On the Existence of Optimal Control of Differential Systems with After-Effect

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Abstract. This paper deals with the study of the optimal control problem for the objective

$$F(x, u, v) = \int_0^T f(x(t), x(t-h), \zeta(t), u(t), v(t), t)dt,$$

with $$x \in X, u \in U, v \in V; X, U$$ and $$V$$ being vector spaces, and $$\zeta(t) = \int_0^h R(t, \tau)x(t-\tau)\,d\tau,$$ subject to the differential equation

$$\frac{dx}{dt}(t) = m(x(t), x(t-h), \zeta(t), u(t), v(t), t) (0 \leq t \leq T),$$

and the constraints

$$g_i(u(t), t) \in S_i, g_2(v(t), t) \in S_2; n_1(x(t), t) \in V_1, n_2(x(t-h), t) \in V_2; n_3(\zeta(t), t) \in V_3 (0 \leq t \leq T),$$

where $$x(t) \in \mathbb{R}^n; \zeta(t) \in \mathbb{R}^n; u(t) \in \mathbb{R}^k; f, m, g_i (i = 1, 2), n_i (1 \leq i \leq 3)$$ and the entries to $$r(t, \tau): \mathbb{R}_+ \times \mathbb{R}_+ \to L(X, X)$$ are continuously differentiable functions. It is assumed that boundary conditions $$x(0) = x(T) = 0$$ are imposed. $$S_i (i = 1, 2)$$ and $$V_i (1 \leq i \leq 3)$$ are convex cones. The existence of a time-optimal control in analytic linear systems is also investigated via an extension of the bang-bang principle.

Key words: differential systems with after-effect, optimal control.

1. Introduction. Notation and General Preliminaries on Optimal Control for Differential Systems with After-Effect

Linear systems with after effect have been studied recently (Minyuk, 1983; Churakova, 1969; Metel’skii, 1978; Kurzhanski, 1966). The controllability and observability of such systems in Euclidean and Hilbert spaces have been studied in (Minyuk, 1983; Churakova, 1969). In (Metel’skii, 1978), a general analysis of the system’s properties of such systems has been stated. These systems present more complicated theoretical analysis of such properties due to the delay. A vector of an Euclidean space is never a state for such systems and a function space is needed to establish the formalism. In (M. de la Sen, 1988), the properties of such systems in the linear and time-invariant case have been related to Algebraic Systems Theory of standard linear differential systems as given in (Kailath, 1980). This paper presents a formulation of the optimal control problem (OC) using a Lagrangean theory as stated in (Craven, 1978) for the linear case. The paper’s body is organized in two sections. In Section 2, the optimization of a Hamiltonian functional is discussed by assuming for the optimal control for a convex cost functional to exist. Section 3 studies the conditions for the existence of a unique optimal control. Finally, conclusions end the paper.
1.1. Notation

The usual symbols for logical and set operations will be used, namely ⇒ (implies), ⇔ (if and only if, also written iff), ∀ (for all), ∃ (there exists); and ∈ (belongs to), ∪ (union), ∩ (intersection); \ (set difference; $S \setminus T$ is the set of elements in $S$ but not in $T$); ⊆ (inclusion), ⊊ (inclusion allowing =), ∅ (empty set); × (cartesian product), int $(S)$ and $\partial(S)$ denote, respectively, the interior and boundary of the set $S$.

- Superscripts $T$ and * indicate transposition and conjugate transposition of an operator.
- $U, V, W, X, Y, Z$ are real vector spaces.
- $B(x, \varepsilon)$ is the $n$-open ball of center $x$ and radius $\varepsilon > 0$ for $x \in X \subseteq \mathbb{R}^n$. $\bar{B}(x, \varepsilon)$ is the closure of $B(x, \varepsilon)$.
- $C(I)$: space of all continuous real functions on the interval $I$, with the uniform norm $\|x\|_\infty = \sup_{t \in I} |x(t)|$.
- $L^p(I)$: the space of functions whose $p$th powers are Lebesgue integrable on $I$, with finite seminorm $\|x\|_p = \left[ \int_I |x|^p \, dt \right]^{1/p}$.

When the space is required to be complete (as are $C(I)$ and $L^2(I)$), a Banach space will be specified.

For spaces of a vector space $X$ and $\alpha \in \mathbb{R}$,

$\alpha S = \{ \alpha s : s \in S \}$, and $S + T = \{ s + t : s \in S, t \in T \}$

- A set $S \subseteq X$ is a convex cone if $S + S \subseteq S$ and $(\forall \alpha \in \mathbb{R}_+) \alpha S \subseteq S$.
- The vector space of all continuous functionals on $X$ is the dual space of $X$, and denoted by $X'$.
- $L(X, Y)$ is the space of all continuous linear maps from $X$ into $Y$.
- For the dual space $X'$ of $X$, a weak * neighborhood of $p \in X'$ is any set $N(p) = \{ y \in X' : |y(x_i) - p(x_i)| < \varepsilon \ (i = 1, 2, \ldots, r) \}$ specified by finitely many points $x_1, x_2, \ldots, x_r$ in $X$ and a positive real constant $\varepsilon$. $Q \subseteq X'$ is weak * closed if every $p \in X' \setminus Q$ has a weak * neighborhood $N(p)$ which does not meet $Q$.
- $\langle . , . \rangle$ stands for the inner product in Euclidean spaces.

1.2. Optimal Control

Consider an optimal control problem, to minimize

$$ F(x, u, v) = \int_0^T f(x(t), x(t - h), \zeta(t), u(t), v(t), t) \, dt $$

$$ \zeta(t) = \int_0^h R(t, \tau)x(t - \tau) \, d\tau, $$

subject to the differential equation

$$ \frac{d}{dt} x(t) = \dot{x}(t) = m(x(t), x(t - h), \zeta(t), u(t), v(t), t) \quad (0 \leq t \leq T) $$

Piecewise continuous functions from $I \times \mathbb{R}$ to vector functions, and the constraints:

$$g_1(u(t), t) \in S_1; \quad g_2(v(t), t) \in S_2;$$
$$n_1(x(t), t) \in V_1; \quad n_2(x(t-h), t) \in V_2; \quad n_3(\zeta(t), t) \in V_3 \quad (0 \leq t \leq T). \quad (1.4)$$

Assume that $x(t), \zeta(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^k$; $f, m, g_i, n_j, \zeta(t)$ $(1 \leq i \leq 2; 1 \leq j \leq 3)$ are continuously differentiable functions in $\mathbb{R}^+$; and that boundary conditions $x(0) = 0$, $x(t) = \varphi(t)$, $\forall t \in [T, T+h]$ are imposed with $\varphi: \mathbb{R}^+ \to \mathbb{R}^n$ being an absolutely continuous differentiable function. Let $S_i \subseteq \mathbb{R}^n$ and $V_j \subseteq \mathbb{R}^{h_j}$ $(1 \leq i \leq 2; 1 \leq j \leq 3)$ be convex cones.

Appropriate vector spaces of functions must be specified for the subsequent formulation. Let $x \in X$, $u \in U$, $v \in V$, the spaces of piecewise continuous functions from $I = [0, T]$ into $\mathbb{R}^n$ or $\mathbb{R}^k$, with the uniform $\| \cdot \|_{\infty}$. Denote by $W$ the space of piecewise continuous functions from $J$ to $\mathbb{R}^n$, with the uniform norm. $x \in X$ is the Lebesgue integral of a function $x \in W$, and $x(0) = 0$, $x(t) = \varphi(t)$, $\forall t \in [T, T+h]$. Then, $x(t) = \int_0^t w(s) \, ds$ is expressed as $w = Dx$, $D = d/dt$, except at discontinuities of $w$, the linear map $D: X \to W$ is made continuous, by giving $X$ the norm $\| x \| = \| x \|_{\infty} + \| Dx \|_{\infty}$.

The differential equation for $x(t)$, with initial condition, expressed as

$$x(t) = \int_0^t m(x(s), x(s-h), \zeta(s), u(s), v(s), s) \, ds \quad (x, \zeta \in X, u \in U), \quad (1.5)$$

may then be written as $Dx = M(x, u, v)$ where the map $M: X \times X \times U \times V \to L(\mathbb{R}, W)$ is defined by $M(x, u, v)(t) = m(x(t), x(t-h), \zeta(t), u(t), v(t), t); \zeta(t) = \int_0^t \varphi(t, \tau) x(t-\tau) \, d\tau$. (The differential equation has now been slightly extended to allow a finite number of points where $x$ is not differentiable).

Let $Q_i$, $P_j$ denote the spaces of piecewise continuous functions from $I$ into $\mathbb{R}^n$ resp. $\mathbb{R}^{h_j}$ $(1 \leq i \leq 2; 1 \leq j \leq 3)$. Define the convex cones $K_i \subseteq Q_i; J_j \subseteq P_j$ as

$$K_i = \{ q_i \in Q_i; \forall t \in I, q_i(t) \in S_i \} \quad (1 \leq i \leq 2)$$
$$J_j = \{ p_j \in P_j; \forall t \in I, p_j(t) \in V_j \} \quad (1 \leq j \leq 3) \quad (1.6)$$

Define the maps $G_1: U \to Q_1; G_2: U \to Q_2$; and $N_i: X \to P_j$ $(1 \leq j \leq 3)$ by $G_1(u)(t) = g_1(u(t), t); G_2(v)(t) = g_2(v(t), t); N_1(x)(t) = n_1(x(t), t); N_2(x)(t) = n_2(x(t-h), t); N_3(x)(t) = n_3(\zeta(t), t) \quad (t \in I)$. Then, the constraints (1.4) are expressed as $G_i(u) \in K_i; N_j(x) \in J_j$ $(1 \leq i \leq 2; 1 \leq j \leq 3)$. Now, the problem (1.1)–(1.2) subject to (1.4) is expressible as

$$(OC): \text{Minimize} \quad \{ F(x, u, v); Dx = M(x, u, v), G_i(u) \in K_i, N_j(x) \in J_j; \quad x \in X, u \in U \quad (1 \leq i \leq 2; 1 \leq j \leq 3) \}.$$
Since $m$ is continuously Fréchet-differentiable,

$$m (x(t) + z(t), x(t - h), \zeta(t), u(t), v(t), t) = m_x (x(t), x(t - h), \zeta(t), u(t), v(t), t) z(t) + \sigma(t),$$

$$(1.7)$$

$$m (x(t), x(t - h) + z(t), \zeta(t), u(t), v(t), t) = m_x (x(t), x(t - h), \zeta(t), u(t), v(t), t) z(t) + \sigma_{-h}(t),$$

$$m (x(t), x(t - h), \zeta(t) + z(t), u(t), v(t), t) = m_x (x(t), x(t - h), \zeta(t), u(t), v(t), t) z(t) + \sigma_{-h}(t),$$

if $x(t), x(t - h), \zeta(t), x(t) + z(t), x(t - h) + z(t), \zeta(t) + z(t) \in X \times [0, T]$, where $m_x$, $m_x(h)$ denote partial derivatives, and $||\sigma(t)|| \leq \varepsilon_1 ||z(t)||_{\infty}$, $||\sigma_{-h}(t)|| \leq \varepsilon_2 ||z(t)||_{\infty}$, $||\sigma_{-h}(t)|| \leq \varepsilon_3 ||z(t)||_{\infty}$ if $||z||_{\infty} < \delta(\varepsilon)$, $\varepsilon = \max(\varepsilon_i; 1 \leq i \leq 3)$.

Consequently, $M$ is partially differentiable with respect to the above elements in $X$, with partial derivatives $M_x(x, u, v)$, $M_{x,-h}(x, u, v)$ given by $M_x(x, u, v) z(t) = m_x(x(t), x(t - h), \zeta(t), u(t), v(t), t) z(t); M_{x,-h}(x, u, v) z(t) = m_x(x(t), x(t - h), \zeta(t), u(t), v(t), t) z(t); M_{x,-h}(x, u, v) z(t) = m_x(x(t), x(t - h), \zeta(t), u(t), v(t), t) z(t)$.

Also, $F(x, u, v)$ is Fréchet differentiable with respect to the above elements in $X$, with partial derivatives $F_{x,-h}(x, u, v)$ given by $F_{x,-h}(x, u, v) z = \int_0^T f_x(x(t), x(t - h), \zeta(t), u(t), v(t), t) z(t) dt; F_{x,-h}(x, u, v) z = \int_0^T f_x(x(t), x(t - h), \zeta(t), u(t), v(t), t) z(t) dt; F_{x,-h}(x, u, v) z = \int_0^T f_x(x(t), x(t - h), \zeta(t), u(t), v(t), t) z(t) dt$; here

$$\|F(x + z, u, v) - F(x, u, v) - F_x(x, u, v) z\| \leq \varepsilon_1 \|z\|_{\infty}, \text{ if } \|z\|_{\infty} < \delta(\varepsilon_1/T);$$

$$\|F(x_{-h} + z, u, v) - F(x_{-h}, u, v) - F_{x_{-h}}(x, u, v) z\| \leq \varepsilon_2 \|z\|_{\infty}, \text{ if } \|z\|_{\infty} < \delta(\varepsilon_2/T);$$

$$\|F(\zeta + h, u, v) - F(x, u, v) - F_{\zeta}(x, u, v) z\| \leq \varepsilon_3 \|z\|_{\infty}, \text{ if } \|z\|_{\infty} < \delta(\varepsilon_3/T),$$

and similarly for the other functions.

For the problem (OC), define a Lagrangean

$$L (x, u, v, \tau, \varpi, \varpi_j; 1 \leq i \leq 2, 1 \leq j \leq 3) = \tau F(x, u, v) - \lambda \bar{D} (Dx - M(x, u, v)) - \varpi G_1(u) - \varpi G_2(u)$$

$$- \sum_{j=1}^3 \varpi_j N_j(x),$$

$$\begin{align*}
L (x, u, v, \tau, \varpi, \varpi_j; 1 \leq i \leq 2, 1 \leq j \leq 3) = \tau F(x, u, v) - \lambda \bar{D} (Dx - M(x, u, v)) - \varpi G_1(u) - \varpi G_2(u) \\
- \sum_{j=1}^3 \varpi_j N_j(x),
\end{align*}$$

$$\begin{align*}
L (x, u, v, \tau, \varpi, \varpi_j; 1 \leq i \leq 2, 1 \leq j \leq 3) = \tau F(x, u, v) - \lambda \bar{D} (Dx - M(x, u, v)) - \varpi G_1(u) - \varpi G_2(u) \\
- \sum_{j=1}^3 \varpi_j N_j(x),
\end{align*}$$

$$\begin{align*}
L (x, u, v, \tau, \varpi, \varpi_j; 1 \leq i \leq 2, 1 \leq j \leq 3) = \tau F(x, u, v) - \lambda \bar{D} (Dx - M(x, u, v)) - \varpi G_1(u) - \varpi G_2(u) \\
- \sum_{j=1}^3 \varpi_j N_j(x),
\end{align*}$$

$$\begin{align*}
L (x, u, v, \tau, \varpi, \varpi_j; 1 \leq i \leq 2, 1 \leq j \leq 3) = \tau F(x, u, v) - \lambda \bar{D} (Dx - M(x, u, v)) - \varpi G_1(u) - \varpi G_2(u) \\
- \sum_{j=1}^3 \varpi_j N_j(x),
\end{align*}$$
where \( \tau \in \mathbb{R}_+, \lambda \in W', \pi_i \in Q'_i, \pi_j \in P'_j (1 \leq i \leq 2; 1 \leq j \leq 3) \) are Lagrange multipliers not all zero.

**Definition 1.1** (Craven, 1978). The dual cone (or polar cone) of \( S \) is the convex cone

\[
S^* = \{ y \in X': (\forall s \in S) y(s) \geq 0 \}.
\]

(1.10)

If \( S \subseteq \mathbb{R}^n \), then \( S^* \subseteq \mathbb{R}^n \).

Two standard abstract minimization problems are

(P1) : Minimize \( \{ f(x): -g(x) \in S, -h(s) \in T \} \),

(P2) : Minimize \( \{ f(x): -h(x) \in T \} \),

where \( X, Y, Z \) are Banach spaces; \( X_0 \) is an open subset of \( X \); \( S \subseteq Y \) is a convex cone; with \( \text{int } S \neq \emptyset \); \( T \subseteq Z \) is a closed convex cone; and the functions \( f: X_0 \to \mathbb{R} \); \( g: X_0 \to Y \); \( h: X_0 \to Z \) are Fréchet differentiable.

The following Definition and Lemma are well-known (Craven, 1978).

**Definition 1.2.** The system \(-h(x) \in T\) is called locally solvable at the point \( a \), if for some \( \delta > 0 \), whenever the direction \( d \) satisfies \( h(a) + h'(a)d \in -T \) and \( \|d\| < \delta \), there exists a solution \( x = a + \alpha d + \eta(\alpha) \) to \(-h(x) \in T\), valid for sufficiently small \( \alpha > 0 \), where \( \|\eta(\alpha)\|/\alpha \to 0 \) as \( \alpha \downarrow 0 \) (for brevity \( \eta(0) = 0(\alpha) \)).

**Lemma 1.1.** (1) (Fritz – John theorem). For (P1), let the constraint \(-h(x) \in T\) be locally solvable at \( a \in X_0 \); let \([h'(a)](h(a))^T(T^*)\) be weak * closed. Then, a necessary condition for (P1) to attain a local minimum at \( x = a \) is

(FJ): \( \tau f'(a) + v g'(a) + \lambda h'(a) = 0; \quad v g(a) = 0; \quad \lambda h(a) = 0; \quad \tau \in \mathbb{R}_+, v \in S^*, \lambda \in T^* \),

where \( \tau \) and \( v \) are not both zero and define the modified Lagrangean \( \tau f(x) + v g(x) + \lambda h(x) \).

(2) (Kuhn – Tucker theorem). For (P2), let the constraint \(-h(x) \in T\) be locally solvable at \( a \in X_0 \); let \([h'(a)](h(a))^T(T^*)\) be weak * closed. Then, a necessary condition for (P2) to attain a local minimum at \( x = a \) is

(KT): \( f'(a) + \lambda h'(a) = 0; \quad \lambda h(a) = 0, \lambda \in T^* \).

(3) (KT) holds with the hypothesis of local solvability are placed by that of regularity at \( a \).
Representations will be required for the vectors in certain dual spaces. A \( y \in [C(I)]' \) may be represented by a function \( \lambda(\cdot) \) by \( y(x) = \int_x \lambda(t) dt \). Suppose that \( \lambda \) in (1.9) can be represented by a function \( \lambda(\cdot) \), where

\[
(\forall z \in X) \quad \lambda(Dz) = \int_0^{T+h} \lambda(z)Dz(t) dt. \tag{1.11}
\]

Assume that \( \varphi(t) = 0, t \in [T, T + h] \). Integrating by parts, and using the boundary conditions \( z(0) = z(T) = 0 \),

\[
Dz = -\int_0^{T+h} [D\lambda(t)] z(t) dt. \tag{1.12}
\]

From Lemma 1.1 under the involved suitable hypothesis necessary conditions for (OC) to attain a minimum at \((x, x(-h), \zeta, u, v) = (x^*, x(-h)^*, \zeta^*, u^*, v^*) = q^* \) are that the Fréchet derivative of \( L \) in (1.9) is zero at \( q^* \) so that:

\[
\begin{align*}
\tau F_x &- \overline{\lambda}(0 - M_x) - \overline{\pi}_1 N_1(x) = 0, \\
\tau F_{x(-h)} + \overline{\lambda} M_{x(-h)} - \overline{\pi}_2 N_2(x) & = 0, \\
\tau F_\zeta + \overline{\lambda} M_\zeta - \overline{\pi}_3 N_3(x) & = 0, \\
\tau F_u + \overline{\lambda} M_u - \overline{\pi}_4 G_1(v) & = 0, \\
\tau F_v + \overline{\lambda} M_v - \overline{\pi}_5 G_2(v) & = 0.
\end{align*} \tag{1.13}
\]

Using Eq. (1.12) into (1.13) for each \( z \in X \), and since \( z(t) = 0, t \in [T, T + h] \), one gets

\[
\int_0^T \left\{ \tau f_x(q^*, t) + \lambda(t) m_x(q^*, t) + \lambda'(t) \\
- \nu(t) n_1_x(q^*, t) \right\} z(t) dt = 0, \tag{1.14}
\]

where \( \overline{\pi} \) is represented by the function \( \nu(\cdot) \), and the Fréchet derivatives \( F(\cdot) \) are represented by integrals as above. Therefore \{ \cdot \} = 0 in (1.14).

A minimum of (OC) at \( q^* \) implies five transversality conditions, namely:

\[
\begin{align*}
\overline{\pi}_1 G_1(u^*) & ; \quad \overline{\pi}_2 G_2(v^*) = 0; \quad \overline{\pi}_1 N_1(x^*) = 0, \\
\overline{\pi}_2 N_2(x(-h)^*) & = 0; \quad \overline{\pi}_3 N_3(\zeta^*) = 0. \tag{1.15}
\end{align*}
\]

The problem (OC) of Eqs. (1.1)–(1.4) has a local minimum \( q^* \), under the necessary conditions of Lemma 1.1, with the transversality conditions (1.15). It is also useful to
investigate sufficient conditions for the existence of a local minimum for the lagrangean theory for problem (OC). For this purpose, the following concept and result are useful. A function \( g: X \to \mathbb{R} \) is convex if, for all \( x, y \) and \( 0 < \lambda < 1 \), \( \lambda g(x) + (1 - \lambda)g(y) - g(\lambda x + (1 - \lambda)y) \in \mathbb{R}^+ \). \( g \) is convex iff its epigraph \( \{(x, y) \in X \times \mathbb{R}: y \geq g(x)\} \) is a convex set. Let \( g: X \to Y \), \( g \) is \( S \)-convex, where \( S \) is a convex cone in \( Y \), if for all \( x, y \) and \( 0 < \lambda < 1 \), \( \lambda g(x) + (1 - \lambda)g(y) - g(\lambda x + (1 - \lambda)y) \in S \). If \( g \) is Fréchet differentiable, then \( g \) is \( S \)-convex (\( S \) being a closed convex cone in \( Y \)) iff, for all \( z \), \( x \in X, f(x) - f(z) - f'(z)(x - z) \in S \).

For sufficient Lagrangean conditions, the following result (Craven, 1978) is useful.

**Lemma 1.2.** (1) Let \((FJ)\) hold with \( \tau = 1 \), \( f \) convex, \( g \) \( S \)-convex, and \( h \) affine (linear plus a constant). Then (P1) is minimized at \( x = a \).

(2) Let \((FJ)\) hold with \( \nu \) not both zero, \( f \) convex, \( g(int U) \)-convex, where \( U \subset Y \) is a convex cone such that \( S \subset U \) and \( V \subset U^* \), and \( h \) \( T \)-convex (or affine). Then, \((P1)\) is minimized at \( a \).

Lemmas 1.1 and 1.2 together with the transversality conditions \((1.15)\) lead to the following main result for problems (OC) for system (1.1) to (1.4).

**Theorem 1.1.** (1) Eqs. \((1.16)\) below are necessary conditions for the differential system of the optimal control of problem (OC), namely (OCDE), if \( G \) being the set of systems \( G_i(u) \in K_i, N_j(x) \in J_j \) \((1 \leq i \leq 2; 1 \leq j \leq 3)\) is locally solvable at a point \( q^* = (x^*, x^*_{(-h)}, \zeta^*, u^*, v^*) \in \text{int}(X \times X \times X \times U \times V) \), the corresponding convex cone being a closed convex cone (and the others having nonempty interiors) with \( \rho(q^*) \in G \) fulfilling that \( \{g'(a)g(a)\}^T(G^*) \) is weak *-closed.

(2) In proposition (1) the hypothesis of local solvability may be substituted by that of the regularity at \( a \).

(3) Propositions (1)–(2) stand also under sufficient conditions if, furthermore, equations \((1.16)\) hold with \( \tau = 1 \), \( f \) convex, \( G \) affine and the remaining \( G_i, N_j \) not being \( G \), being \( Gi \) or \( N_j \)-convex \((1 \leq i \leq 2; 1 \leq j \leq 3)\), respectively. An alternative sufficient condition is \( G \) – convex, \( f \) convex and \( n(\_), n(\_); g(\_), g(\_); \text{int}(N) \) or \( \text{int}(G) \)-convex with either \( \tau \) or one of \( \mu_1, \nu_j \) \((1 \leq i \leq 2; 1 \leq j \leq 3)\) being non zero.

\[
\begin{align*}
\tau f_x(q^*, t) + \lambda(t)m_x(q^*, t) + \lambda'(t) - \nu_1(t)n_{1x}(q^*, t) &= 0, \\
\tau f_x(-h)(q^*, t) + \lambda(t)m_x(-h)(q^*, t) - \lambda'(t) - \nu_2(t)n_{2x(-h)}(q^*, t) &= 0, \\
\tau f_3(q^*, t) + \lambda(t)m_3(q^*, t) + \lambda'(t) - \nu_3(t)n_{3x}(q^*, t) &= 0, \\
\tau f_w(q^*, t) + \lambda(t)m_w(q^*, t) - \mu_1(t)g_1(q^*, t) &= 0, \\
\tau f_v(q^*, t) + \lambda(t)m_v(q^*, t) - \mu_2(t)g_2(q^*, t) &= 0, \\
\nu_1(t)n_1(q^*, t) &= 0; \quad \mu_1(t)g_1(q^*, t) = 0; \\
\nu_2(t)n_2(q^*, (-h), t) &= 0; \quad \mu_2(t)g_2(q^*, t) = 0; \\
\nu_3(t)n_3(q^*, t) &= 0.
\end{align*}
\]

\[
(1.16) \quad \text{(OCDE)}
\]
1.3. Comments and Relationships with Standard Linear Systems

The formulation of Section 1.2 has been generalized in a standard way that being appli-
cable to standard linear systems (see Minyuk, 1983; Churakova, 1969; Metel’kii, 1978;
Kurzanski, 1966; De la Sen, 1988; Kailath, 1980; Craven, 1978) by including, using
topological concepts, inequality constraints in the objective functional. The main con-
clusion is that the problem of equality (differential system equations) inequality (state,
delayed-state, input and disturbance vectors) constraints may be formulated using the
concept of $S$-convex functions in a $S$ closed convex cone for constraints in the levels
of the above magnitudes may be directly generalized from the standard linear cases us-
ring a Lagrangean theory for the existence of necessary conditions for convex objective
functionals, for local solvability – and sufficient (Fritz-John/Kühn-Tücker) conditions for
such a solvability, so that the problem reduces to the solution of a standard system of
differential equations, including transversal differential constraints (OCDE).

2. Time-Optimal Control, Bang-Bang Principle

The general problem to be considered in this section is the following. Let a control system
given by the differential equation:

$$
\dot{x}(t) = A(t)x(t) + A_1(t)x(t-h) + \int_{h}^{0} R(t, \tau)x(t - \tau) \, d\tau + B(t)u(t) + f(t)
$$

subjected to the initial conditions $x_0(t) = x_s$ and $x(t) = \varphi(t)$, $t \in [t_0 - h, t_0)$. This sys-
tem is a (linear) particular case with state convolution influencing the state time-derivative
(see De Guzmán (1980) for the standard linear case).

The following assumptions are made.

(1) $A(t)$ is a real $n \times n$ matrix function, piecewise analytic in $[t_0, T]$, namely $[t_0, T]
may be divided into intervals $[t_i, t_{i+1}]$ with $t_0 < t_1 < t_2 < \ldots < T$ and the entries
of $A(t)$ are analytic in each $(t_i, t_{i+1})$. At the points $t_i$, $A(t)$ is continuous either on its
left-side or in its right-side.

(2) $A_1(t)$ and $B(t)$ are, respectively, $n \times n$ and $n \times m$ real matrix functions being
piecewise analytic in $[t_0, T]$.

(3) $R(t, \tau)$ is a $n \times n$ real matrix function being piecewise analytic in $[t_0, T] \times [t_0, T]$.

(4) $f(t)$ is a $n$-vector of disturbances being a set of piecewise continuous functions $F
on [t_0, T]$.

(5) $u(t)$ is a $m$-vector piecewise continuous function defined in $[t_0, T]$ which takes
values in a constraint compact subset of $\mathbb{R}^m$, $\mathcal{L} = \{u \in \mathbb{R}^m: |u_i| \leq 1; i = 1, 2, \ldots, m\}$.

The set $\mathcal{U}$ of such functions is the family of admissible controls.

(6) The pair $(x_0, \varphi(t))$ such that $\varphi: [t_0 - h, t_0] \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ with $\varphi(\cdot)$ being piecewise
continuous is the set of initial conditions in the initial-value problem (2.1).

(7) $x(t)$ is the state $n$-vector in a function space $X$ of elements $x: [0, t] \times X \times X \times
\mathcal{U} \times F \rightarrow X$. 

An absolutely continuous solution (2.1) is in (De la Sen, 1988).

\[ x(t) = F(t, t_0)x_0 + \int_{t_0}^{t} F(t, s + h)A_1(s + h)\varphi(s) \, ds \\
+ \int_{t_0}^{t} \left[ \int_{t_0-s}^{h} F(t, \tau + s)R(\tau + s, \tau) \, d\tau \right] \varphi(s) \, ds \\
+ \int_{t_0}^{t} F(t, \tau)B(\tau)u(\tau) \, d\tau + \int_{t_0}^{t} F(t, \tau)f(\tau) \, d\tau, \]  

(2.2)

\[ \frac{\partial F(t, \tau)}{\partial \tau} = -F(t, \tau)A(\tau) - F(t, \tau + h)A_1(\tau + h) \\
- \int_{t_0}^{h} F(t, s + \tau)R(s + \tau, s) \, ds, \]  

(2.3)

\( F(t, t) = I; \ F(t, \tau) \equiv 0, \ \tau > t. \)

The attainable set \( K(t) \) for \( t \geq t_0 \) is

\[ K(t) = \{ x(t, u): \ u \in U \}. \]

For each \( t \geq t_0 \), a compact set \( G(t) \) which continuously varies with \( t \) is given. This set is the target set. It is assumed that \( x_0 \not\in G(t_0), \ \varphi(t) \not\in G(t), t \in [t_0 - h, t_0]. \) If there is \( t > t_0 \) such that

\[ K(t) \cap G(t) \neq \emptyset, \]  

(2.4)

this control drives \( x_0, \ \varphi(t), t \in [t_0 - h, t_0] \) to the target in time \( t \). \( u^* \in U \) is a time-optimal control if there is \( t^* > t_0 \) such that \( x(t^*, u) \in G(t^*) \) and \( G(t) \cap K(t) = \emptyset \) for all \( t \) with \( t_0 \leq t \leq t^* \).

The following result is useful (Craven, 1978) to establish the bang-bang principle for system (2.1).

**Lemma 2.1** [(preliminary)] 1. Let \( g: [t_0, t] \to \mathbb{R}^n \) a piecewise analytic function on \( [t_0, t] \). Let \( \mathcal{V} \) be the set of functions from \( [t_0, t] \) to \([-1, 1]\) nonoscillant (see Craven, 1978) and piecewise continuous. Define the sets

\[ \mathcal{V}_0 = \{ v \in \mathcal{V}: \ \forall s \in [t_0, t], \ |v(s)| = 1 \} , \]

\[ K = \left\{ \int_{t_0}^{t} g(s)v(s) \, ds: v \in \mathcal{V} \right\} , \]  

(2.5)

\[ K_0 = \left\{ \int_{t_0}^{t} g(s)v(s) \, ds: v \in \mathcal{V}_0 \right\} . \]

Then, \( K = K_0 \) and it is a compact and convex set.

2. Let \( Y(s) \) be a piecewise analytic \( n \times m \)-matrix function defined on \( [t_0, t] \). Let \( \mathcal{V} \) be the set of piecewise \( m \)-vector functions on \( [t_0, t] \) taking values in \( \mathcal{L} \) and being
nonoscillant. Define sets \( V_0, K \) and \( K_0 \) as in proposition (1) with the changes \( g \to Y \) and \( v \to v_1 \) (in Definition of \( V_0 \) only). Then, proposition 1 of the Lemma stands.

(3) Let \( \mathcal{F} \) a family of compact sets \( \mathbb{R}^n \). Introduce a metric \( \rho \) as follows:

\[
\rho(K_1, K_2) = \inf \{ \varepsilon > 0 : K_1 \varepsilon \supseteq K_2, K_2 \varepsilon \supseteq K_1 \}
\]

for each \( K_1, K_2 \in \mathcal{F} \), where if \( A \) is a compact set, then

\[
A_{\varepsilon} = U \{ \overline{B}(x, \varepsilon) : x \in A \}.
\]

Then if \( P \) and \( Q \) are two compact and convex sets \( \rho(P, Q) = \rho(\partial(P), \partial(Q)) \).

Now, Lemma 2.1 is used in the subsequent formulation to establish the bang-bang principle for the linear system with after-effect (2.1).

2.1. Bang-Bang Principle

\( K(t) = K_0(t) \) is a compact and convex set is an immediate consequence of Lemma 2.1. The continuity is proved as follows. Let \( t_2 \in [t_0, \infty) \). It is now proved that given a real constant \( \varepsilon > 0 \). There is \( \delta > 0 \) such that if \( |t_3 - t_2| \leq \delta \), then \( \rho(K(t_3), K(t_2)) \leq \varepsilon \); so that:

a) for all \( x(t_3, u) \in K(t_3) \), there is \( x(t_2; \overline{u}) \) such that \( |x(t_3, u) - x(t_2; \overline{u})| \leq \varepsilon \);

b) for all \( x(t_2, \overline{u}) \in K(t_2) \), there is \( x(t_3; \overline{u}) \in K(t_3) \) such that \( |x(t_2, u) - x(t_3, \overline{u})| \leq \varepsilon \).

For some \( \overline{u}, \overline{\overline{u}} \in U \) and each \( u \in U \). Assume now \( t_3 > t_2 \), then one gets from (2.2)

\[
|x(t_3, u) - x(t_2; \overline{u})| = \left| \int_{t_2}^{t_3} F(t_3, s + h)A_1(s + h)\varphi(s) \, ds \right|
\]

\[
+ \int_{t_2}^{t_3} \int_{t_0}^{s} [F(t_3, \tau + s)R(s + \tau, \tau) \, d\tau] \varphi(s) \, ds
\]

\[
+ \int_{t_0}^{t_3} F(t_3, \tau)B(\tau)u(\tau) \, d\tau + \int_{t_0}^{t_3} F(t_3, \tau)f(\tau) \, d\tau
\]

\[
- \left[ F(t_2, t_0) + \int_{t_0}^{t_2} F(t_2, s + h)A_1(s + h)\varphi(s) \, ds \right]
\]

\[
+ \int_{t_0}^{t_2} \left. \int_{t_0}^{\tau} F(t_2, \tau + s)R(\tau + s, \tau) \, d\tau \right| \varphi(s) \, ds
\]

\[
+ \int_{t_0}^{t_2} F(t_2, \tau)B(\tau)u(\tau) \, d\tau + \int_{t_0}^{t_2} F(t_2, \tau)f(\tau) \, d\tau \right|
\]
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\[
= \left( F(t_3, t_0) - F(t_2, t_0) \right) x_0
\]

\[+
\int_{t_0}^{t_2} [F(t_3, s + h) - F(t_2, s + h)] A_1(s + h)\varphi(s) \, ds
\]

\[+
\int_{t_0-h}^{t_0} \left[ \int_{t_0-s}^{h} (F(t_3, \tau + s) - F(t_2, \tau + s)) R(s + \tau, \tau) \, d\tau \right]
\]

\[+
\int_{t_0}^{t_2} [F(t_3, \tau) (B(\tau)u(\tau) + f(\tau)) - F(t_2, \tau) (B(\tau)\bar{u}(\tau) + f(\tau))] \, d\tau
\]

\[+
\int_{t_2}^{t_3} F(t_3, s + h) A_1(s + h)\varphi(s) \, ds
\]

\[+
\int_{t_2}^{t_3} F(t_3, \tau) (B(\tau)u(\tau) + f(\tau)) \, d\tau \right].
\]

Theorem 2.1 (Bang-Bang principle). The function \( K: t \in [t_0, T] \to K(t) \in \mathcal{F} \) (\(K(t)\) being a compact and convex set) is continuous in the metric \( \rho \) defined in \( \mathcal{F} \) (in Lemma 2.1). Furthermore, if \( \mathcal{U}_0 = \{ u \in \mathcal{U} \colon \forall s \in [t_0, T], \ V_i = 1, 2, \ldots, m, \ |u_i(s)| = 1 \}, \ K_0(t) = \{ x(t, u) \colon u \in \mathcal{U}_0 \}, \) then \( K(t) = K_0(t) \).

The above theorem states the reachable set may be generated continuously using bang-bang control in the same way as in the standard linear case without after-effect (De Guzmán, 1980). The following (result-) corollaries may be established without difficulty:

(A) If there is one \( u_0^* \in \mathcal{U}_0 \) which is optimal, then it is optimal in \( \mathcal{U} \).

(B) If there is an optimal control in \( \mathcal{U}_0 \), then there is a bang-bang optimal control.
2.2. Existence of the Optimal Control. Pontryagin’s Maximum Principle

The next result is referred to the existence of optimal control assumed that the target is attainable. It follows by direct extension of results for the linear case by using Lemma 2.1.

**Theorem 2.2.** Assume there is \( t \geq 0 \) such that \( K(t) \cap G(t) \neq \emptyset \) for system (2.1). Then, there is an optimal control \( u^* \) which makes to verify \( x(t^*, u^*) \in \partial K(t^*) \).

\( u \in U \) is extremal iff \( x(t, u) \in \partial K(t) \). Theorem 2.2 establishes that optimal control is necessarily extremal. Next theorem gives a necessary and sufficient condition for an optimal control to exist.

**Theorem 2.3** (Pontryagin’s principle). A control \( u^* \) is an extremal for system (2.1) in \([t_0, t^*]\) iff there is a \( n \)-vector function \( \lambda(t) \) being identically zero whose time-derivative exists in \([t_0, T^*] \) and verifies:

a) \( \lambda'(t) = -\frac{\partial F^*(t, \tau)}{\partial \tau} \lambda(t); \) b) for all \( t \in [t_0, t^*] \), it follows that

\[
\langle \lambda(t), B(t)u^*(t) \rangle = \max \{ \langle \lambda(t), B(t) \rangle : u \in U \}.
\]

**Proof.** Assume that \( x(t^*, u) \in \partial K(t^*) \). Since \( K(t^*) \) is convex and compact, there is a tangent hyperplane \( \Pi \) of \( K(t^*) \) in \( x(t^*, u^*) \) such that the outward normal \( \lambda(t^*) \) in \( x(t^*, u^*) \) verifies the equation of \( \Pi \), \( \langle \lambda - x(t^*, u^*), \lambda(t^*) \rangle = 0 \). The points of \( K(t^*) \) are in the affine half-space determined by \( \Pi \) given by \( \langle \lambda - x(t^*, u^*), \lambda(t^*) \rangle < 0 \).

Define the function \( \lambda: [t_0, t^*] \to \mathbb{R}^n \) by \( \lambda(t) = F(t, t^*)\lambda(t^*) \) where \( F(t, \tau) \) is given by Eq. 2.3. Assume that there is \( t_1 \in [t_0, t^*] \) such that \( \langle \lambda(t_1), B(t_1)u^*(t_1) \rangle < \max \{ \langle \lambda(t_1), B(t_1)u : u \in U \} \).

By continuity, \( \langle \lambda(s), B(s)u^*(s) \rangle < \max \{ \langle \lambda(s), B(s)u : u \in U \} \) for all \( s \) in an open interval of \([t_0, t^*] \). Define now a function \( \overline{\lambda}: [t_0, t^*] \to U \) such that

\[
\langle \lambda(s), B(s)\overline{\lambda}(s) \rangle = \max \{ \langle \lambda(s), B(s)u \rangle : u \in U \} \tag{2.8}
\]

which is defined for each \( s \in [t_0, t^*] \) by taking the vector \( \overline{\lambda}(s) \) with the same direction as that of \( B^*(s)\lambda(s) \) and maximum modulus with the constraint \( \overline{\lambda}(s) \in U \). If \( B^*(s)\lambda(s) = 0 \), then \( \overline{\lambda}(s) \) is arbitrary piecewise continuous in \( U \). \( \overline{\lambda} \) is admissible. Then,

\[
\langle \lambda(t^*), x(t^*, \overline{\lambda}) \rangle = \langle \lambda(t^*), F(t^*, t_0)x_0 \rangle \\
+ \int_{t_0-h}^{t_0} F(t^*, s + h)A_1(s + h)\varphi(s) \, ds \\
+ \int_{t_0-h}^{t_0} \left[ \int_{t_0-s}^{h} F(t^*, \tau + s)R(\tau + s, \tau) \, d\tau \right] \varphi(s) \, ds
\]
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\[ + \int_{t_0}^{t^*} F(t^*, \tau) (B(\tau)\varphi(\tau) + f(\tau)) \, d\tau \]

\[ > \left\langle \lambda(t^*), F(t^*, t_0)\varphi(0) + \int_{t_0}^{t^*} F(t^*, s + h)A_1(s + h)\varphi(s) \, ds \right\rangle \]

\[ + \int_{t_0}^{t^*} \left[ \int_{t_0-s}^{h} F(t^*, \tau + s)R(\tau + s, \tau) \, d\tau \varphi(s) \, ds \right] \]

\[ + \int_{t_0}^{t^*} F(t^*, \tau) (B(\tau)u^*(\tau) + f(\tau)) \, d\tau \]

\[ = \langle \lambda(t^*), x(t^*, u^*) \rangle. \] (2.9)

Namely, \( \langle \lambda(t^*), x(t^*, \overline{\pi}) - x(t^*, u^*) \rangle > 0 \) so that \( x(t^*, u^*) \not\in K(t^*) \) which is a contradiction. Then (b) holds. By hypothesis, \( x(t^*, u^*) \in \partial K(t^*) \) from Theorem 2.2.

If \( x(t^*, u^*) \in \text{int} \{ K(t^*) \} \), consider \( x(t^*, \overline{\pi}) \in K(t^*) \) such that \( \langle \lambda(t^*), x(t^*, u^*) - x(t^*, \overline{\pi}) \rangle < 0 \). Such points exist since there is a ball of center \( x(t^*, u^*) \) contained in \( K(t^*) \). From hypothesis, \( \langle \lambda(t), B(t)u^*(t) \rangle > \langle \lambda(t), B(t)\overline{\pi}(t) \rangle \), all \( t \in [t_0, t^*] \). Then, \( \langle \lambda(t^*), x(t^*, u^*) \rangle \geq \langle \lambda(t^*), x(t^*, \overline{\pi}) \rangle \).

The above proof implies that once \( \lambda(t^*) \) is fixed, \( \lambda(s) \) results fixed, and condition b) fixes for \( \forall s \in [t_0, t^*] \) such that \( B^*(s)\lambda(s) = 0 \).

3. Conclusion

This paper has dealt with the study of optimal control for systems with after-effect. The first part has studied the extension of an optimal control for the nonlinear case and arbitrary convex nonlinear functionals by extending Lagrangean theory. The second part has dealt with the derivation of the Pontryagin’s maximum principle for time-optimal control of analytic linear systems with after-effect. In both cases, it has been found that well-known results for the standard linear case can be directly extended without difficulty.

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Apie diferencialinį sistemų su liekamuojų poveikų optimalų valdymą

Manuel de la SEN

Straipsnyje nagrinėjamas sistemų su liekamuojų poveikų optimalus valdymas. Pirmoje dalyje analizuojamas optimalus valdymas netiesiniu atveju panaudojant Lagranžo teoriją. Antroje dalyje Pontriagino maksimumo principas pritaikomas tiesinių sistemų su liekamuojų poveikų optimaliam valdymui.