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Recursive Algorithms of Time Series Observations Recognition

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Abstract. The paper presents new method for sequential classification of the time series observations. Methods and algorithms of sequential recognition are obtained on the basis of the recursive equations for sufficient statistics. These recursive equations allow to construct algorithms of current classification of observable sequences in the rate of entering its values into the on-line operation. Classification algorithms are realized in the form of computer programs, including personal computers. They allow to build multi-channel conveyer computational structures for the sequential recognizers of time series observations.

Key words: time series, processes classes, samples, sequential classification.

1. Introduction

Recognition of random processes based on a description of classes by dynamic model in the form of differential and difference stochastic equations was considered in a paper by Shpilewski (1971) and by Petrow and Shpilewski (1974). Further, the Markov models gained a wide dissemination in practice (Shpilewski, 1980). The proposed approach to the recognition of stationary random processes appeared to be extremely fruitful. It allowed to obtain constructive methods of recognition of random processes with continuous and discrete time, to work out dynamic classification algorithms in the recognition of sequences in current time.

Sequential recognition of M classes time series observation are considered in the given paper. A practically important class of stochastic models for describing a discrete random processes are (Box, Jenkins, 1970):

– autoregressive processes of order p, $AR(p)$

$$y[n] = \sum_{i=1}^{p} a_m[i]y[n-i] + v[n], \tag{1}$$

– moving average processes of order q, $MA(q)$

$$y[n] = v[n] - \sum_{i=1}^{q} b_m[i]v[n-i], \tag{2}$$

– autoregressive-moving average processes of order (p, q), $ARMA(p, q)$

$$y[n] - \sum_{i=1}^{p} a_m[i]y[n-i] = v[n] - \sum_{i=1}^{q} b_m[i]v[n-i], \tag{3}$$

where $n = 1, 2, \ldots, N$. Parameters of the dynamic models (1), (2), (3) are defined in the learning regime by the realizations of known status. For this purpose the methods of the least squares, maximal likelihood, Bayes or Yule-Wolker equations are used.

Moving average and autoregressive-moving average processes are not the Markov processes, therefore, methods that have been developed earlier for Markov processes cannot be applied to the $MA(q)$ and $ARMA(p, q)$.

2. Statement of the Problem

Let on a measurable space (Ω, F) a stochastic processes $Y(n)$ with a discrete time

$$Y(n) = [y[1], y[2], y[3], \ldots, y[n], \ldots] \tag{4}$$

are given. Let $\{F_n\}$, $n > 0$ be a non decreasing family of σ-algebra $F_n = \sigma\{y[\nu], \nu \prec n\}$ and $F = \sigma\{UF_n\}$ in relation to which $Y[1:n]$ is measurable. With respect to the observable process $Y(n)$, M alternative hypotheses

$$H = \{h_1, h_2, h_3, \ldots, h_M\}$$

are introduced. If p_m is an a-priori probability of the hypothesis h_m, then

$$p_m > 0, \quad \sum_{m=1}^{M} p_m = 1.$$

M probability measures P_m, $m = 1, 2, \ldots, M$, are linked with the hypothesis h_m and $P = \sum p_m P_m$. Let $P[1:n]$, $P_m[1:n]$ be restriction of measures P, P_m on σ-algebra F_n.

The problem of multialternative recognition of the observable sample

$$y[1:n] = [y[1], y[2], y[3], \ldots, y[n]] \tag{5}$$

rests on breaking the sample space A (respectively Ω) into nonintersecting sets A_m (or Ω into ω_m) from the condition of the minimum of the error probability

$$P_{er}(n) = \min_{A_m} \sum_{m=1}^{M} P_{er,m}(n) \left\{ y[1:n] \notin A_m, h_m \right\}, \tag{6}$$

for the decision rule

$$h_k\colon \; y[1:n] \in A_k. \tag{7}$$

3. General Solution

The construction of abstract sets A_m in the space of realisations may by obtained using decision rules: the observed realisation (2) belongs to the class h_m, for which the a-posterior probability of class h_m is maximal

$$h_k\colon \; P\left(h_k/y[1:n]\right) = \max_m P\left(h_m/y[1:n]\right). \tag{8}$$

The a-posterior probability of the class h_m is equal

$$P\left(h_m/y[1:n]\right) = p_m f\left(y[1:n]/h_m\right)/f\left(y[1:n]\right).$$

Then the decision rule (8) will take the form

$$h_k\colon \; p_k f\left(y[1:n]/h_k\right) = \max_m p_m f\left(y[1:n]/h_m\right). \tag{9}$$

The probability density function $f(y[1:n]/h_m)$ in (9) for discrete random processes $y[1:n]$ (1) for each class h_m we present in the form

$$f\left(y[1:n]/h_m\right) = f\left(y[1]/h_m\right) \prod_{i=1}^{n-1} f\left(y[i+1]/y[1:i], h_m\right), \tag{10}$$

or in the recursive form

$$f\left(y[1:n+1]/h_m\right) = f\left(y[1:n]/h_m\right) f\left(y[n+1]/y[1:n], h_m\right) \tag{11}$$

with the initial conditions $f(y[1]/h_m)$.

The recursive equations for the likelihood function $L_m(n)$ follow from (11)

$$L_m(n+1) = L_m(n) f\left(y[n+1]/y[1:n], h_m\right) \tag{12}$$

with the initial conditions $L_m(1) = f(y[1]/h_m)$.

The Bayesian decision rule (8) take the form

$$h_k\colon\ p_k L_k(n) = \max_m p_m L_m(n). \tag{13}$$

The logarithm of likelihood function $l_m(n) = \ln(L_m(n))$ satisfies the recursive equations

$$l_m(n+1) = l_m(n) + \ln\left(f\left(y[n+1]/y[1:n], h_m\right)\right) \tag{14}$$

with the initial conditions $l_m(1) = \ln(f(y[1]/h_m))$.

The Bayesian decision rule (8) take the form

$$h_k\colon\ \ln p_k + l_k(n) = \max_m \left(\ln p_m + l_m(n)\right). \tag{15}$$

The logarithm of likelihood function ratio $u_m(n) = \ln(L_m(n)/L_M(n))$ satisfies the recursive equations

$$\begin{aligned} u_m(n+1) \;=\; & u_m(n) + \ln\left(f\left(y[n+1]/y[1:n], h_m\right)\right) \\ & - \ln\left(f\left(y[n+1]/y[1:n], h_M\right)\right) \end{aligned} \tag{16}$$

with the initial conditions $u_m(1) = \ln(f(y[1]/S_m)/f(y[1]/h_M)$.

The Bayesian decision rule (8) take the form

$$h_k\colon\ \ln p_k + u_k(n) = \max_m \left(\ln p_m + u_m(n)\right). \tag{17}$$

The considered general mathematical statement of processes recognition problem and standard solution extremely simplify the problem and conceal those difficulties, which arise while solving practical problem. In order to build the constructive methods for the recognition of real processes two moments have an essential value:

– in what form the measures P_m, $m = 1, 2, \ldots, M$ are set for description of real processes;
– what degree of indeterminacy is in setting of the set measure.

The methods of recognition of the discrete random processes based on the use of time series models in the form of stochastic recurrent equation (1), (2), (3) for the description of the classes is considered in the given paper.

4. Recursive Algorithms of *AR(p)* Observations Recognition

Autoregressive processes of order p $AR(p)$

$$y[n] = \sum_{i=1}^{p} a_m[i] y[n-i] + v[n] \tag{18}$$

with initial condition, $y[0] = y[-1] = .. = y[1-p] = 0$, where $v[n]$ is white noise process normal distribution, having mean zero and constant variance

$$E\left(v[n]\right) = 0, \quad \mathrm{var}(v[n]) = \sigma^2$$

are taken as discrete random processes of m-th class model. The stationarity conditions for autoregressive processes may be expressed by saying that the roots of characteristic equation

$$z^p - \sum_{i=1}^{p} a_m[i] z^{p-i} = 0$$

for the processes (18) must lie inside the unit circle.

The sequential classification of AR observations we shall get on the basis of the following algorithm.

Algorithm 1

The discrete random processes described by equation (14) are general Markov. A transient probability density functions in (10), (11), (12) are normal and have the form

$$f\left(y[n+1]/y[1:n], S_m\right) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{ -\left(y[n+1] - \sum_{i=0}^{p-1} a_m[i] y[n-i]\right)^2 /(2\sigma^2)\right\}. \tag{19}$$

Substituting (19) into (12) we have the recursive equations for $L_m(n)$.

The logarithm of likelihood function $l_m(n)$ satisfies the recursive equations

$$l_m(n+1) = l_m(n) - \left(y[n+1] - \sum_{i=0}^{p-1} a_m[i] y[n-i]\right)^2 /(2\sigma^2) - \log(2\pi\sigma^2)/2. \tag{20}$$

The logarithm of likelihood function ratio $u_m(n)$ in decision rule (13) satisfies the recursive equations

$$u_m(n+1) = u_m(n) - \left(y[n+1] - \sum_{i=0}^{p-1} a_m[i] y[n-i]\right)^2 /(2\sigma^2) + \left(y[n+1] - \sum_{i=0}^{p-1} a_M[i] y[n-i]\right)^2 /(2\sigma^2). \tag{21}$$

Recursive equations (20), (21) and decision rules (15), (17) give us the recursive recognition of m classes $AR(p)$ observations.

EXAMPLE 1. The discrete random processes (4) of the three classes h_1, h_2, h_3 are described by the AR models (22) of order $p = 6$.

$$\begin{aligned} y[n] &= +1,2y[n-1] - 1.06y[n-2] + 0.85y[n-3] \\ &\quad -0.31y[n-4] + 0.19y[n-5] - 0.06y[n-6] + v[n], \\ y[n] &= +0.6y[n-1] - 0.26y[n-2] - 0.34y[n-3] \\ &\quad +0.81y[n-4] - 0.07y[n-5] + 0.01y[n-6] + v[n], \\ y[n] &= -1,6y[n-1] - 0.91y[n-2] + 0.39y[n-3] \\ &\quad +0.80y[n-4] + 0.37y[n-5] + 0.05y[n-6] + v[n]. \end{aligned} \tag{22}$$

The sufficient statistics u_1, u_2, u_3 are calculated using the recursive equations (21). The class of the observable signal we determine by decision rule (17).The process of dynamical recognition of the observable realisations are presented on the Figs. 1–3.

5. Recursive Algorithms of *MA(q)* Observations Recognition

Moving average processes of order q $MA(q)$

$$y[n] = v[n] - \sum_{i=1}^{q} b_m[i]v[n-i] \tag{23}$$

with initial condition $v[0] = 0$, $v[-1] = 0, \ldots, v[-q] = 0$ are taken as discrete random processes m classes model. The invariability conditions for moving average processes may be expressed by saying that the roots of characteristic equation for the processes (23)

$$z^q - \sum_{i=1}^{q} b_m[i]z^{q-i} = 0$$

must lie inside the unit circle. Moving average processes described by equation (23) are non-Markov processes. We give recursive equations for sufficient statistics (7) on the basis of the algorithm.

Algorithm 2
The conditional probability density function $f(y[n+1]/y[1:n], h_m)$ is normal with

$$E\,(y[n+1]/y[1:n], h_m) = -e_m[1:n]y[1:n]',$$
$$\text{var}\,(y[n+1]/y[1:n], h_m) = \sigma^2,$$

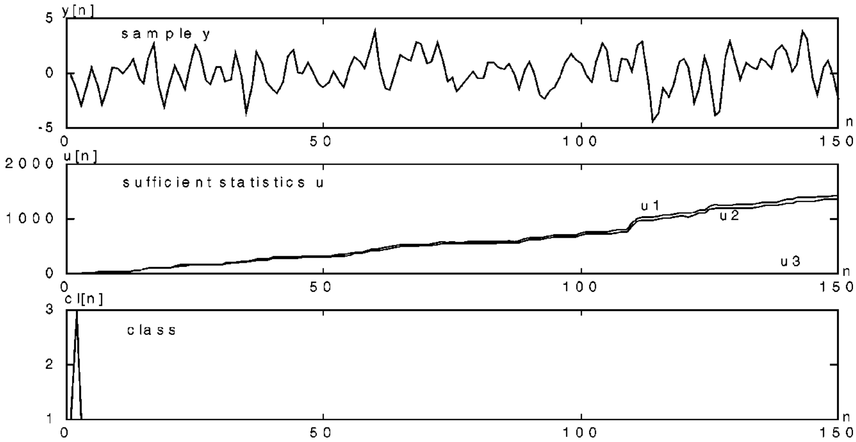

Fig. 1. Current recognition of the AR 1^{st} class observations.

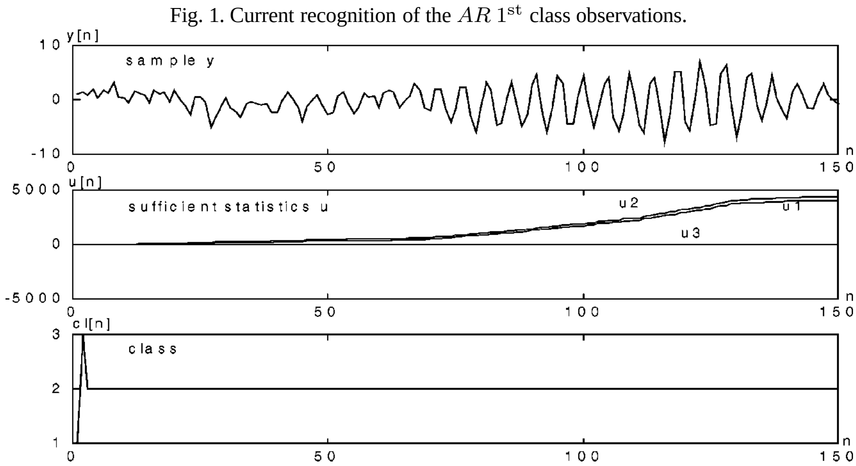

Fig. 2. Current recognition of the AR 2^{nd} class observations.

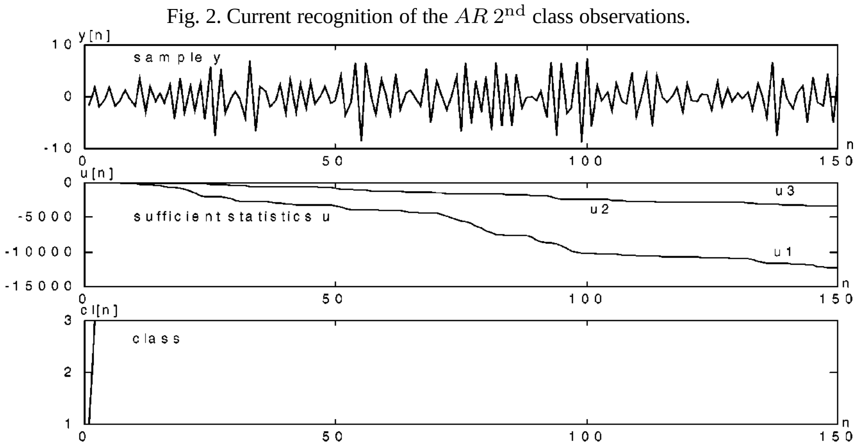

Fig. 3. Current recognition of the AR 3^{rd} class observations.

where $\{')$ sign of transposition, $E(y[1]) = 0$ and $e[1 : n]$ satisfy the system of the recursive equations for each m

$$\begin{aligned} n = 1 : N&, \\ & Cq = Cq + D * Bq; \\ & D = Cq[1]; \\ & Cq = [Cq[2 : q, 0]]; \\ & e[1 : n] = [D, e]; \end{aligned} \tag{24}$$

with initial condition $\texttt{e} = []$, $\texttt{Bq[i]} = \texttt{b[i]}$, $\texttt{Cq[i]} = \texttt{0}$, $\texttt{i} = \texttt{1}, \texttt{2}, \ldots, \texttt{q}$, $\texttt{D} = \texttt{1}$.

A transient probability density functions in (10), (11), (12) are normal and has the form

$$\begin{aligned} & f\left(y[n+1]/y[1:n], h_m\right) \\ & = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{ -\left(y[n+1] + e_m[1:n]y[1:n]'\right)^2 / (2\sigma^2) \right\}. \end{aligned} \tag{25}$$

The logarithm of likelihood function $l_m(n)$ satisfies the recursive equations

$$l_m(n+1) = l_m(n) - \left(y[n+1] + e_m[1:n]y[1:n]'\right)^2 / (2\sigma^2) - \log(2\pi\sigma^2)/2. \tag{26}$$

The logarithm of likelihood function ratio $u_m(n)$ in decision rule (13) satisfies the recursive equations

$$\begin{aligned} u_m(n+1) = \; & u_m(n) - \left(y[n+1] + e_m[1:n]y[1:n]'\right)^2 / (2\sigma^2) \\ & + \left(y[n+1] + e_M[1:n]y[1:n]'\right)^2 / (2\sigma^2). \end{aligned} \tag{27}$$

Recursive equations (26), (27) and decision rules (15), (17) give us the recursive recognition of m classes $MA(q)$ observations.

EXAMPLE 2. Moving average processes of order 2, $MA(2)$

$$\begin{aligned} & y[n] = v[n] - 0.3v[n-1] + 0.4v[n-2], \\ & y[n] = v[n] - 0.4v[n-1] - 0.3v[n-2], \\ & y[n] = v[n] + 1.0v[n-1] + 0.3v[n-2], \\ & y[n] = v[n] + 0.2v[n-1] - 0.5v[n-2] \end{aligned} \tag{28}$$

with initial condition $v[0] = 0$, $v[-1] = 0$, $v[-2] = 0$ are taken as discrete random processes of four classes h_1, h_2, h_3, h_4 model. The sufficient statistics u_1, u_2, u_3, u_4 are calculated using the recursive equations (27). The class of the observable signal we determine by decision rule (17). The process of dynamical recognition of the observable realizations are presented on the Figs. 4–7.

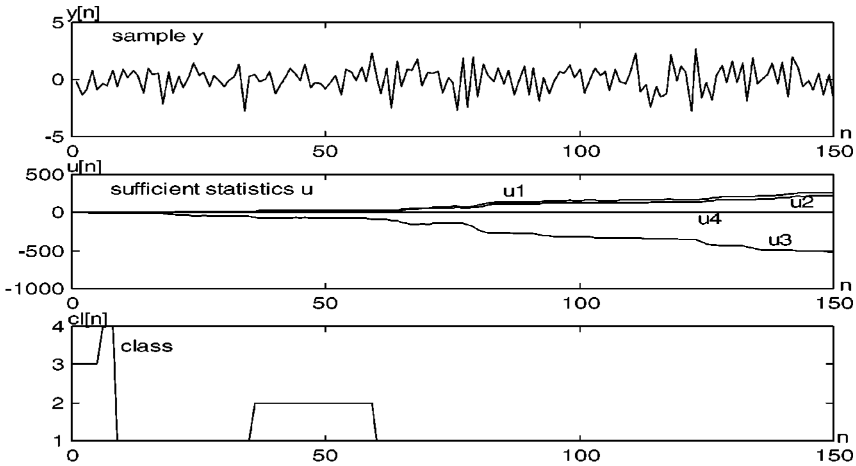

Fig. 4. Current recognition of the MA 1^{st} class observations.

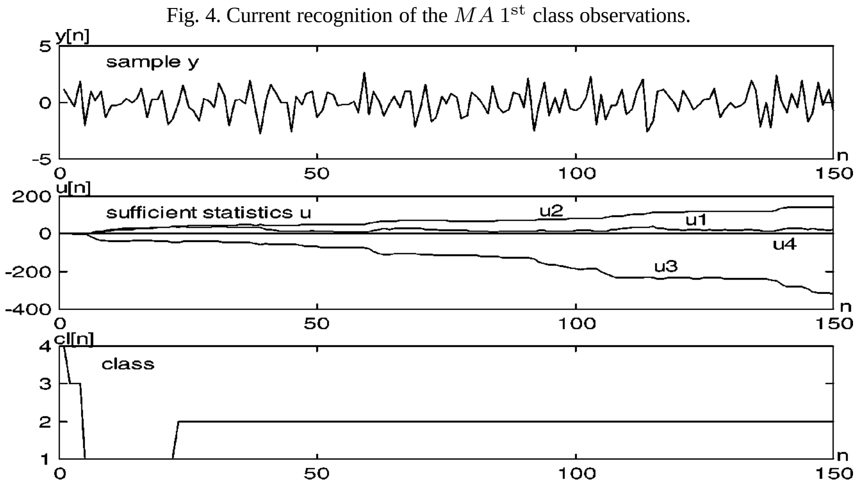

Fig. 5. Current recognition of the MA 2^{nd} class observations.

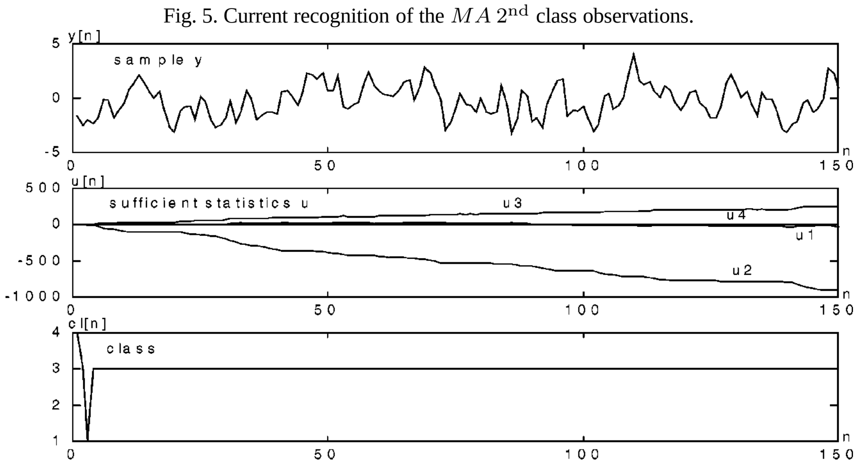

Fig. 6. Current recognition of the MA 3^{rd} class observations.

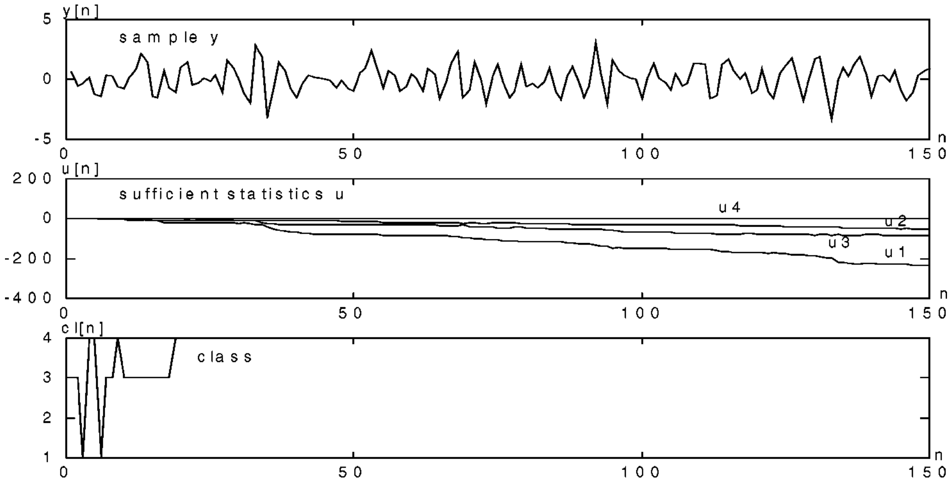

Fig. 7. Current recognition of the MA 4^{th} class observations.

6. Recursive Algorithms of *ARMA(p,q)* Observations Recognition

Autoregressive-moving average processes of order (p, q)

$$y[n] - \sum_{i=1}^{p} a_m[i]y[n-i] = v[n] - \sum_{i=1}^{q} b_m[i]v[n-i] \tag{29}$$

with initial condition $y[0] = y[-1] = .. = y[1-p] = 0$ and $v[0] = 0$, $v[-1] = 0, \ldots, v[-q] = 0$ are taken as discrete random processes m classes model. The stationarity and invertibility conditions for autoregressive-moving average processes may be expressed by saying that the roots of characteristic equation for the processes (18), (23) must lie inside the unit circle. We give recursive equations for sufficient statistics (7) on the basis of the Algorithm 3.

Algorithm 3

The conditional probability density function $f(y[n+1]/y[1:n], h_m)$ is normal with

$$\begin{aligned} &E\left(y[n+1]/y[1:n], h_m\right) = -c_m[1:n]y[1:n]', \\ &\text{var}(y[n+1]/y[1:n], h_m) = \sigma^2, \end{aligned} \tag{30}$$

where $E(y[n+1]/y[1:n], h_m)$ is linear function of observed sample $y[1:n]$, $\{'\}$ sign of transposition and $c[1:i]$ satisfy the system of recursive equation

$$\begin{aligned} &d[1:r] = g[1:r] + b[1:r]d[1]; \\ &g[1:r] = [d[2:i], 0]; \\ &c[1:i] = [d[1], c[1:i]]; \end{aligned} \tag{31}$$

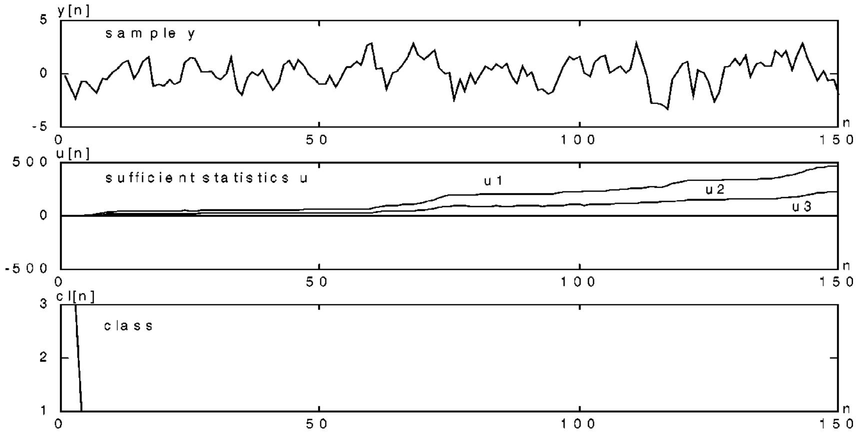

Fig. 8. Current recognition of the $ARMA$ 1^{st} class observations.

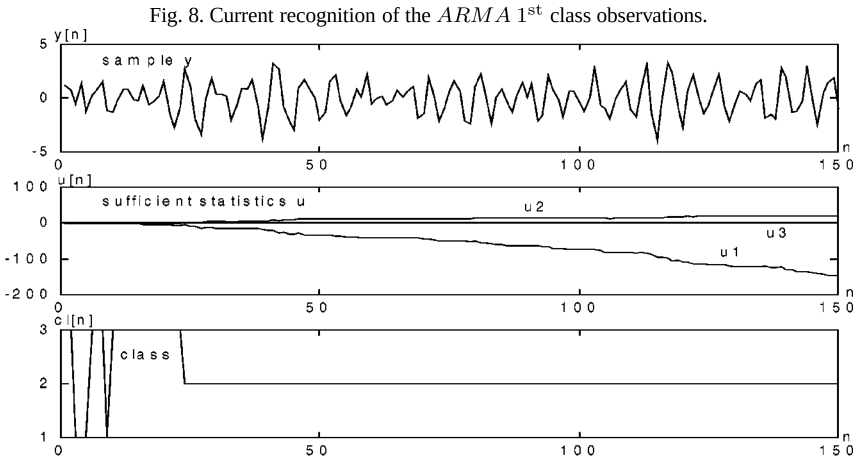

Fig. 9. Current recognition of the $ARMA$ 2^{nd} class observations.

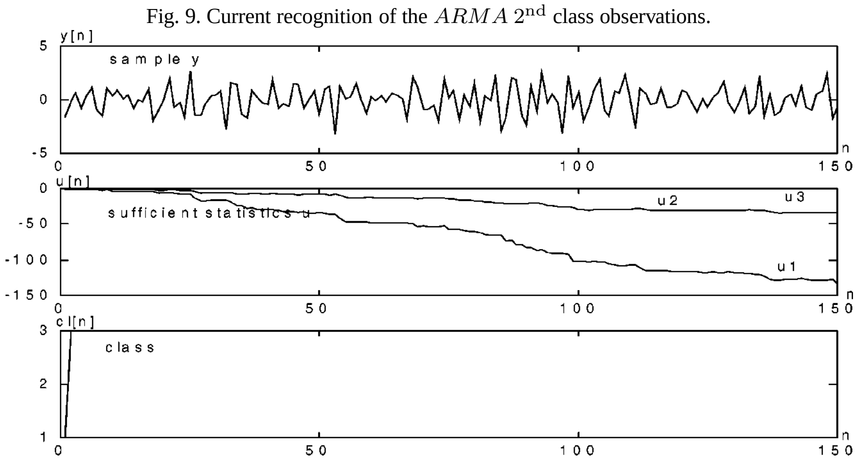

Fig. 10. Current recognition of the $ARMA$ 3^{rd} class observations.

where $r = \max(p, q)$, with the initial conditions for the recursive equations:

$$E(y[1]) = 0, g[1:r] = a[1:r], d[1] = 1, c[1] = [].$$

A transient probability density functions in (10), (11), (12) are normal and has the form

$$f\left(y[n+1]/y[1:n], h_m\right) \\ = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\left(y[n+1] + c_m[1:n]y[1:n]'\right)^2/(2\sigma^2)\right\}. \quad (32)$$

The logarithm of likelihood function $l_m(n)$ satisfies the recursive equations

$$l_m(n+1) = l_m(n) - \left(y[n+1] + c_m[1:n]y[1:n]'\right)^2/(2\sigma^2) - \log(2\pi\sigma^2)/2. \quad (33)$$

The logarithm of likelihood function ratio $u_m(n)$ in decision rule (13) satisfies the recursive equations

$$u_m(n+1) = u_m(n) - \left(y[n+1] + c_m[1:n]y[1:n]'\right)^2/(2\sigma^2) \\ + \left(y[n+1] + c_M[1:n]y[1:n]'\right)^2/(2\sigma^2). \quad (34)$$

Recursive equations (33), (34) and decision rules (15), (17) give us the recursive recognition of m classes $ARMA(p, q)$ observations.

EXAMPLE 3. Autoregressive-moving average processes of order (2, 1)

$$\begin{aligned} &y[n] - 0.4y[n-1] - 0.3y[n-2] = v[n] + 0.3v[n-1], \\ &y[n] - 0.5y[n-1] + 0.7y[n-2] = v[n] - 0.5v[n-1], \\ &y[n] - 0.3y[n-1] + 0.3y[n-2] = v[n] - 0.7v[n-1] \end{aligned} \quad (35)$$

with initial condition $y[0] = y[-1] = 0$ and $v[0] = 0$ are taken as discrete random processes three classes h_1, h_2, h_3 model. The sufficient statistics u_1, u_2, u_3 are calculated using the recursive equations (27). The class of the observable signal we determine by decision rule (17). The process of dynamical recognition of the observable realisations are presented on the Figs. 8–10.

7. Conclusion

This paper is intended to introduce the reader to aspects of the theory and practice of recursive recognition of the discrete random processes observations. The time series models as $AR(p)$, $MA(q)$, $ARMA(p, q)$ are used for a discrete random processes description. The recursive equations for sufficient statistics are obtained. These recursive equations allow to construct algorithms of current classification of observable sequences in the rate of entering its values into the on-line operation. The recursive algorithms for non Markov processes are obtained. Classification algorithms are realised in the form of computer programs.

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Laiko eilučių stebėjimų atpažinimo rekursyvūs algoritmai

Edward SHPILEWSKI

Straipsnyje pateikiamas laiko eilučių realizacijų nuoseklaus klasifikavimo metodai. Nuoseklaus atpažinimo metodai ir algoritmai yra gauti pakankamos statistikos rekurentinių lygčių pagrindu. Šios rekurentinės lygtys leidžia konstruoti nuoseklių ateinančių stebėjimo sekų klasifikavimo algoritmus. Klasifikavimo algoritmas realizuotas kompiuterio programomis. Algoritmas leidžia konstruoti daugiakanales skaičiavimo struktūras laiko eilučių nuosekliam atpažinimui.