ANALYTIC SOLUTION OF HOMOGENEOUS POINT-DELAY SYSTEMS. THE FIRST ORDER CASE

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Abstract. New formulae providing exact explicit solutions for a class of scalar delayed systems are given.

Key words: point-delayed system, stability.

1. Introduction. The stability of a linear delay-differential system with a point delay in its state has been studied in different works (Hmamed, 1985; Hmamed, 1986a; Hmamed 1986b; Mori, 1986; Bourles, 1987; Mori and Kokame, 1989; De la Sen, 1992a; De la Sen, 1992b). However, the main difficulty when dealing with delayed systems in the time-domain is the non availability of exact explicit solutions for any \( t > 0 \) and the absence of direct stability tests of easy testing for such systems. If an exact explicit solution could be obtained for a delayed system for any \( t > 0 \), the study of its stability would become direct. In this note a simple general method to obtain such a solution is proposed, and an exact explicit solution for any \( t \) is provided for the linear scalar invariant case with one point-delay.

In many works exact formulae for solving delayed systems have been given. The most general of those results can be found in De la Sen (1988). However, in the literature none of the exact given solutions for systems with state-delays is explicit. This means that the exact solutions always include a subsidiary delayed differential equation that is not directly solvable to compute the fundamental matrix (De la Sen, 1988; Burton, 1985). Let us consider a first-order homogeneous and invariant linear differential system with one point delay in its state, as shown in the next equation:

\[
\dot{x}(t) = a_0 x(t) + a_1 x(t - h),
\]
with initial conditions \( x(t) = 0 \) \( \forall t < 0 \); \( x(0) = \varphi(0) = x_0 \), where \( a_0, a_1 \in \mathbb{R}^* \) and \( h \in \mathbb{R}^+ \). For \( t \) such that \( 0 \leq t < h \) system (1) is reduced to the non-delayed system \( \dot{x}(t) = a_0 x(t) \) with initial condition \( x(0) = x_0 \) which can be solved as an ordinary first-order differential equation with a known forcing term in the case when \( \varphi(t) \) is nonzero on \( [-h, 0) \), (Burton, 1985). Since the global stability is independent of the chosen initial conditions, it is sufficient the choice of \( \varphi(t) = 0 \) on \( [-h, 0) \) with \( \varphi(0) = x_0 \) to investigate the stability of (1). Again, note that for \( t \) such that \( h < t \leq 2h \) Eq. 1 becomes

\[
\dot{x}(t) = a_0(t) x(t) + a_1(t) \varphi(t - h) .
\]

Once again, it could be possible to solve (2) as an ordinary first-order differential equation with a known forcing term, and find an exact explicit solution \( \phi_2(t) \) of \( x(t) \) for \( t \) such that \( h < t \leq 2h \). By following successively this simple method, it would be possible to find the exact explicit solution for any interval. However, the problem is to find a general formula for the \( n \) term, and then to solve exactly and explicitly the delayed differential equation and then checking directly its stability by using the above solution. This goal is pursued in the next section for a class of first-order linear delayed systems by following the so-called method of steps (Hale, 1977).

2. Study of a class of first-order homogeneous linear systems with one point-delay. Let us consider a first-order homogeneous and invariant linear differential system with one point delay in its state, as shown in Eq. 1, subject to initial conditions \( x(t) = 0 \) \( \forall t < 0 \); \( x(0) = \varphi(0) = x_0 \), where \( a_0, a_1 \in \mathbb{R}^* \) and \( h \in \mathbb{R}^+ \). For \( t \) such that \( 0 \leq t < h \) system (1) is reduced to the non-delayed system \( \dot{x}(t) = a_0 x(t) \) with initial condition \( x(0) = x_0 \) and with exact explicit solution \( \phi_1(t) = x_0 e^{a_0 t} \). For successive intervals is possible to apply the method described in Section 1. The exact explicit solutions for the second and third intervals are the following ones:

For \( t \) such that \( h \leq t < 2h \),

\[
x(t) = \phi_2(t) = x_0 e^{a_0 t} \left[ 1 - \frac{a_1 h}{ea_0 h} + \frac{a_1 t}{ea_0 h} \right] .
\]

For \( t \) such that \( 2h \leq t < 3h \),

\[
x(t) = \phi_3(t) = x_0 e^{a_0 t} \left[ 1 - \frac{a_1 h}{ea_0 h} + \frac{2a_1^2 h^2}{e^2 a_0 h} + \frac{a_1 t}{ea_0 h} - \frac{2a_1^2 h t}{e^2 a_0 h} + \frac{a_1^2 T^2}{2e^2 a_0 h} \right]
\]
Define \( b = a_1e^{-ah} \). After some simple algebraic manipulations, it is possible to rewrite (3) - (4) as follows.

\[
\begin{align*}
\phi_1(t) &= x_0e^{at}, \\
\phi_2(t) &= \phi_1(t)\left[1 + b(t - h)\right], \\
\phi_3(t) &= \phi_1(t)\left[1 + b\left(t - h\right) + \frac{b}{2}(t - 2h)^2\right].
\end{align*}
\]

It is seen below that the exact explicit solution for the \( n \)-th interval is

\[
\phi_n(t) = x_0e^{at}\left(1 + \sum_{i=1}^{n-1} (t - ih)^i \prod_{j=1}^{i} \left(\frac{a_1e^{-ah}}{j}\right)\right).
\]

This solution is formalized in the following main result whose proof is given by using complete induction in Appendix A.

**Theorem (Main result).** Consider system (1). An exact explicit unique solution for that system on \([0, \infty)\), subject to the initial conditions \( \varphi(t) = 0 \), \( t \in [-h, 0) \), and \( \varphi(0) = x(0) = x_0 \), is built in closed-form by using truncated functions as follows

\[
\Omega(t) = \sum_{i=1}^{\infty} \phi_{IT}(t), \quad \text{all } t \geq 0
\]

subject to the initial conditions \( \varphi(t) = 0 \), \( t \in [-h, 0) \), and \( \varphi(0) = x(0) = x_0 \), where

\[
\begin{align*}
\phi_{IT}(t) &= x_0e^{at}(U(t) - U(t - h)), \\
\phi_{IT}(t) &= x_0e^{at}\left(1 + \sum_{k=1}^{i-1} (t - kh)^k \prod_{j=1}^{k} \left(\frac{a_1e^{-ah}}{j}\right)\right) \\
&\times (U(t - (i - 1)h) - U(t - ih)) \\
(\Rightarrow \phi_{IT}(t) = 0 \quad \text{for } \quad t < (i - 1)h \quad \text{or} \quad t \geq ih \quad \text{for} \quad i \geq 2)
\end{align*}
\]

where \( U(t) \) is the unity step function at \( t = 0 \).

Conditions for stability of system (1) could be directly obtained by taking the limit when \( n \) tends to infinity in Eq. 8.
Analytic solution of homogeneous point-delay systems.

Numerical example. Let us consider the following scalar delayed system

$$\dot{x}(t) = -6x(t) + x(t-2), \quad (11)$$

with initial conditions $x(t) \equiv 0 \; \forall t < 0$; $x(0) = \varphi(0) = 3$. For $t$ such that $0 \leq t < 2$ system (11) is reduced to the non-delayed system $\dot{x}(t) = -6x(t)$ with initial condition $x(0) = 3$ and with exact explicit solution $\phi_1(t) = 3e^{-6t}$. For $t$ such that $2 \leq t < 4$,

$$x(t) = \phi_2(t) = 3e^{-6t} \left[ 1 - \frac{2}{e^{-12}} + \frac{t}{e^{-12}} \right], \quad (12)$$

and note that $\phi_1(2) = \phi_2(2)$. According to the Theorem, the exact explicit solution for the $n$-th interval is

$$\phi_n(t) = 3e^{-6t} \left( 1 + \sum_{i=1}^{n-1} \left( (t-i2)^{\frac{i}{j=1}} \left( \frac{e^{12}}{j} \right) \right) \right). \quad (13)$$

3. Conclusions. The simple method presented in this note has been used to find an explicit exact solution for a class of delay-differential systems with one point delay which does not depend on a time-delayed fundamental matrix.

APPENDIX A. Proof of the main result by complete induction. For $n = 1$ ($\Rightarrow$ $0 \leq t < h$), the proposed solution (9) and system (1) become, respectively, to

$$\Omega(t) = \phi_1(t) = x_0e^{at}(U(t) - U(t-h)) = x_0e^{at} = \phi_1(t), \quad (A1)$$
$$\dot{x}(t) = a_0x(t), \; x(0) = x_0. \quad (A2)$$

For $n = 2$ ($\Rightarrow$ $h \leq t < 2h$), the proposed solution (9) and system (1) become, respectively, to

$$\Omega(t) = \phi_1(t) + \phi_2(t) = x_0e^{at}[1 + a_1e^{-ah}(t-h)], \quad (A3)$$
$$\dot{x}(t) = a_0x(t) + a_1\phi_1(t-h) = a_0x(t) + a_1x_0e^{at}(t-h). \quad (A4)$$

By direct substitution it can be seen that (A1) is a solution for (A2) and (A3) is a solution for (A4). Therefore, the theorem is demonstrated for $n = 1, 2$. Let
us suppose that the proposed solution (9) is valid for both \( n - 2 \) and for \( n - 1 \), this is, for \((n - 3)h \leq t < (n - 2)h\) and for \((n - 2)h \leq t < (n - 1)h\). Thus

\[
\phi_{n-2}(t) = x_0 e^{a_0 t} \left( 1 + \sum_{i=1}^{n-3} \left( (t - ih)^{j-1} \prod_{j=1}^{i} \left( \frac{a_1 e^{-a_0 h}}{j} \right) \right) \right) \tag{A5}
\]

is a solution for system (1) for any \( t \) such that \((n - 3)h \leq t < (n - 2)h\) and

\[
\phi_{n-1}(t) = x_0 e^{a_0 t} \left( 1 + \sum_{i=1}^{n-2} \left( (t - ih)^{j-1} \prod_{j=1}^{i} \left( \frac{a_1 e^{-a_0 h}}{j} \right) \right) \right) \tag{A6}
\]

is a solution for system (1) for any \( t \) such that \((n - 2)h \leq t < (n - 1)h\). In particular, \( \phi_{n-1}(t) \) must satisfy the state equation (1) for such interval:

\[
\dot{\phi}_{n-1}(t) = a_0 \phi_{n-1}(t) + a_1 \phi_{n-2}(t - h) \tag{A7}
\]

Note that, by differentiating \( \phi_{n-1}(t) \), one obtains

\[
\dot{\phi}_{n-1}(t) = a_0 \phi_{n-1}(t) + x_0 e^{a_0 t} \left[ \sum_{i=1}^{n-2} \left( i(t - ih)^{j-1} \prod_{j=1}^{i} \left( \frac{a_1 e^{-a_0 h}}{j} \right) \right) \right]. \tag{A8}
\]

Substitute (A8) in (A7) to yield

\[
a_1 \phi_{n-2}(t - h) = x_0 e^{a_0 t} \left[ \sum_{i=1}^{n-2} \left( i(t - ih)^{j-1} \prod_{j=1}^{i} \left( \frac{a_1 e^{-a_0 h}}{j} \right) \right) \right]
\Rightarrow a_1 x_0 e^{a_0 (t-h)} \left[ 1 + \sum_{i=1}^{n-3} \left( (t - (i + 1)h)^{j-1} \prod_{j=1}^{i} \left( \frac{a_1 e^{-a_0 h}}{j} \right) \right) \right]
\Rightarrow x_0 e^{a_0 t} \left[ \sum_{i=1}^{n-2} \left( i(t - ih)^{j-1} \prod_{j=1}^{i} \left( \frac{a_1 e^{-a_0 h}}{j} \right) \right) \right]. \tag{A9}
\]

The proposed solution (9) for the \( n \)-th interval, \((n - 1)h \leq t < nh\), becomes

\[
\Omega(t) = \phi_n(t) = x_0 e^{a_0 t} \left( 1 + \sum_{i=1}^{n-1} \left( (t - ih)^{j-1} \prod_{j=1}^{i} \left( \frac{a_1 e^{-a_0 h}}{j} \right) \right) \right) \tag{A10}
\]

and therefore it must be demonstrated that (A10) satisfies the state equation as follows

\[
\dot{\phi}_n(t) = a_0 \phi_n(t) + a_1 \phi_{n-1}(t - h). \tag{A11}
\]
Note that the right-hand side of (A11) can be rewritten as

\[
a_0 \phi_n(t) + a_1 \phi_{n-1}(t - h) = a_0 \phi_n(t) + a_1 x_0 e^{a_0(t - h)} \left[ 1 + \sum_{i=1}^{n-2} \left( (t - (i + 1)h)i \prod_{j=1}^{i} \left( \frac{a_1 e^{-ah}}{j} \right) \right) \right]. (A12)
\]

By adding and subtracting the left-hand and the right-hand sides of identity (A9) to (A12) one obtains the next chain of identities

\[
a_0 \phi_n(t) + a_1 \phi_{n-1}(t - h) = a_0 \phi_n(t) + a_1 x_0 e^{a_0(t - h)} \left[ 1 + \sum_{i=1}^{n-2} \left( (t - (i + 1)h)i \prod_{j=1}^{i} \left( \frac{a_1 e^{-ah}}{j} \right) \right) \right] - a_1 x_0 e^{a_0(t - h)} \left[ 1 + \sum_{i=1}^{n-3} \left( (t - (i + 1)h)i \prod_{j=1}^{i} \left( \frac{a_1 e^{-ah}}{j} \right) \right) \right] + x_0 e^{a_0(t - h)} \left[ \sum_{i=1}^{n-2} \left( i(t - ih)i - 1 \prod_{j=1}^{i} \left( \frac{a_1 e^{-ah}}{j} \right) \right) \right] = a_0 \phi_n(t) + a_1 x_0 e^{a_0(t - h)}(t - (n - 2 + 1)h)\prod_{j=1}^{n-2} \left( \frac{a_1 e^{-ah}}{j} \right) + x_0 e^{a_0(t - h)} \left[ \sum_{i=1}^{n-2} \left( i(t - ih)i - 1 \prod_{j=1}^{i} \left( \frac{a_1 e^{-ah}}{j} \right) \right) \right] = a_0 \phi_n(t) + x_0 e^{a_0(t - n - 1)}(t - (n - 1)h)\prod_{j=1}^{n-2} \left( \frac{a_1 e^{-ah}}{j} \right) + x_0 e^{a_0(t - h)} \left[ \sum_{i=1}^{n-2} \left( i(t - ih)i - 1 \prod_{j=1}^{i} \left( \frac{a_1 e^{-ah}}{j} \right) \right) \right] = a_0 \phi_n(t) + x_0 e^{a_0(t - n - 1)}(t - (n - 1)h)\prod_{j=1}^{n-1} \left( \frac{a_1 e^{-ah}}{j} \right) + x_0 e^{a_0(t - h)} \left[ \sum_{i=1}^{n-2} \left( i(t - ih)i - 1 \prod_{j=1}^{i} \left( \frac{a_1 e^{-ah}}{j} \right) \right) \right] = a_0 \phi_n(t) + x_0 e^{a_0(t - n)}(t - (n - 1)h)\prod_{j=1}^{n-1} \left( \frac{a_1 e^{-ah}}{j} \right) + x_0 e^{a_0(t - h)} \left[ \sum_{i=1}^{n-2} \left( i(t - ih)i - 1 \prod_{j=1}^{i} \left( \frac{a_1 e^{-ah}}{j} \right) \right) \right] = a_0 \phi_n(t) + x_0 e^{a_0(t - h)} \left[ \sum_{i=1}^{n-1} \left( (i - 1)(t - ih)i - 1 \prod_{j=1}^{i} \left( \frac{a_1 e^{-ah}}{j} \right) \right) \right] = \dot{\phi}_n(t), \quad (A13)
\]
and hence Eq. 9 satisfies (1) for any positive integer $n$ and it is then a solution for (1) subject to initial conditions $x(t) \equiv 0 \forall t < 0; \; x(0) \phi(0) = x_0$. The unicity follows from the fact that the right-hand side of (1) is globally Lipschitz (Burton, 1985). Note also that the existing unicity of the solution is maintained if $\phi: [-h, 0]$ is any absolutely continuous function with possible isolated bounded steps (Burton, 1985).

REFERENCES


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PIRMOSIOS EILĖS HOMOGENINIŲ SISTEMŲ SU VĖLINIMU ANALITINIS SPRENDIMAS
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