$. Therefore, Lemma 2 implies that $e^{\Delta x} \mathcal{F}_\xi(x) \to \infty$ as $x \to \infty$, for some positive $\Delta$.

Choosing $y = \Delta/\delta > 0$ in (16), we obtain:
\[
\frac{\mathcal{F}_\eta(\delta x)}{\sqrt{x} \mathcal{F}_\xi(x)} \leq \frac{E e^{y\eta}}{\sqrt{e^{\delta y x} \mathcal{F}_\xi(x)}} = \frac{1}{\sqrt{e^{\Delta x} \mathcal{F}_\xi(x)}} e^{(\Delta/\delta) \eta} \quad x \to \infty
\]
because $E e^{\varepsilon \eta}$ is finite for an arbitrarily positive $\varepsilon$ according to the main condition of Corollary 1. The statement of Corollary 1 follows now from Theorem 1.

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References


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