On fixed points for $\alpha-\eta-\psi$-contractive multi-valued mappings in partial metric spaces

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Abstract. Recently, Samet et al. introduced the notion of $\alpha-\psi$-contractive type mappings and established some fixed point theorems in complete metric spaces. Successively, Asl et al. introduced the notion of $\alpha_*$-contractive multi-valued mappings and gave a fixed point result for these multi-valued mappings. In this paper, we establish results of fixed point for $\alpha_*$-admissible mixed multi-valued mappings with respect to a function $\eta$ and common fixed point for a pair $(S, T)$ of mixed multi-valued mappings, that is, $\alpha_*$-admissible with respect to a function $\eta$ in partial metric spaces. An example is given to illustrate our result.

Keywords: partial metric space, $\alpha-\eta-\psi$-contractive condition, $\alpha_*$-admissible pair with respect to a function $\eta$, fixed point, common fixed point.

1 Introduction

The study of iterative schemes for various classes of contractive and nonexpansive mappings is a central topic in metric fixed point theory. This research started with the work of Banach [6] who proved a classical theorem, known as the Banach contraction principle, for the existence of a unique fixed point for a contraction. The importance of this result is also in the fact that it gives the convergence of an iterative scheme to a unique fixed point. Since Banach’s result, there has been a lot of activity in this area and many developments have been taken place (see also [26]). Some authors have also provided results dealing with the existence and approximation of fixed points of certain classes of contractive multi-valued mappings [7, 8, 12, 17, 21, 22].

Let $(X, d)$ be a metric space and let $CB(X)$ denote the collection of all nonempty closed and bounded subsets of $X$. For $A, B \in CB(X)$, define

$$H(A, B) := \max \left\{ \sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A) \right\},$$

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where \( d(x, A) := \inf \{ d(x, a) : a \in A \} \) is the distance of a point \( x \) to the set \( A \). It is known that \( H \) is a metric on \( CB(X) \), called the Hausdorff metric induced by the metric \( d \).

**Definition 1.** Let \((X, d)\) be a metric space. An element \( x \) in \( X \) is said to be a fixed point of a multi-valued mapping \( T : X \to CB(X) \) if \( x \in Tx \).

We recall that \( T : X \to CB(X) \) is said to be a multi-valued contraction mapping if there exists \( k \in [0, 1) \) such that
\[
H(Tx, Ty) \leq kd(x, y) \quad \text{for all } x, y \in X.
\]

The study of fixed points for multi-valued contractions using the Hausdorff metric was initiated by Nadler [17] who proved the following theorem.

**Theorem 1.** (See [17].) Let \((X, d)\) be a complete metric space and \( T : X \to CB(X) \) be a multi-valued contraction mapping. Then there exists \( x \in X \) such that \( x \in Tx \).

Later on, an interesting and rich fixed point theory was developed. The theory of multi-valued mappings has application in control theory, convex optimization, differential equations and economics (see also [11,15]). On the other hand, Matthews [16] introduced the concept of a partial metric as a part of the study of denotational semantics of dataflow networks. He gave a modified version of the Banach contraction principle, more suitable in this context (see also [2,3,10,13,19,20,27]). In fact, (complete) partial metric spaces constitute a suitable framework to model several distinguished examples of the theory of computation and also to model metric spaces via domain theory (see, [9,14,16,23,25,28]). More recently, Aydi et al. [5] introduced a notion of partial Hausdorff metric type, associated to a partial metric, and proved an analogous to the well known Nadler’s fixed point theorem [17] in the setting of partial metric spaces. Very recently, Romaguera [24] introduced the concept of mixed multi-valued mappings, so that both a self mapping \( T : X \to X \) and a multi-valued mapping \( T : X \to CBp(X) \) (the family of all non-empty, closed and bounded subsets of a partial metric space \( X \)), are mixed multi-valued mappings. In this paper, we establish results of fixed point for \( \alpha_\ast \)-admissible mixed multi-valued mappings with respect to a function \( \eta \). Also, we prove results of common fixed point for a pair \((S,T)\) of multi-valued mappings, that is, \( \alpha_\ast \)-admissible with respect to a function \( \eta \) in the setting of partial metric spaces.

In the sequel, the letters \( \mathbb{R} \) and \( \mathbb{N} \) will denote the set of all real numbers and the set of all positive integer numbers, respectively.

### 2 Preliminaries

First, we recall some definitions of partial metric spaces that can be found in [10,16,18,19,23]. A partial metric on a nonempty set \( X \) is a function \( p : X \times X \to [0, +\infty) \) such that for all \( x, y, z \in X \):

\[
\begin{align*}
(p1) \quad & x = y \iff p(x, x) = p(x, y) = p(y, y); \\
(p2) \quad & p(x, x) \leq p(x, y);
\end{align*}
\]
Remark 1. (See [4].) Let $\{\text{Cauchy sequence}\}$ in $X$ be a family of open $p$-balls $B_p(x, \epsilon) = \{y \in X: p(x, y) < p(x, x) + \epsilon\}$ \begin{equation} B_p(x, \epsilon) = \{y \in X: p(x, y) < p(x, x) + \epsilon\} \end{equation} for all $x \in X, \epsilon > 0$.

Let $(X, p)$ be a partial metric space. A sequence $\{x_n\}$ in $(X, p)$ converges to a point $x \in X$ if and only if $p(x, x) = \lim_{n \to +\infty} p(x, x_n)$.

A sequence $\{x_n\}$ in $(X, p)$ is called a Cauchy sequence if there exists (and is finite) $\lim_{n,m \to +\infty} p(x_n, x_m)$. A partial metric space $(X, p)$ is said to be complete if every Cauchy sequence $\{x_n\}$ in $X$ converges, with respect to $\tau_p$, to a point $x \in X$ such that $p(x, x) = \lim_{n,m \to +\infty} p(x_n, x_m)$.

A sequence $\{x_n\}$ in $(X, p)$ is called 0-Cauchy if $\lim_{n,m \to +\infty} p(x_n, x_m) = 0$. We say that $(X, p)$ is 0-complete if every 0-Cauchy sequence in $X$ converges, with respect to $\tau_p$, to a point $x \in X$ such that $p(x, x) = 0$.

Now, we recall the definition of partial Hausdorff metric and some properties that can be found in [1]. Let $CB^p(X)$ be the family of all nonempty, closed and bounded subsets of the partial metric space $(X, p)$, induced by the partial metric $p$. Note that closedness is taken from $(X, \tau_p)$ and boundedness is given as follows: $A$ is a bounded subset in $(X, p)$ if there exist $x_0 \in X$ and $M \geq 0$ such that for all $a \in A$, we have $a \in B_p(x_0, M)$, that is, $p(x_0, a) < p(x_0, x_0) + M$.

For $A, B \in CB^p(X)$ and $x \in X$, define

\[ p(x, A) = \inf \{p(x, a): a \in A\}, \]
\[ \delta_p(A, B) = \sup \{p(a, B): a \in A\}, \]
\[ \delta_p(B, A) = \sup \{p(b, A): b \in B\}. \]

Remark 1. (See [4].) Let $(X, p)$ be a partial metric space and $A$ any nonempty set in $(X, p)$, then
\[ a \in \bar{A} \text{ if and only if } p(a, A) = p(a, a), \] \begin{equation} a \in \bar{A} \text{ if and only if } p(a, A) = p(a, a), \end{equation} where $\bar{A}$ denotes the closure of $A$ with respect to the partial metric $p$. Note that $A$ is closed in $(X, p)$ if and only if $A = \bar{A}$.

In the following proposition, we bring some properties of the mapping $\delta_p : CB^p(X) \times CB^p(X) \to [0, +\infty)$.

Proposition 1. (See [1, Prop. 2.2].) Let $(X, p)$ be a partial metric space. For any $A, B, C \in CB^p(X)$, we have the following:

(i) $\delta_p(A, A) = \sup \{p(a, a) : a \in A\}$;
(ii) $\delta_p(A, A) \leq \delta_p(A, B)$;
(iii) $\delta_p(A, B) = 0$ implies that $A \subseteq B$;
(iv) $\delta_p(A, B) \leq \delta_p(A, C) + \delta_p(C, B) - \inf_{c \in C} p(c, c)$.

Let $(X, p)$ be a partial metric space. For $A, B \in CB^p(X)$, define

$$H_p(A, B) = \max \{\delta_p(A, B), \delta_p(B, A)\}.$$

In the following proposition, we bring some properties of the mapping $H_p$.

**Proposition 2.** (See [1, Prop. 2.3].) Let $(X, p)$ be a partial metric space. For all $A, B, C \in CB^p(X)$, we have:

(h1) $H_p(A, A) \leq H_p(A, B)$;
(h2) $H_p(A, B) = H_p(B, A)$;
(h3) $H_p(A, B) \leq H_p(A, C) + H_p(C, B) - \inf_{c \in C} p(c, c)$.

**Corollary 1.** (See [1, Cor. 2.4].) Let $(X, p)$ be a partial metric space. For $A, B \in CB^p(X)$ the following holds:

$$H_p(A, B) = 0 \text{ implies that } A = B.$$

**Remark 2.** The converse of Corollary 1 is not true in general as shown by the following example.

**Example 1.** (See [1, Ex. 2.6].) Let $X = [0, 1]$ be endowed with the partial metric $p : X \times X \to [0, +\infty)$ defined by

$$p(x, y) = \max\{x, y\} \text{ for all } x, y \in X.$$

From (i) of Proposition 1, we have

$$H_p(X, X) = \delta_p(X, X) = \sup\{x : 0 \leq x \leq 1\} = 1 \neq 0.$$

In view of Proposition 2 and Corollary 1, we call the mapping $H_p : CB^p(X) \times CB^p(X) \to [0, +\infty)$, a partial Hausdorff metric induced by $p$.

**Remark 3.** It is easy to show that any Hausdorff metric is a partial Hausdorff metric. The converse is not true (see Example 1).

### 3 Main results

In [24], Romaguera introduced the concept of mixed multi-valued mappings as follows.

**Definition 2.** Let $(X, p)$ be a partial metric space. $T : X \to X \cup CB^p(X)$ is called a mixed multi-valued mapping on $X$ if $T$ is a multi-valued mapping on $X$ such that for each $x \in X$, $Tx \in X$ or $Tx \in CB^p(X)$.

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As said above, both a self mapping $T : X \to X$ and a multi-valued mapping $T : X \to CB^p(X)$, are mixed multi-valued mappings. This approach is motivated, in part, by the fact that $CB^p(X)$ may be empty.

Now, we consider the family
$$\Psi = \{(\psi_1, \ldots, \psi_5) : \psi_i : [0, +\infty) \to [0, +\infty), \ i = 1, \ldots, 5\}$$

such that:
(i) $\psi_2$, $\psi_5$ are nondecreasing and $\psi_4$ is increasing;
(ii) $\psi_1(t), \psi_2(t), \psi_3(t) \leq \psi_4(t)$ for all $t > 0$;
(iii) $\psi_4(s + t) \leq \psi_4(s) + \psi_4(t)$ for all $s, t > 0$;
(iv) $\psi_1(t), \psi_2(t), \psi_5(t)$ are continuous in $t = 0$ and $\psi_1(0) = \psi_2(0) = \psi_5(0) = 0$;
(v) $\sum_{n=1}^{\infty} \psi_n(t) < +\infty$ for all $t > 0$.

The following lemma is obvious.

**Lemma 1.** If $(\psi_1, \ldots, \psi_5) \in \Psi$, then $\psi_4(t) < t$ for all $t > 0$.

Let $(X, p)$ be a partial metric space and $\alpha, \eta : X \times X \to [0, +\infty)$ be two functions with $\eta$ bounded. In the sequel we denote
$$\alpha_*(A, B) = \inf_{x \in A, y \in B} \alpha(x, y) \quad \text{and} \quad \eta_*(A, B) = \sup_{x \in A, y \in B} \eta(x, y)$$

for every $A, B \subset X$.

**Definition 3.** Let $(X, p)$ be a partial metric space, $T : X \to X \cup CB^p(X)$ a mixed multi-valued mapping and $\alpha : X \times X \to [0, +\infty)$ a function. We say that $T$ is an $\alpha_*$-admissible mixed multi-valued mapping if
$$\alpha(x, y) \geq 1 \quad \text{implies} \quad \alpha_*(Tx, Ty) \geq 1, \ x, y \in X.$$ 

**Definition 4.** Let $(X, p)$ be a partial metric space, $S, T : X \to X \cup CB^p(X)$ be two mixed multi-valued mappings and $\alpha, \eta : X \times X \to [0, +\infty)$ be two functions with $\eta$ bounded. We say that the pair $(S, T)$ is $\alpha_*$-admissible with respect to $\eta$ if:
$$\alpha(x, y) \geq \eta(x, y) \quad \text{implies} \quad \alpha_*(Sx, Ty) \geq \eta_*(Sx, Ty), \ x, y \in X.$$ 

We say that $T$ is an $\alpha_*$-admissible mixed multi-valued mapping with respect to $\eta$ if the pair $(T, T)$ is $\alpha_*$-admissible with respect to $\eta$.

If we take, $\eta(x, y) = 1$ for all $x, y \in X$, then the definition of $\alpha_*$-admissible mixed multi-valued mapping with respect to $\eta$ reduces to Definition 3.

The following theorem is one of our main results.

**Theorem 2.** Let $(X, p)$ be a 0-complete partial metric space and let $T : X \to X \cup CB^p(X)$ be a mixed multi-valued mapping. Assume that there exist $(\psi_1, \ldots, \psi_5) \in \Psi$
and two functions \( \alpha, \eta : X \times X \to [0, +\infty) \) with \( \eta \) bounded, such that
\[
\inf_{u \in TX} \eta(x, u) \leq \alpha(x, y) \quad \text{implies}
H(Tx, Ty) \leq \max \left\{ \frac{\psi_1(p(x, y)), \psi_2(p(x, Tx))}{2}, \psi_3(p(y, Ty)), \frac{\psi_4(p(x, Ty)) + \psi_5(p(y, Tx) - p(y, y))}{2} \right\}
\]
(3)
for all \( x, y \in X \). Also suppose that the following assertions hold:

(i) \( T \) is an \( \alpha_\ast \)-admissible mixed multi-valued mapping with respect to \( \eta \);
(ii) there exist \( x_0 \in X \) and \( x_1 \in TX_0 \) such that \( \alpha(x_0, x_1) \geq \eta(x_0, x_1) \);
(iii) for a sequence \( \{x_n\} \subset X \) such that \( \alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1}) \) for all \( n \in \mathbb{N} \) and \( x_n \to x \) as \( n \to +\infty \), then either
\[
\inf_{u_n \in Ty_n} \eta(y_n, u_n) \leq \alpha(y_n, x) \quad \text{or} \quad \inf_{v_n \in Ty_n} \eta(z_n, v_n) \leq \alpha(z_n, x)
\]
holds for all \( n \in \mathbb{N} \), where \( \{y_n\} \) and \( \{z_n\} \) are two given sequences such that \( y_n \in Tx_n \) and \( z_n \in Ty_n \) for all \( n \in \mathbb{N} \).

Then \( T \) has a fixed point.

Proof. By (ii) there exist \( x_0 \in X \) and \( x_1 \in TX_0 \) such that \( \alpha(x_0, x_1) \geq \eta(x_0, x_1) \). This implies that \( \alpha(x_0, x_1) \geq \eta(x_0, x_1) \geq \inf_{y \in TX_0} \eta(x_0, y) \). If \( x_0 = x_1 \) or \( x_1 \in TX_1 \), then \( x_1 \) is a fixed point of \( T \). Assume that \( x_1 \notin TX_1 \) and that \( TX_1 \) is not a singleton. Therefore, from (3), we have
\[
0 < p(x_1, TX_1) \leq H(Tx_0, TX_1)
\]
\[
\leq \max \left\{ \frac{\psi_1(p(x_0, x_1)), \psi_2(p(x_0, TX_0))}{2}, \psi_3(p(x_1, TX_1)), \frac{\psi_4(p(x_0, TX_1)) + \psi_5(p(x_1, TX_0) - p(x_1, x_1))}{2} \right\}
\]
\[
\leq \max \left\{ \frac{\psi_1(p(x_0, x_1)), \psi_2(p(x_0, x_1))}{2}, \psi_3(p(x_1, TX_1)) \right\}
\]
\[
\leq \max \left\{ \psi_4(p(x_0, x_1)), \psi_4(p(x_1, TX_1)) \right\}
\]
\[
= \max \left\{ \psi_4(p(x_0, x_1)), \psi_4(p(x_1, TX_1)) \right\}.
\]
Now, if
\[
\max \left\{ \psi_4(p(x_0, x_1)), \psi_4(p(x_1, TX_1)) \right\} = \psi_4(p(x_1, TX_1)),
\]
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then
$$0 < p(x, Tx) \leq H(Tx, Tx) \leq \psi_4(p(x, Tx)) < p(x, Tx),$$
which is a contradiction. Hence,
$$0 < p(x, Tx) \leq H(Tx, Tx) \leq \psi_4(p(x_0, x_1)).$$
If \( q > 1 \), then
$$0 < p(x, Tx) < qH(Tx, Tx) \leq q\psi_4(p(x_0, x_1)).$$
So there exists \( x_2 \in Tx \) such that
$$0 < p(x, x_2) < qH(Tx, Tx) \leq q\psi_4(p(x_0, x_1)).$$
If \( Tx = \{x_2\} \) is a singleton, again by (3), we get
$$0 < p(x, x_2) \leq H(Tx, Tx) \leq \psi_4(p(x_0, x_1))$$
and so (4) holds.

Note that \( x_1 \neq x_2 \). Also, since \( T \) is \( \alpha_*-\text{admissible} \) with respect to \( \eta \), we have
$$\alpha_*(Tx, Tx) \supseteq \eta_*(Tx, Tx).$$
This implies
$$\alpha(x_1, x_2) \supseteq \alpha_*(Tx, Tx) \supseteq \eta_*(Tx, Tx) \supseteq \eta(x_1, x_2) \supseteq \inf_{y \in Tx} \eta(x_1, y).$$
Therefore, from (3), we have
$$H(Tx, Tx) \leq \max \left\{ \psi_1(p(x, x_2)), \psi_2(p(x, Tx)), \psi_3(p(x,Tx)), \psi_4(p(x, Tx)) \right\} \leq \psi_4(p(x, x_2)) \leq \psi_4(p(x_1, x_2)) \leq \psi_4(p(x_0, x_1)).$$

Put \( t_0 = p(x_0, x_1) > 0 \). Then from (4), we deduce that \( p(x_1, x_2) < q\psi_4(t_0) \). Now, since \( \psi_4 \) is increasing, we deduce \( \psi_4(p(x_1, x_2)) < \psi_4(q\psi_4(t_0)) \). Put
$$q_1 = \frac{\psi_4(q\psi_4(t_0))}{\psi_4(p(x_1, x_2))} > 1.$$
If \( x_2 \in Tx \), then \( x_2 \) is a fixed point of \( T \). Hence, we suppose that \( x_2 \notin Tx \). Then
$$0 < p(x_2, Tx) \leq H(Tx, Tx) < q_1H(Tx, Tx).$$
So there exists \( x_3 \in Tx \) (obviously \( x_3 = Tx \) if \( Tx \) is a singleton) such that
$$0 < p(x_2, x_3) < q_1H(Tx, Tx).$$
and from (5), we get

\[ 0 < p(x_2, x_3) < q_1 H(Tx_1, Tx_2) \leq q_1 \psi_4(p(x_1, x_2)) = \psi_4(q\psi_4(t_0)). \]

Again, since \( \psi_4 \) is increasing, then \( \psi_4(p(x_2, x_3)) < \psi_4(q\psi_4(t_0)). \) Put

\[ q_2 = \frac{\psi_4(q\psi_4(t_0))}{\psi_4(p(x_2, x_3))} > 1. \]

If \( x_3 \in Tx_3, \) then \( x_3 \) is a fixed point of \( T. \) Hence, we assume that \( x_3 \notin Tx_3. \) Then

\[ 0 < p(x_3, Tx_3) \leq H(Tx_2, Tx_3) < q_2 H(Tx_2, Tx_3). \]

So there exists \( x_4 \in Tx_3 \) (obviously \( x_4 = Tx_3 \) if \( Tx_3 \) is a singleton) such that

\[ 0 < p(x_3, x_4) < q_2 H(Tx_2, Tx_3). \] (6)

Clearly, \( x_2 \neq x_3. \) Again, since \( T \) is \( \alpha_* \)-admissible with respect to \( \eta, \)

\[ \alpha(x_2, x_3) \geq \alpha_*(Tx_1, Tx_2) \geq \eta_*(Tx_1, Tx_2) \geq \eta(x_2, x_3) \geq \inf_{y \in Tx_2} \eta(x_2, y). \]

From (3), we have

\[
H(Tx_2, Tx_3) \leq \max \left\{ \psi_1(p(x_2, x_3)), \psi_2(p(x_2, Tx_2)), \psi_3(p(x_3, Tx_3)), \right.
\]

\[
\left. \frac{\psi_4(p(x_2, Tx_3)) + \psi_5(p(x_3, Tx_2) - p(x_3, x_3))}{2} \right\}
\]

\[ \leq \psi_4(p(x_2, x_3)). \] (7)

Thus from (6) and (7), we deduce that

\[ 0 < p(x_3, x_4) < q_2 H(Tx_2, Tx_3) < q_2 \psi_4(p(x_2, x_3)) = \psi_4(q\psi_4(t_0)). \]

By continuing this process, we obtain a sequence \( \{x_n\} \subset X \) such that \( x_n \in Tx_{n-1}, \)

\[ x_n \neq x_{n-1}, \alpha(x_{n-1}, x_n) \geq \eta(x_{n-1}, x_n) \text{ and } p(x_n, x_{n+1}) \leq \psi_4^{k-1}(q\psi_4(t_0)) \text{ for all } n \in \mathbb{N}. \]

Now for all \( m > n, \) we can write

\[ p(x_n, x_m) \leq \sum_{k=n}^{m-1} p(x_k, x_{k+1}) \leq \sum_{k=n}^{m-1} \psi_4^{k-1}(q\psi_4(t_0)). \]

Therefore, \( \{x_n\} \) is a 0-Cauchy sequence. Since, \( (X, p) \) is a 0-complete partial metric space, then there exists \( z \in X \) such that \( p(x_n, z) \rightarrow p(z, z) = 0 \) as \( n \rightarrow +\infty. \) Then from (iii), either

\[ \inf_{u_n \in Ty_n} \eta(y_n, u_n) \leq \alpha(y_n, z) \quad \text{or} \quad \inf_{v_n \in Tz_n} \eta(z_n, v_n) \leq \alpha(z_n, z) \]

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holds for all \( n \in \mathbb{N} \), where \( \{y_n\} \) and \( \{z_n\} \) are two given sequences such that \( y_n \in Tx_n \) and \( z_n \in Ty_n \) for all \( n \in \mathbb{N} \). Here \( x_{n-1} \in Tx_{n-2} \) and \( x_n \in Tx_{n-1} \).

Therefore, either

\[
\inf_{u_n \in Tx_{n-1}} \eta(x_{n-1}, u_n) \leq \alpha(x_{n-1}, z) \quad \text{or} \quad \inf_{v_n \in Tx_n} \eta(x_n, v_n) \leq \alpha(x_n, z)
\]

holds for all \( n \in \mathbb{N} \). If \( p(z, Tz) > 0 \), from (3), we have

\[
p(z, Tz) \leq H(Tx_{n-1}, Tz) + p(x_n, z) - p(x_n, x_n)
\]

\[
\leq \max \left\{ \psi_1(p(x_{n-1}, z)), \psi_2(p(x_{n-1}, Tx_{n-1})), \psi_3(p(z, Tz)), \frac{\psi_4(p(x_{n-1}, Tz)) + \psi_5(p(z, Tx_{n-1}))}{2} \right\} + p(x_n, z)
\]

or

\[
p(z, Tz) \leq H(Tx_n, Tz) + p(x_{n+1}, z) - p(x_{n+1}, x_{n+1})
\]

\[
\leq \max \left\{ \psi_1(p(x_n, z)), \psi_2(p(x_n, Tx_n)), \psi_3(p(z, Tz)), \frac{\psi_4(p(x_n, Tz)) + \psi_5(p(z, Tx_n))}{2} \right\} + p(x_{n+1}, z)
\]

\[
\leq \max \left\{ \psi_1(p(x_n, z)), \psi_2(p(x_n, x_{n+1})), \psi_3(p(z, Tz)), \frac{\psi_4(p(x_n, z) + p(z, Tz)) + \psi_5(p(z, x_{n+1}))}{2} \right\} + p(x_{n+1}, z)
\]

for all \( n \in \mathbb{N} \). Taking limit as \( n \to +\infty \) in the above inequalities, we get

\[
p(z, Tz) \leq \psi_4(p(z, Tz)) < p(z, Tz)
\]

a contradiction. Thus \( p(z, Tz) = 0 \). If \( Tz \) is a singleton, then \( z = Tz \). If \( Tz \) is not a singleton, from \( p(z, Tz) = 0 = p(z, z) \), by Remark 1, we deduce \( z \in Tz \). Thus \( z \) is a fixed point of \( T \).

If in Theorem 2, we assume \( \eta(x, y) = 1 \) for all \( x, y \in X \), then we obtain the following corollary.

**Corollary 2.** Let \( (X, p) \) be a \( 0 \)-complete partial metric space and let \( T : X \to X \cup CB^p(X) \) be a mixed multi-valued mapping. Assume that there exist \( (\psi_1, \ldots, \psi_5) \in \Psi \) where

and a function \( \alpha : X \times X \to [0, +\infty) \), such that
\[
H(Tx, Ty) \leq \max \left\{ \psi_1(p(x, y)), \psi_2(p(x, Tx)), \psi_3(p(y, Ty)), \psi_4(p(x, Ty)) + \psi_5(p(y, Tx) - p(y, y)) \right\}
\]
for all \( x, y \in X \) with \( \alpha(x, y) \geq 1 \). Also suppose the following assertions hold:

(i) \( T \) is an \( \alpha \)-admissible mixed multi-valued mapping;
(ii) there exist \( x_0 \in X \) and \( x_1 \in Tx_0 \) such that \( \alpha(x_0, x_1) \geq 1 \);
(iii) for a sequence \( \{x_n\} \subset X \) such that \( \alpha(x_n, x_{n+1}) \geq 1 \) for all \( n \in \mathbb{N} \) and \( x_n \to x \) as \( n \to +\infty \), then either
\[
\alpha(y_n, x) \geq 1 \quad \text{or} \quad \alpha(z_n, x) \geq 1
\]
holds for all \( n \in \mathbb{N} \) where \( \{y_n\} \) and \( \{z_n\} \) are two given sequences such that \( y_n \in Tx_n \) and \( z_n \in Ty_n \) for all \( n \in \mathbb{N} \).

Then \( T \) has a fixed point.

Example 2. Let \( X = \{1, 2, 3, 4\} \) and \( p : X \times X \to [0, +\infty) \) be defined by \( p(1, 1) = p(2, 2) = p(4, 4) = 1/6 \), \( p(3, 3) = 0 \), \( p(1, 2) = p(1, 4) = p(2, 4) = p(3, 4) = 1/2 \), \( p(1, 3) = 1/4 \), \( p(2, 3) = 1/3 \) and \( p(x, y) = p(y, x) \) for all \( x, y \in X \). Let \( T : X \to CB^p(X) \) be defined by \( T1 = \{3\}, T2 = \{1\}, T3 = \{3\} \) and \( T4 = \{1, 4\} \). Clearly, \( (X, p) \) is a 0-complete partial metric space and \( Tx \) is a bounded closed subset of \( X \) for all \( x \in X \). Let \( \alpha : X \times X \to [0, +\infty) \) be defined by \( \alpha(1, 1) = \alpha(1, 3) = \alpha(2, 3) = \alpha(3, 3) = \alpha(3, 1) = \alpha(3, 2) = 1 \) and \( \alpha(x, y) = 0 \) otherwise. Now, let \( \psi_1, \psi_2, \psi_3, \psi_4, \psi_5 : [0, +\infty) \to [0, +\infty) \) be defined by \( \psi_1(t) = t/2, \psi_2(t) = 2t/3, \psi_3(t) = t/2, \psi_4(t) = 3t/4 \) and \( \psi_5(t) = 5t/6 \), then \( \psi_1, \psi_2, \psi_3, \psi_4, \psi_5 \in \Psi \).

Now, we have:
\[
H(T1, T1) = H(\{3\}, \{3\}) = 0 \leq \psi_1(p(1, 1)),
H(T1, T3) = H(\{3\}, \{3\}) = 0 \leq \psi_1(p(1, 3)),
H(T2, T3) = H(\{1\}, \{3\}) = 0.25 \leq \psi_3(p(2, \{1\})),
H(T3, T3) = H(\{3\}, \{3\}) = 0 \leq \psi_1(p(3, 3)).
\]

This implies
\[
H(Tx, Ty) \leq \max \left\{ \psi_1(p(x, y)), \psi_2(p(x, Tx)), \psi_3(p(y, Ty)), \psi_4(p(x, Ty)) + \psi_5(p(y, Tx) - p(y, y)) \right\}
\]
for all \( x, y \in X \) with \( \alpha(x, y) \geq 1 \). \( T \) is an \( \alpha \)-admissible mixed multi-valued mapping and \( x_0 = 1 \) satisfies condition (ii). Now, we note that for a sequence \( \{x_n\} \subset X \) such that

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\(\alpha(x_n, x_{n+1}) \geq 1\) for all \(n \in \mathbb{N}\) and \(x_n \to x\) as \(n \to +\infty\), we have \(x = 3\) and this ensures that (iii) holds. Thus, by Corollary 2 the mixed multi-valued mapping \(T\) has a fixed point.

We note that
\[
H(T2, T4) = \frac{1}{2} > \max \left\{ \psi_1(p(2, 4)), \psi_2(p(2, T2)), \psi_3(p(4, T4)), \psi_4(p(2, T4)) + \psi_5(p(4, T2) - p(4, 4)) \right\}.
\]

4 Common fixed point results

Let \((X, p)\) be a partial metric space, let \(\alpha, \eta : X \times X \to [0, +\infty)\) be two functions with \(\eta\) bounded and let \(S, T : X \to 2^X\) be two multi-valued mappings on \(X\). We denote
\[
\Gamma(Sx, Ty) = \min \left\{ \inf_{u \in Sx} \eta(x, u), \inf_{v \in Ty} \eta(y, v) \right\} = \Gamma(Ty, Sx).
\]

Let \(\Phi = \{ (\psi_1, \ldots, \psi_5) : \psi_i : [0, +\infty) \to [0, +\infty), \, i = 1, \ldots, 5 \} \) such that:

(i) \(\psi_2, \psi_3\) are nondecreasing and \(\psi_4, \psi_5\) are increasing;
(ii) \(\psi_1(t), \psi_2(t), \psi_3(t) \leq \min \{ \psi_4(t), \psi_5(t) \}\) for all \(t > 0\);
(iii) \(\psi_1(s + t) \leq \psi_1(s) + \psi_1(t)\) (\(i = 4, 5\)) for all \(s, t > 0\);
(iv) \(\psi_1(t), \psi_2(t)\) and \(\psi_3(t)\) are continuous in \(t = 0\) and \(\psi_1(0) = \psi_2(0) = \psi_3(0) = 0\);
(v) \(\sum_{n=1}^{\infty} \psi_5^n(t) < +\infty\) for all \(t > 0\);
(vi) \(\psi_4(t) < t\) for all \(t > 0\);
(vii) \(\psi_4(\psi_5(t)) = \psi_5(\psi_4(t))\) for all \(t > 0\).

The following theorem is our main result on the existence of common fixed point for multi-valued mappings.

Theorem 3. Let \((X, p)\) be a 0-complete partial metric space and let \(S, T : X \to X \cup CB^1(X)\) be two mixed multi-valued mappings on \(X\). Assume that there exist \((\psi_1, \ldots, \psi_5) \in \Phi\) and two functions \(\alpha, \eta : X \times X \to [0, +\infty)\) with \(\eta\) bounded such that
\[
H(Sx, Ty) \leq \max \left\{ \psi_1(p(x, y)), \psi_2(p(x, Sx)), \psi_3(p(y, Ty)), \frac{\psi_4(p(x, Ty) - p(x, x)) + \psi_5(p(y, Tx) - p(y, y))}{2} \right\}
\]
for all \(x, y \in X\) with \(\alpha(x, y) \geq \Gamma(Sx, Ty)\). Also suppose the following assertions hold:

(i) the pair \((S, T)\) is \(\alpha, \eta\)-admissible with respect to \(\eta\);
(ii) there exist \(x_0 \in X\) and \(x_1 \in Sx_0\) such that \(\alpha(x_0, x_1) \geq \eta(x_0, x_1)\);
(iii) \(\alpha(x, x) \geq \Gamma(Sx, Tx)\) for all \(x \in X\), which is a fixed point of \(S\) or \(T\);

(iv) for a sequence \( \{x_n\} \subset X \) such that \( \alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1}) \) for all \( n \in \mathbb{N} \) and \( x_n \to x \) as \( n \to +\infty \), then either

\[
\inf_{u_n \in Sx_n} \eta(y_n, u_n) \leq \alpha(y_n, x) \quad \text{or} \quad \inf_{v_n \in Tx_n} \eta(z_n, v_n) \leq \alpha(z_n, x)
\]

holds for all \( n \in \mathbb{N} \) where \( \{y_n\} \) and \( \{z_n\} \) are two given sequences such that \( y_n \in Tx_n \) and \( z_n \in Sx_n \) for all \( n \in \mathbb{N} \).

Then \( S \) and \( T \) have a common fixed point.

**Proof.** From (iii) and (9) it follows that the mixed multi-valued mappings \( S \) and \( T \) have the same fixed points. Let \( x_0 \in X \) and \( x_1 \in Sx_0 \) be such that \( \alpha(x_0, x_1) \geq \eta(x_0, x_1) \), then

\[
\alpha(x_0, x_1) \geq \eta(x_0, x_1) \geq \inf_{u \in Sx_0} \eta(x_0, u) \geq \Gamma(Sx_0, Tx_1).
\]

If \( x_0 = x_1 \), then \( x_0 \) is a common fixed point of \( S \) and \( T \). The same holds if \( x_1 \in Tx_1 \). Hence, we assume that \( x_0 \neq x_1 \) and \( x_1 \notin Tx_1 \). Assume that \( Tx_1 \) is not a singleton, from (9), we have

\[
0 < p(x_1, Tx_1) \leq H(Sx_0, Tx_1)
\leq \max \left\{ \psi_1(p(x_0, x_1)), \psi_2(p(x_0, Sx_0)), \psi_3(p(x_1, Tx_1)), \psi_4(p(x_0, Tx_1) - p(x_0, x_0)) + \psi_5(p(x_1, Sx_0) - p(x_1, x_1)) \right\} \frac{1}{2}
\leq \max \left\{ \psi_1(p(x_0, x_1)), \psi_2(p(x_0, x_1)), \psi_3(p(x_1, Tx_1)), \psi_4(p(x_0, x_1) - p(x_1, x_1) - p(x_0, x_0)) \right\} \frac{1}{2}
\leq \max \left\{ \psi_1(p(x_0, x_1)), \psi_2(p(x_0, x_1)), \psi_3(p(x_1, Tx_1)), \max \left\{ \psi_4(p(x_0, x_1)), \psi_4(p(x_1, Tx_1)) \right\} \right\}
= \max \left\{ \psi_4(p(x_0, x_1)), \psi_4(p(x_1, Tx_1)) \right\}.
\]

Now, if \( \max \{\psi_4(p(x_0, x_1)), \psi_4(p(x_1, Tx_1))\} = \psi_4(p(x_1, Tx_1)) \), then

\[
0 < p(x_1, Tx_1) \leq H(Sx_0, Tx_1) \leq \psi_4(p(x_1, Tx_1)) < p(x_1, Tx_1),
\]

which is a contradiction. Hence,

\[
\max \left\{ \psi_4(p(x_0, x_1)), \psi_4(p(x_1, Tx_1)) \right\} = \psi_4(p(x_0, x_1)).
\]

If \( q > 1 \), then

\[
0 < p(x_1, Tx_1) \leq H(Sx_0, Tx_1) < qH(Sx_0, Tx_1)
\]
and hence there exists \( x_2 \in Tx_1 \) such that
\[
0 < p(x_1, x_2) < qH(Sx_0, Tx_1) \leq \psi_4(p(x_0, x_1)).
\] (10)

If \( Tx_1 = \{x_2\} \) is a singleton, again by (9), we get
\[
0 < p(x_1, x_2) \leq H(Sx_0, Tx_1) \leq \psi_4(p(x_0, x_1))
\]
and so (10) holds. Note that \( x_1 \neq x_2 \). Also, since the pair \((S, T)\) is \( \alpha_* \)-admissible with respect to \( \eta \), then \( \alpha_*(Sx_0, Ty) \geq \eta_*(Sx_0, Ty) \). This implies
\[
\alpha(x_1, x_2) \geq \alpha_*(Sx_0, Tx_1) \geq \eta_*(Sx_0, Tx_1) \geq \eta(x_1, x_2)
\]
\[
\geq \inf_{y \in Tx_1} \eta(x_1, y) \geq H(Sx_2, Tx_1).
\]

If \( x_2 \in Sx_2 \), then \( x_2 \) is a common fixed point of \( S \) and \( T \). Assume that \( x_2 \notin Sx_2 \) and that \( Sx_2 \) is not a singleton, from (9), we have
\[
0 < p(x_2, Sx_2) \leq H(Sx_2, Tx_1)
\]
\[
\leq \max \left\{ \psi_1(p(x_2, x_1)), \psi_2(p(x_2, Sx_2)), \psi_3(p(x_1, Tx_1)), \right. \\
\left. \psi_4(p(x_2, Tx_1) - p(x_2, x_2)) + \psi_5(p(x_1, Sx_2) - p(x_1, x_1)) \right\}
\]
\[
\leq \max \left\{ \psi_1(p(x_1, x_2)), \psi_2(p(x_2, Sx_2)), \psi_3(p(x_1, x_2)), \right. \\
\left. \psi_5(p(x_1, x_2) + p(x_2, Sx_2) - p(x_2, x_2) - p(x_1, x_1)) \right\}
\]
\[
\leq \max \{ \psi_5(p(x_1, x_2)), \psi_5(p(x_2, Sx_2)) \}.
\]

Now, if \( \max \{ \psi_5(p(x_1, x_2)), \psi_5(p(x_2, Sx_2)) \} = \psi_5(p(x_2, Sx_2)) \), then
\[
0 < p(x_2, Sx_2) \leq H(Sx_2, Tx_1) \leq \psi_5(p(x_2, Sx_2)) < p(x_2, Sx_2),
\]
which is a contradiction. Hence,
\[
0 < p(x_2, Sx_2) \leq H(Sx_2, Tx_1) \leq \psi_5(p(x_1, x_2)).
\] (11)

The same is worth also if \( Sx_2 \) is a singleton. Put \( t_0 = p(x_0, x_1) \). Then from (10), we have \( p(x_1, x_2) < q\psi_4(t_0) \) where \( t_0 > 0 \). Now, since \( \psi_5 \) is increasing, then \( \psi_5(p(x_1, x_2)) < \psi_5(q\psi_4(t_0)) \). Put
\[
q_1 = \frac{\psi_5(q\psi_4(t_0))}{\psi_5(p(x_1, x_2))} > 1.
\]

Since \( x_2 \in Tx_1 \) or \( x_2 = Tx_1 \), we have
\[
0 < p(x_2, Sx_2) \leq H(Sx_2, Tx_1) < q_1H(Sx_2, Tx_1)
\]
and hence there exists \( x_3 \in Sx_2 \) or \( x_3 = Sx_2 \) such that

\[
0 < p(x_2, x_3) \leq q_1 H(Sx_2, Tx_1).
\]

Now, from (11), we deduce

\[
0 < p(x_2, x_3) < q_1 H(Sx_2, Tx_1) \leq q_1 \psi_5(p(x_1, x_2)) = \psi_5(q\psi_4(t_0)).
\]

Clearly, \( x_2 \neq x_3 \). Again, since the pair \((S, T)\) is \( \alpha \)-admissible with respect to \( \eta \), then

\[
\alpha(x_2, x_3) \geq \alpha(x(Tx_1, Sx_2) \geq \eta(Tx_1, Sx_2) \geq \eta(x_2, x_3)
\]

\[
\geq \inf_{y \in Sx_2} \eta(x_2, y) \geq I(Sx_2, Tx_3).
\]

If \( x_3 \in Tx_3 \) or \( x_3 = Tx_3 \), then \( x_3 \) is a common fixed point of \( S \) and \( T \). Assume that \( x_3 \notin Tx_3 \). Now, from (9) we deduce

\[
0 < p(x_3, Tx_3) \leq H(Sx_2, Tx_3)
\]

\[
\leq \max \{ \psi_1(p(x_2, x_3)), \psi_2(p(x_2, Sx_2)), \psi_3(p(x_3, Tx_3)),
\]

\[
\frac{\psi_4(p(x_2, Tx_3) - p(x_2, x_2)) + \psi_5(p(x_3, Sx_2) - p(x_3, x_3))}{2}
\]

\[
\leq \max \{ \psi_1(p(x_2, x_3)), \psi_2(p(x_2, x_3)), \psi_3(p(x_3, Tx_3)),
\]

\[
\frac{\psi_4(p(x_2, x_3) + p(x_3, Tx_3) - p(x_3, x_3) - p(x_2, x_2))}{2}
\]

\[
\leq \max \{ \psi_4(p(x_2, x_3)), \psi_4(p(x_3, Tx_3)) \}.
\]

If \( \max \{ \psi_4(p(x_2, x_3)), \psi_4(p(x_3, Tx_3)) \} = \psi_4(p(x_3, Tx_3)) \), then

\[
0 < p(x_3, Tx_3) \leq H(Sx_2, Tx_3) \leq \psi_4(p(x_3, Tx_3)) < p(x_3, Tx_3),
\]

which is a contradiction. Hence,

\[
\max \{ \psi_4(p(x_2, x_3)), \psi_4(p(x_3, Tx_3)) \} = \psi_4(p(x_2, x_3))
\]

and so

\[
0 < p(x_3, Tx_3) \leq H(Sx_2, Tx_3) \leq \psi_4(p(x_2, x_3)). \tag{12}
\]

Again, since \( \psi_4 \) is increasing, we deduce that

\[
\psi_4(p(x_2, x_3)) < \psi_4(\psi_5(q\psi_4(t_0))).
\]

Put

\[
q_2 = \frac{\psi_4(\psi_5(q\psi_4(t_0)))}{\psi_4(p(x_2, x_3))} > 1.
\]

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Then

\[ 0 < p(x_3, Tx_3) \leq H(Sx_3, Tx_3) < q_2H(Sx_3, Tx_3) \]

and hence there exists \( x_4 \in Tx_3 \) or \( x_4 = Tx_3 \) such that

\[ 0 < p(x_3, x_4) < q_2H(Sx_3, Tx_3) \leq q_2\psi_4(p(x_2, x_3)). \]  

(13)

Now, from (12) and (13), we deduce that

\[ 0 < p(x_3, x_4) < q_2H(Sx_3, Tx_3) \leq q_2\psi_4(p(x_2, x_3)) = \psi_4(q\psi_4(t_0)). \]

By continuing this process, we obtain a sequence \( \{x_n\} \) in \( X \) such that \( x_{2n} \in Tx_{2n-1} \), \( x_{2n+1} \in Sx_{2n} \), and

\[
\begin{align*}
p(x_{2n-1}, x_{2n}) &\leq (\psi_4\psi_5)^{n-1}(q\psi_4(t_0)) \\
p(x_{2n}, x_{2n+1}) &\leq \psi_5[(\psi_4\psi_5)^{n-1}(q\psi_4(t_0))].
\end{align*}
\]

Now, for all \( m > n \), we can write

\[
p(x_{2n}, x_{2m}) \leq \sum_{k=n}^{m-1} p(x_{2k}, x_{2k+1}) + \sum_{k=n}^{m-1} p(x_{2k+1}, x_{2k+2}) \leq \sum_{k=n}^{m-1} \psi_5^k(q\psi_4(t_0)) + \sum_{k=n}^{m-1} \psi_5^k(q\psi_4(t_0)) \leq 2 \sum_{k=n}^{m-1} \psi_5^k(q\psi_4(t_0)).
\]

Since \( \sum_{k=1}^{+\infty} \psi_5^k(q\psi_4(t_0)) < +\infty \), we get \( \lim_{n \to +\infty} p(x_{2n}, x_{2m}) = 0 \). Similarly, we obtain

\[
\begin{align*}
\lim_{n \to +\infty} p(x_{2n+1}, x_{2m+1}) &= 0, \\
\lim_{n \to +\infty} p(x_{2n+1}, x_{2m}) &= 0, \\
\lim_{n \to +\infty} p(x_{2n}, x_{2m+1}) &= 0.
\end{align*}
\]

This implies that \( \lim_{n,m \to +\infty} p(x_n, x_m) = 0 \) and so \( \{x_n\} \) is a 0-Cauchy sequence. Since \( (X, p) \) is a 0-complete partial metric space, then there exists \( z \in X \) with \( p(z, z) = 0 \) such that \( x_n \to z \) as \( n \to +\infty \). Then from (ii) either

\[
\inf_{u \in S_{y_n}} \eta(y_n, u) \leq \alpha(y_n, z) \text{ or } \inf_{v \in T_{x_n}} \eta(z_n, v) \leq \alpha(z_n, z)
\]

holds for all \( n \in \mathbb{N} \), where \( \{y_n\} \) and \( \{z_n\} \) are two given sequences such that \( y_n \in Tx_n \) and \( z_n \in Sy_n \) for all \( n \in \mathbb{N} \). Here \( x_{2n} \in Tx_{2n-1} \) and \( x_{2n+1} \in Sx_{2n} \). Therefore, either

\[
\begin{align*}
\inf_{u \in S_{x_{2n}}} \eta(x_{2n}, u) &\leq \alpha(x_{2n}, z) \text{ or } \inf_{v \in T_{x_{2n+1}}} \eta(x_{2n+1}, v) \leq \alpha(x_{2n+1}, z)
\end{align*}
\]

holds for all \( n \in \mathbb{N} \). So from (9) and \( p(z, z) = 0 \) we have
\[0 < p(z, Tz) \leq H(Sx_{2n}, Tz) + p(x_{2n+1}, z) - p(x_{2n+1}, x_{2n+1})\]

\[\leq \max\left\{\psi_1(p(x_{2n}, z)), \psi_2(p(x_{2n}, Sx_{2n})), \psi_3(p(z, Tz)), \frac{\psi_4(p(x_{2n}, Tz) - p(x_{2n}, Sx_{2n})) + \psi_5(p(z, Sx_{2n}))}{2}\right\}\]

\[+ p(x_{2n+1}, z)\]

or

\[0 < p(z, Sz) \leq H(Tx_{2n+1}, Sz) + p(x_{2n+2}, z) - p(x_{2n+2}, x_{2n+2})\]

\[\leq \max\left\{\psi_1(p(x_{2n+1}, z)), \psi_2(p(z, Sz)), \psi_3(p(x_{2n+1}, Tx_{2n+1})), \frac{\psi_4(p(z, Tx_{2n+1}) + \psi_5(p(x_{2n+1}, Sz) - p(x_{2n+1}, x_{2n+1}))}{2}\right\}\]

\[+ p(x_{2n+2}, z)\]

for all \(n \in \mathbb{N}\). Taking limit as \(n \to +\infty\) in above inequalities we get

\[p(z, Tz) \leq \psi_4(p(z, Tz)) \quad \text{or} \quad p(z, Sz) \leq \psi_5(p(z, Sz))\]

and hence \(p(z, Tz) = 0\) or \(p(z, Sz) = 0\). This implies that \(z\) is a fixed point of \(T\) or \(S\), and hence \(z\) is a common fixed point of the mixed multi-valued mappings \(S\) and \(T\). \(\square\)

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**References**


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