Value Distribution of General Dirichlet Series. VI

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Abstract. In the paper a limit theorem in the sense of weak convergence of probability measures on the complex plane for a new class of general Dirichlet series is obtained.

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1 Introduction

Let \( s = \sigma + it \) be a complex variable, and let \( \mathbb{N}, \mathbb{R} \) and \( \mathbb{C} \) denote the sets of all positive integers, real and complex numbers, respectively. The series of the form

\[
\sum_{m=1}^{\infty} a_m e^{-\lambda_m s},
\]

where \( a_m \in \mathbb{C} \), and \( \{\lambda_m\} \) is an increasing sequence of real numbers, \( \lim_{m \to \infty} \lambda_m = +\infty \), is called a general Dirichlet series. If \( \lambda_m = \log m \), then we obtain an ordinary Dirichlet series

\[
\sum_{m=1}^{\infty} \frac{a_m}{m^s}.
\]

Suppose that series (1) converges absolutely for \( \sigma > \sigma_a \) and has the sum \( f(s) \). Then \( f(s) \) is an analytic function in the region \( \{ s \in \mathbb{C} : \sigma > \sigma_a \} \).

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In [1] limit theorems on the complex plane for the function \( f(s) \) were obtained. Suppose that \( f(s) \) is meromorphically continuable to the half-plane \( \{ s \in \mathbb{C} : \sigma > \sigma_1 \} \), \( \sigma_1 < \sigma \), and that all poles of \( f(s) \) in this region are included in a compact set. Moreover, we require that, for \( \sigma > \sigma_1 \), the estimates
\[
f(s) = O(|t|^a), \quad |t| \geq t_0 > 0, \quad a > 0
\]
and
\[
\int_0^T |f(\sigma + it)|^2 dt = O(T), \quad T \to \infty,
\]
should be satisfied. Denote by \( \mathcal{B}(S) \) the class of Borel sets of the space \( S \), and let, for \( T > 0 \),
\[
\nu_T(\ldots) = \frac{1}{T} \operatorname{meas}\{ t \in [0, T]: \ldots \},
\]
where \( \operatorname{meas}\{ A \} \) is the Lebesgue measure of a measurable set \( A \subset \mathbb{R} \), and in place of dots a condition satisfied by \( t \) is to be written. On \( (\mathbb{C}, \mathcal{B}(\mathbb{C})) \) define the probability measure \( P_T(A) \) by
\[
P_{T,\sigma}(A) = \nu_T(f(\sigma + it) \in A).
\]
The first result of [1] is the following theorem.

**Theorem A.** Suppose that for the function \( f(s) \) conditions (2) and (3) are satisfied. Then on \( (\mathbb{C}, \mathcal{B}(\mathbb{C})) \) there exists a probability measure \( P_{\sigma} \) such that the measure \( P_{T,\sigma} \) converges weakly to \( P_{\sigma} \) as \( T \to \infty \).

For the identification of the limit measure \( P_{\sigma} \) in Theorem A some additional conditions are necessary. Also, for the definition of \( P_{\sigma} \) we need some topological structure. Let \( \gamma = \{ s \in \mathbb{C} : |s| = 1 \} \) be the unit circle on the complex plane, and let
\[
\Omega = \prod_{m=1}^{\infty} \gamma_m,
\]
where \( \gamma_m = \gamma \) for all \( m \in \mathbb{N} \). With the product topology and pointwise multiplication the infinite-dimensional torus \( \Omega \) is a compact topological Abelian group.
Therefore, on \((\Omega, B(\Omega))\) the probability Haar measure \(m_H\) exists, and we obtain a probability space \((\Omega, B(\Omega), m_H)\). Let \(\omega(m)\) denote the projection of \(\omega \in \Omega\) to the coordinate space \(\gamma_m\). Suppose, that the exponents \(\lambda_m\) satisfy the inequality

\[
\lambda_m \geq c(\log m)^\delta
\]

with some positive constants \(c\) and \(\delta\). Then in [1] it was proved (Lemma 3) that, for \(\sigma > \sigma_1\),

\[
f(\sigma, \omega) = \sum_{m=1}^{\infty} a_m \omega(m) e^{-\lambda_m \sigma}
\]

is a complex-valued random variable defined on the probability space \((\Omega, B(\Omega), m_H)\). Then [1] contains the following statement.

**Theorem B.** Suppose that the system of exponents \(\{\lambda_m\}\) is linearly independent over the field of rational numbers, satisfies inequality (4), and for the function \(f(s)\) conditions (2) and (3) are satisfied. Then the probability measure \(P_{T,\sigma}\) converges weakly to the distribution of the random variable \(f(\sigma, \omega)\) as \(T \to \infty\).

Condition (4) restricts the choice of sequence of exponents \(\{\lambda_m\}\) for which Theorem B is true. The aim of this note is to replace condition (4) by a certain average condition. Suppose that, for \(\sigma > \sigma_1\), the series

\[
\sum_{m=1}^{\infty} |a_m|^2 e^{-2\lambda_m \sigma \log^2 m}
\]

converges. Later, it will be proved that the convergence of series (6) is a sufficient condition that \(f(\sigma, \omega)\) defined by (5) should be a complex-valued random variable for \(\sigma > \sigma_1\).

**Theorem 1.** Suppose that the system of exponents \(\{\lambda_m\}\) is linearly independent over the field of rational numbers, series (6) converges, and for the function \(f(s)\) conditions (2) and (3) are satisfied. Then the probability measure \(P_{T,\sigma}\) converges weakly to the distribution of the random variable \(f(\sigma, \omega)\) as \(T \to \infty\).
2 The random variable $f(\sigma, \omega)$

In this section we will prove that, if series (6) converges, then $f(\sigma, \omega)$, $\sigma > \sigma_1$, is a complex-valued random variable. For this, we will use Rademacher’s theorem on series of pairwise orthogonal random variables, for the proof, see, for example, [2]. Denote by $E\xi$ the expectation of the random element $\xi$.

**Lemma 2** [2]. Suppose that $\{X_m\}$ is a sequence of pairwise orthogonal random variables and that $\sum_{m=1}^{\infty} E|X_m|^2 \log^2 m < \infty$. Then the series $\sum_{m=1}^{\infty} X_m$ converges almost surely.

**Theorem 2.** Let $\sigma > \sigma_1$. Then $f(\sigma, \omega)$ is a complex-valued random variable defined on the probability space $(\Omega, B(\Omega), m_H)$.

**Proof.** Let, for $\sigma > \sigma_1$,

$$\xi_m(\omega) = a_m \omega(m)e^{-\lambda_m \sigma}.$$  

Then $\{\xi_m : m \in \mathbb{N}\}$ is a sequence of pairwise orthogonal complex-valued random variables. Since by (7)

$$E|\xi_m|^2 = |a_m|^2 e^{-2\lambda_m \sigma}$$

and series (6) converges, we have that, for $\sigma > \sigma_1$,

$$\sum_{m=1}^{\infty} E|\xi_m|^2 \log^2 m < \infty.$$  

Therefore, by Lemma 2 for $\sigma > \sigma_1$ the series

$$\sum_{m=1}^{\infty} a_m \omega(m)e^{-\lambda_m \sigma}$$

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converges for almost all \( \omega \in \Omega \) with respect to the Haar measure \( m_H \). This shows that \( f(\sigma, \omega) \), for \( \sigma > \sigma_1 \), is a random variable on \( (\Omega, B(\Omega), m_H) \).

Clearly, there exists general Dirichlet series with small exponents, for which series (6) converges. For example, if \( \lambda_m = \log \log^2 m \) and \( a_m = O(1/m) \), then series (1) converges absolutely for \( \sigma \geq 1 \). Suppose that it is analytically continuable to some region \( \sigma > \sigma_1 \) with \( \sigma_1 < 1 \). Then we have that

\[
\sum_{m=1}^{\infty} |a_m|^2 e^{-2\lambda_m \sigma} \log^2 m \ll \sum_{m=1}^{\infty} \frac{\log^{2-2\sigma} m}{m^2} < \infty.
\]

3 Limit theorems for Dirichlet polynomials

In the sequel we suppose that the system of exponents \( \{\lambda_m\} \) is linearly independent over the field of rational numbers. Let, for \( n \in \mathbb{N} \) and fixed \( \hat{\omega} \in \Omega \),

\[
g_{N,n}(s) = \sum_{m=1}^{N} a_m v(m, n) e^{-\lambda_m s}
\]

and

\[
g_{N,n}(s, \omega) = \sum_{m=1}^{N} a_m v(m, n) \omega(m) e^{-\lambda_m s},
\]

where \( v(m, n) = \exp\{-e^{(\lambda_m - \lambda_n)\sigma_2}\}, \sigma_2 > 0 \), and on \((\mathbb{C}, B(\mathbb{C}))\) define two probability measures

\[
P_{T,N,n,\sigma}(A) = \nu_T(g_{N,n}(\sigma + it) \in A)
\]

and

\[
\tilde{P}_{T,N,n,\sigma}(A) = \nu_T(g_{N,n}(\sigma + it, \hat{\omega}) \in A).
\]

**Theorem 3.** On \((\mathbb{C}, B(\mathbb{C}))\) there exists a probability measure \( P_{N,n} \), such that the probability measures \( P_{T,N,\sigma} \) and \( \tilde{P}_{T,N,\sigma} \) both converge weakly to \( P_{N,n,\sigma} \) as \( T \to \infty \).

**Proof.** For the proof of Theorem 3, we introduce one more probability measure

\[
Q_T(A) = \nu_T\left((e^{it\lambda_m})_{m \in \mathbb{N}} \in A\right), \quad A \in B(\Omega).
\]
The dual group of $\Omega$ is isomorphic to 
\[
\bigoplus_{m \in \mathbb{N}} \mathbb{Z}_m,
\]
where $\mathbb{Z}_m = \mathbb{Z}$ for all $m \in \mathbb{N}$, $k = \{k_m : m \in \mathbb{N}\} \in \bigoplus_{m \in \mathbb{N}} \mathbb{Z}_m$, where only a finite number of integers $k_m$ are non-zero, acts on $\Omega$ by 
\[
 k \to \omega^k = \prod_{m=1}^{\infty} \omega^{k_m}(m), \quad \omega \in \Omega.
\]

Therefore, the linear independence of $\{\lambda_m\}$ shows that the Fourier transform $g_T(k)$ of the measure $Q_T$ is of the form 
\[
g_T(k) = \int_{\Omega} \left( \prod_{m=1}^{\infty} \omega^{k_m}(m) \right) dQ_T = \frac{1}{T} \int_0^T \left( \prod_{m=1}^{\infty} e^{it\lambda_m k_m} \right) dt
\]
\[
= \begin{cases} 
1, & \text{if } k_m = 0 \text{ for all } m \in \mathbb{N}, \\
\frac{\exp\{iT \sum_{m=1}^{\infty} \lambda_m k_m\} - 1}{iT \sum_{m=1}^{\infty} \lambda_m k_m}, & \text{otherwise}.
\end{cases}
\]

Hence we find that 
\[
\lim_{T \to \infty} g_T(k) = \begin{cases} 
1, & \text{if } k_m = 0 \text{ for all } m \in \mathbb{N}, \\
0, & \text{otherwise},
\end{cases}
\]
and a continuity theorem for probability measures on locally compact group, see, for example, [3], implies that the probability measure $Q_T$ converges weakly to the measure $m_H$.

Let $h : \Omega \to \mathbb{C}$ be given by the formula 
\[
h(\{\omega(m) : m \in \mathbb{N}\}) = \sum_{m=1}^{\infty} a_m v(m, n)e^{-\lambda_m \sigma} \omega(m).
\]

The function $h$ is continuous and satisfies 
\[
h(\{e^{i\lambda_m t} : m \in \mathbb{N}\}) = g_{N,n}(\sigma + it).
\]
Therefore, Theorem 5.1 of [4] and the weak convergence of the probability measure $Q_T$ show that $P_{T,N,n,\sigma} = Q_T h^{-1}$ converges weakly to $m_H h^{-1}$ as $T \to \infty$.

Now define $h_1 : \Omega \to \Omega$ by

$$h_1(\{\omega(m) : m \in \mathbb{N}\}) = (\{\omega(m) \hat{\omega}^{-1}(m) : m \in \mathbb{N}\}),$$

where $\hat{\omega}$ is a fixed element of $\Omega$. Then we have that $g_{N,n}(\sigma + it, \hat{\omega}) = \sum_{m=1}^{N} a_m v(m, n) e^{-\lambda_m (\sigma + it)} \hat{\omega}^{-1}(m) = h(h_1(\{e^{i\lambda_m t} : m \in \mathbb{N}\})).$

Hence, similarly as above, we obtain that the probability $P_{T,N,n,\sigma} = Q_T (hh_1)^{-1}$ converges weakly to the measure $m_H (hh_1)^{-1} = (m_H h_1^{-1})h^{-1} = m_H h^{-1}$, because the Haar measure $m_H$ is invariant with respect to translations by points from $\Omega$. The theorem is proved.

Note that in [1] an another proof of Theorem 3 based on the study of a finite-dimensional torus has been given.

4 Limit theorems for absolutely convergent Dirichlet series

In this section we construct a general Dirichlet series related to series (1) which converges absolutely for $\sigma > \sigma_1$. We also correct some inaccuracies of [1].

We take $\sigma_2 > \sigma_a - \sigma_1 > 0$ and define, for $\sigma > \sigma_1$,

$$g_n(s) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} f(s + z) l_n(z) \frac{dz}{z},$$

where

$$l_n(s) = \frac{s}{\sigma_2} \Gamma\left(\frac{s}{\sigma_2}\right) e^{\lambda_n s}.$$

Using Mellin’s inversion formula and the definition of $l_n(s)$, we find that

$$g_n(s) = \sum_{m=1}^{\infty} \frac{a_m e^{-\lambda_m s}}{2\pi i \sigma_2} \int_{\sigma - i\infty}^{\sigma + i\infty} \Gamma\left(\frac{z}{\sigma_2}\right) e^{-(\lambda_m - \lambda_n)z} dz$$

$$= \sum_{m=1}^{\infty} a_m \exp\{-e^{(\lambda_m - \lambda_n)\sigma_2}\} e^{-\lambda_m s} = \sum_{m=1}^{\infty} a_m v(m, n) e^{-\lambda_m s}. \quad (8)$$
It remains to prove that the later series converges absolutely for \( \sigma > \sigma_1 \). Clearly,

\[
g_n(s) = \sum_{m=1}^{\infty} a_n(m) e^{-\lambda_m s}, \tag{9}
\]

where

\[
a_n(m) = \frac{1}{2\pi i \sigma_2} \int_{\sigma_2 - i\infty}^{\sigma_2 + i\infty} \frac{\Gamma\left(\frac{z}{\sigma_2}\right)}{z^m} e^{-(\lambda_m - \lambda_n)s} ds \ll_n e^{-\lambda_m \sigma_2}.
\]

This and (9) yield the absolute convergence of series (8) for \( \sigma > \sigma_1 \).

Define, for \( \omega \in \Omega \),

\[
g_n(s, \omega) = \sum_{m=1}^{\infty} a_m \omega(m) v(m, n) e^{-\lambda_m s}.
\]

The aim of this section is to obtain limit theorems for probability measures

\[
P_{T,n,\sigma}(A) = \nu_T(g_n(\sigma + it) \in A)
\]

and

\[
\hat{P}_{T,n,\sigma}(A) = \nu_T(g_n(\sigma + it, \omega) \in A),
\]

where \( A \in B(\mathbb{C}) \).

**Theorem 4.** Let \( \sigma > \sigma_1 \). Then on \( (\mathbb{C}, B(\mathbb{C})) \) there exists a probability measure \( P_{n,\sigma} \) such that the probability measures \( P_{T,n,\sigma} \) and \( \hat{P}_{T,n,\sigma} \) both converge weakly to \( P_{n,\sigma} \) as \( T \to \infty \).

**Proof.** We will give a shortened proof, because it only in some details differs from that given in [1].

By Theorem 3 the probability measures \( P_{T,N,n,\sigma} \) and \( \hat{P}_{T,N,n,\sigma} \) both converge weakly to the measure \( P_{N,n,\sigma} \) as \( T \to \infty \). We will prove that the family of probability measures \( \{P_{N,n,\sigma}\} \) is tight. By the Chebyshev inequality, for any positive \( M \),

\[
P_{T,N,n,\sigma}\left(\{z \in \mathbb{C} : |z| > M\}\right) = \nu_T\left(|g_{N,n}(\sigma + it)| > M\right) \leq \frac{1}{MT} \int_{0}^{T} |g_{N,n}(\sigma + it)| dt.
\]
Since the series for $g_n(s)$ converges absolutely for $\sigma > \sigma_1$, there exists a constant $C > 0$ such that
\[
\limsup_{T \to \infty} \frac{1}{T} \int_0^T |g_{N,n}(\sigma + it)| dt \leq C.
\]

For arbitrary $\varepsilon > 0$, let $M = C/\varepsilon$. Then we deduce from the last two inequalities that
\[
\limsup_{T \to \infty} P_{T,N,n,\sigma}(\{ z \in \mathbb{C} : |z| > M \}) \leq \varepsilon. \tag{10}
\]

The function $h : \mathbb{C} \to \mathbb{R}$, $z \to |z|$, is continuous and so, by Theorem 3, the probability measure
\[
\nu_T(|g_{N,n}(\sigma + it)| \in A), \quad A \in \mathcal{B}(\mathbb{R}),
\]
converges weakly to $P_{N,n,\sigma} h^{-1}$ as $T \to \infty$. This, the properties of the weak convergence, and (10) imply
\[
P_{N,n,\sigma}(\{ z \in \mathbb{C} : |z| > M \}) \leq \liminf_{T \to \infty} P_{T,N,n,\sigma}(\{ z \in \mathbb{C} : |z| > M \})
\leq \limsup_{T \to \infty} P_{T,N,n,\sigma}(\{ z \in \mathbb{C} : |z| > M \}) \leq \varepsilon.
\]

Define $K_\varepsilon = \{ z \in \mathbb{C} : |z| \leq M \}$. Then $K_\varepsilon$ is a compact set, and
\[
P_{N,n,\sigma}(K_\varepsilon) \geq 1 - \varepsilon
\]
for all $N \in \mathbb{N}$. This shows the tightness of the family $\{P_{N,n,\sigma}\}$. Hence by the Prokhorov theorem, see, for example, [4], the family $\{P_{N,n,\sigma}\}$ is relatively compact.

Now let a random variable $\theta_T$ be defined on a certain probability space $(\hat{\Omega}, \mathcal{B}(\hat{\Omega}), \mathbb{P})$ and uniformly distributed on $[0, T]$. We put
\[
X_{T,N,n}(\sigma) = g_{N,n}(\sigma + i\theta_T).
\]
Then, by Theorem 3,
\[
X_{T,N,n}(\sigma) \overset{D}{\to} X_{N,n}(\sigma), \tag{11}
\]
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where \( D \) means the convergence in distribution, and \( X_{N,n}(\sigma) \) is a complex-valued random variable with the distribution \( P_{N,n,\sigma} \). Moreover, the relative compactness implies the existence of \( \{P_{N_1,n,\sigma}\} \subset \{P_{N,n,\sigma}\} \) such that \( P_{N_1,n,\sigma} \) converges weakly to some measure \( P_{n,\sigma} \) as \( N_1 \to \infty \). Then

\[
X_{N_1,n}(\sigma) \xrightarrow{D} P_{n,\sigma}.
\]

(12)

This, (11), the relation

\[
\lim_{N \to \infty} \limsup_{T \to \infty} \nu_T(\{g_{N,n}(\sigma + it) - g_n(\sigma + it)\} \geq \varepsilon) = 0
\]

and Theorem 4.2 of [4] show that

\[
X_{T,n}(\sigma) \xrightarrow{D} P_{n,\sigma},
\]

(13)

where

\[
X_{T,n}(\sigma) = g_n(\sigma + i\theta_T).
\]

Hence the measure \( P_{T,n,\sigma} \) converges weakly to \( P_{n,\sigma} \) as \( T \to \infty \). By (13) the measure \( P_{n,\sigma} \) does not depends on the choice of \( N_1 \), and we have that

\[
X_{N,n}(\sigma) \xrightarrow{D} P_{n,\sigma}.
\]

(14)

To complete the proof of the theorem it remains to repeat the above arguments for random variables

\[
\hat{X}_{T,N,n}(\sigma) = g_{N,n}(\sigma + i\theta_T, \omega)
\]

and

\[
\hat{X}_{T,n}(\sigma) = g_{n}(\sigma + i\theta_T, \omega),
\]

and to use (14). \( \square \)

5 Proof of Theorem 1

First we observe that the method of the contour integration shows that the function \( f(s) \) is approximated in the mean by the function \( g_n(s) \). More precisely, we have, for \( \sigma > \sigma_1 \),

\[
\lim_{n \to \infty} \limsup_{T \to \infty} \frac{1}{T} \int_0^T \left| f(\sigma + it) - f_n(\sigma + it) \right| dt = 0.
\]

(15)
An analogous assertion is also valid for the function $f(s, \omega)$, namely, for $\sigma > \sigma_1$,

$$\lim_{n \to \infty} \limsup_{T \to \infty} \frac{1}{T} \int_0^T |f(\sigma + it, \omega) - f_n(\sigma + it, \omega)| \, dt = 0. \quad (16)$$

The details of the proof of (15) and (16) can be found in [1].

We introduce one more probability measure on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$. Let

$$\hat{P}_{T, \sigma}(A) = \nu_T(\{f(\sigma + it, \omega) \in A\}), \quad A \in \mathcal{B}(\mathbb{C}).$$

**Theorem 5.** Let $\sigma > \sigma_1$. Then on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ there exists a probability measure $P_\sigma$ as $T \to \infty$.

**Proof.** We argue similarly to the proof of Theorem 4. Repeating the proof of Theorem 4 with $f_n(\sigma + it)$ and $f_n(\sigma + it, \omega)$ in place of $f_{N,n}(\sigma + it)$ and $f_{N,n}(\sigma + it, \omega)$, respectively, and with $f(\sigma + it)$ and $f(\sigma + it, \omega)$ in place of $f_n(\sigma + it)$ and $f_n(\sigma + it, \omega)$, respectively, and using (15), (16) and Theorem 4.2 of [4], we obtain the theorem. \(\square\)

**Proof of Theorem 1.** Define the one parameter group $\{\varphi_t : t \in \mathbb{R}\}$ of measurable measure preserving transformations on $\Omega$ by $\varphi_t(\omega) = a_t \omega, \omega \in \Omega$, where $a_t = \{e^{-i\lambda m_t} : m \in \mathbb{N}\}$. Then the group $\{\varphi_t : t \in \mathbb{R}\}$ is ergodic [1]. Let $A$ be a continuity set of the measure $P_\sigma$ in Theorem 5. Then by Theorem 5

$$\lim_{T \to \infty} \hat{P}_{T, \sigma}(A) = P_\sigma(A). \quad (17)$$

Taking

$$\theta(\omega) = \begin{cases} 1, & \text{if } f(\sigma, \omega) \in A, \\ 0, & \text{if } f(\sigma, \omega) \notin A, \end{cases}$$

we have that $\theta$ is a random variable on $(\Omega, \mathcal{B}(\Omega), m_H)$, and

$$\mathbb{E}(\theta) = \int_{\Omega} \theta \, dm_H = m_H(\omega \in \Omega : f(\sigma, \omega) \in A) = P_f(A), \quad (18)$$

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where $P_f$ is the distribution of $f(\sigma, \omega)$. Moreover, the process $\theta(\varphi_t(\sigma))$ is ergodic, therefore by the classical Birkhoff-Khinchine theorem

$$\mathbb{E}(\theta) = \lim_{T \to \infty} \frac{1}{T} \int_0^T \theta(\varphi_T(\omega)) \, dt$$

for almost all $\omega$ with respect to the measure $m_H$. On the other hand,

$$\frac{1}{T} \int_0^T \theta(\varphi_t(\omega)) \, dt = \hat{P}_{T,\sigma}(A).$$

This, (18) and (19) show that

$$\lim_{T \to \infty} \hat{P}_{T,\sigma}(A) = P_f(A),$$

and in view of (17) we conclude that $P_\sigma(A) = P_f(A)$ for all continuity sets $A$ of $P_\sigma$. Hence $P_\sigma$ coincides with $P_f$, and the theorem is proved.

References


