Tropical Kirchhoff's Effective Conductance Formula

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Abstract

Given an assignment of real weights to the ground elements of a matroid, the min-max weight of an element e is the minimum, over all circuits containing e, of the maximum weight of an element in that circuit with the element e removed. We use this concept to establish structural results for the minimum weight basis problem: detecting the persistence of single elements, determining the new optimal value after the weight of a single element is arbitrarily perturbed, as well as when an element is contracted or deleted. This latter result gives us a tropical $(\min, +, -)$ analogue of the classical arithmetic $(+, \times, /)$ Kirchhoff's effective conductance formula for electrical networks.

1 Introduction

Let $M = (E, \mathcal{I})$ be an arbitrary loopless matroid; being loopless means that no single-element set is dependent and no single element belongs to all bases. The *minimum weight basis problem* is, given an assignment of real weights to the ground elements, to compute the minimum weight of a basis, the latter being the sum of weights of the basis elements. Thank classical results of Rado [14], Gale [7] and Edmonds [6], the *algorithmic* aspect of this problem is well understood: the minimum weight of a basis can be computed by the standard greedy algorithm. In this paper, we contribute three results concerning the *structural* aspects of this problem, the intriguing one being the relation to the classical Kirchhoff's effective conductance formula from 1847.

All three results are based on the concept of the "min-max weight" of a ground element. Given a weighting $x : E \to \mathbb{R}$, the *min-max weight* of a ground element $e \in E$, which we denote by $x[e]$, is the minimum over all circuits containing e, of the maximum weight of an element in this circuit after the element e is removed.

Our first result concerns the so-called *persistence problem*: given a weighting of ground elements, decide whether a given ground element belongs to some, to none, or to all optimal bases (see, for example, [9, 10, 5, 3, 4]). An element is *persistent* if it either belongs to all optimal bases, or is avoided by all optimal bases. Theorem 1 solves this problem in terms of min-max weights of ground elements.

Persistence An element e is *contained* in some optimal basis if and only if $x[e] \geqslant x(e)$, and is *avoided* by some optimal basis if and only if $x[e] \leqslant x(e)$. Hence, an element e is *not* persistent if and only if $x[e] = x(e)$ holds.

Our second result concerns the *postoptimality analysis* problem in optimization (see, for example, [8, 15, 12]) in the case of the minimum weight basis problem in matroids: if the weight of a single ground element is altered, by how much the optimal value (the minimum weight of a basis) changes? Theorem 2 gives an answer.

Postoptimality Let $x : E \to \mathbb{R}$ be a weighting, and $\theta \in \mathbb{R}$ an arbitrary real number. If the weight of a ground element e is changed from $x(e)$ to θ, then the difference between the new optimal value and the old one is $\max\{0, x[e] - x(e)\}$ if $\theta \geqslant x[e]$, and is $\theta - \min\{x[e], x(e)\}$ if $\theta \leqslant x[e]$.

Given a matroid $M = (E, \mathcal{I})$ on a ground set E and a ground element $e \in E$, the independent sets of the matroid $M - e$ obtained by *deleting* the element e are all sets $I \in \mathcal{I}$ with $e \notin I$. The independent sets of the matroid M/e obtained by *contracting* the element e are all sets $I - e$ with $I \in \mathcal{I}$ and $e \notin I$ (we recall these operations in Section 3.4). Given a weighting $x : E \to \mathbb{R}$ of ground elements, how are the minimum weights of bases in the resulting submatroids related to the minimum weight of a basis in the matroid M itself? Theorem 3 answers this question in terms of the min-max weight $x[e]$ of the element e.

Contraction/deletion: If the element e is contracted, then the minimum weight of a basis decreases by exactly $\min\{x(e), x[e]\}$. If the element e is deleted, then the minimum weight of a basis increases by exactly $\max\{0, x[e] - x(e)\}$.

In the special case of graphic matroids, the first claim (which actually motivated the title of this paper) gives us the tropical version of the classical *arithmetic* $(+, \times, /)$ effective conductance formula for electrical networks proved by Kirchhoff [11]. This formula expresses the effective conductance between the endpoints of any edge e in an electrical network G as a *ratio* $\kappa_G(x)/\kappa_{G/e}(x)$ of the spanning tree polynomial of the network itself, divided by the spanning tree polynomial of the network G/e obtained by contracting the edge e.

In the tropical semiring[1] $(\mathbb{R}_+, \min, +)$, the spanning tree polynomial of a graph turns into the minimum weight spanning tree problem, and the *ratio* of polynomials turns into the *difference* between their tropical versions. We thus have a tropical analogue of the Kirchhoff's formula for arbitrary matroids. The tropical analogue of the "effective conductance" of an element e is then $\min\{x(e), x[e]\}$; see Section 3.5 for details.

[1]The difference between the arithmetic $(\mathbb{R}_+, +, \times)$ and the tropical $(\mathbb{R}_+, \min, +)$ semirings is that, in the latter, "addition" means taking the minimum, and "multiplication" means adding the numbers. In particular, tropical polynomials solve minimization problems.

2 Preliminaries

We use the standard matroid terminology as, for example, in Oxley's book [13]. A *matroid* on a finite set E of *ground elements* is a pair $M = (E, \mathcal{I})$, where $\mathcal{I} \subseteq 2^E$ is a nonempty downwards closed collection of subsets of E, called *independent sets*, with the *augmentation property*: whenever I and J are independent sets of cardinalities $|I| < |J|$, there is an element $e \in J \setminus I$ such that the set $I + e$ is independent; as customary, we abbreviate $I \cup \{e\}$ to $I + e$, and we write $J - e$ for $J \setminus \{e\}$.

Maximal under the set inclusion independent sets are *bases*. The augmentation property yields the *basis exchange axiom*: if A and B are bases, then for every element $e \in A$ there is an element $f \in B$ such that $A - e + f$ is also a basis. The following two important refinements of this axiom are known as the *symmetric basis exchange* and the *bijective basis exchange*.

Fact 1 (Brualdi [1], Brylawski [2])**.** *Let A and B be bases.*

(1) *For every $e \in A$ there is an $f \in B$ such that $A - e + f$ and $B - f + e$ are both bases.*
(2) *There is a bijection $\phi : A \to B$ such that, for every $e \in A$, the set $A - e + \phi(e)$ is a basis.*

A subset of E is *dependent* if it is not independent. Minimal under the set inclusion dependent sets are *circuits*. For a ground element e, an *e-circuit* is just a circuit containing this element e. An element e is a *loop* if the set $\{e\}$ is dependent, and is a *coloop* if e belongs to all bases. To avoid "pathological" situations, we will assume that our matroid is *loopless*: no ground element is a loop or a coloop. We will need this assumption to ensure two properties: every circuit has at least two elements, and at least one e-circuit exists for each ground element e.

An important property of circuits is given by the following fact known as the *unique circuit property* (see, for example, Brualdi [1, Lemma 1] or Oxley [13, Proposition 1.1.4]): if B is a basis and $e \notin B$, then $B + e$ contains a unique circuit C, and for every $f \in B$, the set $B - f + e$ is a basis if and only if $f \in C$. Since B is independent, this circuit C is an e-circuit.

The unique circuit $C = C(B, e)$ given by this property is known as the *fundamental circuit of e relative to B*. We will call the independent set $\mathrm{Path}(e, B) := C(e, B) - e \subseteq B$ the *fundamental path of e relative to B*. For an element $f \in B$, the set $\mathrm{Cut}(f, B) := \{e \notin B \colon f \in C(e, B)\}$ is known as the *fundamental cut of f relative to B*. Note the duality: $e \in \mathrm{Cut}(f, B)$ if and only if $f \in \mathrm{Path}(e, B)$. The unique circuit property immediately yields the following property of fundamental paths and cuts.

Fact 2. *Let B be a basis, $f \in B$ and $e \notin B$. Then $B - f + e$ is a basis if and only if $e \in \mathrm{Cut}(f, B)$, which happens if and only if $f \in \mathrm{Path}(e, B)$.*

Let $x : E \to \mathbb{R}$ be a weighting of the ground elements; here and throughout, $\mathbb{R}$ stands for the set of all real numbers, and $\mathbb{R}_+$ for the set of nonnegative real numbers. The *weight* of a set $F \subseteq E$ is the sum $x(F) := \sum_{f \in F} x(f)$ of the weights of its elements. The *minimal* *basis problem* in a matroid $M = (E, \mathcal{I})$ on E is, given an input weighting $x : E \to \mathbb{R}$, to compute the minimum weight of a basis:

$$\tau_M(x) := \min_{B \text{ basis}} \sum_{f \in B} x(f) .$$

We will call a basis B *x-optimal* (or just *optimal*, if the weighting is clear from the context) if $x(B) = \tau_M(x)$ holds, that is, if the basis B is of minimal x-weight.

Given a weighting $x : E \to \mathbb{R}$ of the ground elements, the *min-max weight* of an element $e \in E$, which we will denote by $x[e]$, is the minimum, over all e-circuits C, of the maximum weight of an element in the independent set $C - e$:

$$x[e] := \min_{C \text{ e-circuit}} \max_{f \in C - e} x(f) .$$

Since e is not a loop (the set $\{e\}$ is independent), the set $C - e$ is nonempty for every e-circuit C. Moreover, since e is not a coloop, at least one e-circuit C exists. So, the min-max weight is well-defined. Note that the min-max weight $x[e]$ of e does not depend on the weight $x(e)$ of the element e itself: it only depends on the weights of the remaining elements. So, all three relations $x[e] < x(e)$, $x[e] = x(e)$ and $x[e] > x(e)$ are possible. And, as we will see soon, these relations are decisive for the minimum weight basis problem: they tell us whether the element e is in some, in all or in none of the optimal bases (see Theorem 1). The *accumulated weight* of the element e is

$$x\{e\} := \min \{x(e), x[e]\} .$$

That is, the accumulated weight $x\{e\}$ of an element e is the minimum of the weight $x(e)$ of e itself and of the maximum weight of an element in an e-circuit with the element e removed, whichever of these two numbers is smaller. Note that we always have $x\{e\} \leqslant x(e)$, while $x[e] \leqslant x(e)$ does not need to hold.

Example 1. Recall that the *graphic matroid* (or the *cycle matroid*) $M(G)$ determined by an undirected connected graph $G = (V, E)$ has edges of G as its ground elements. Independent sets are forests, bases are spanning trees of G, and circuits are simple cycles in G. A loop is an edge with equal endpoints, and a coloop is an edge whose deletion increases the number of connected components; such edges are also called *bridges*. The min-max weight $x[e]$ of an edge $e \in E$ is the minimum, over all simple paths in G of length at least two between the endpoints of e, of the maximum weight of an edge in this path. If T is a spanning tree of G, and $e \notin T$ an edge of G, then $\mathrm{Path}(e, T)$ consists of all edges of the unique path in T between the endpoints of e. If $f \in T$, then $\mathrm{Cut}(e, T)$ consists of all edges of $G - e$ lying between the two trees of $T - e$. The accumulated weight $x\{e\}$ is also known as the bottleneck distance between the endpoints of e.

3 Results

Throughout this section, let $M = (E, \mathcal{I})$ be an arbitrary loopless matroid.

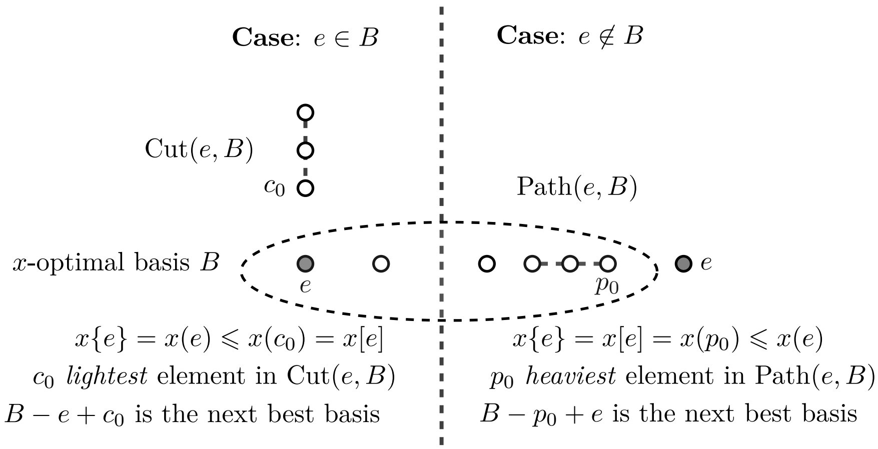


Figure 1: A schematic summary of Lemma 1 depending on whether our element e belongs to an optimal basis B or not.

3.1 Min-max weight and optimal bases

Our main technical result (Lemma 1 below) relates min-max weights of single ground elements with their fundamental paths and cuts relative to any optimal basis.

Lemma 1 (Main lemma). *Let $x : E \to \mathbb{R}$ be a weighting, $e \in E$ a ground element, and B an optimal basis.*

(1) *If $e \in B$, then*

  (a) $x(e) \leqslant x[e] =$ *the minimum weight of an element in* $\mathrm{Cut}(e, B)$*;*

  (b) *the minimum weight of a basis avoiding the element e is $x(B) - x(e) + x[e]$.*

(2) *If $e \notin B$, then*

  (a) $x(e) \geqslant x[e] =$ *the maximum weight of an element in* $\mathrm{Path}(e, B)$*;*

  (b) *the minimum weight of a basis containing the element e is $x(B) - x[e] + x(e)$.*

Proof. Postponed to Section 4. □

Remark 1 (From optimal bases to accumulated weights). Having an optimal basis B, we can compute the accumulated weights $x\{e\} = \min\{x(e), x[e]\}$ of all ground elements $e \in E$ by just looking at the weights of elements of B: if $e \in B$, then $x\{e\} = x(e)$ (by Lemma 1(1)), and if $e \notin B$, then $x\{e\} = x[e]$ is the weight of a heaviest element in $\mathrm{Path}(e, B)$ (by Lemma 1(2)).

The following simple corollary of Lemma 1 shows how such a heaviest element in $\mathrm{Path}(e, B)$ can be found.

Corollary 1. *Let $x : E \to \mathbb{R}$ be a weighting and $B = \{f_1, \ldots, f_r\}$ be an optimal basis with $x(f_1) \leqslant \ldots \leqslant x(f_r)$. If $e \notin B$, then $x[e] = x(f_i)$, where i is the smallest index for which the set $\{f_1, \ldots, f_i, e\}$ is dependent.*

Proof. For $j = 1, \ldots, r$, let $B_j = \{f_1, \ldots, f_j\}$ be the set of the j lightest elements of B; let also $B_0 = \emptyset$. The set $B_0 + e = \{e\}$ is independent because e is not a loop, and the set $B_r + e = B + e$ is dependent, because B is a basis and $e \notin B$. So, there is a unique index $i \in \{1, \ldots, r\}$ such that the set $B_{i-1} + e$ is independent but $B_i + e$ is dependent. Our goal is to show that $x[e] = x(f_i)$ holds for this i.

Since B_i is independent but $B_i + e$ is dependent, the set $B_i + e$ contains an e-circuit C. Since $C \subseteq B_i + e \subseteq B + e$, the uniqueness of the fundamental circuits yields $C = C(e, B)$; hence, $\mathrm{Path}(e, B) = C(e, B) - e \subseteq B_i$. Since the set $B_{i-1} + e$ is independent, the last element f_i of B_i must be contained in $\mathrm{Path}(e, B)$. Thus, f_i is a heaviest element of $\mathrm{Path}(e, B)$, and Lemma 1(2) gives $x[e] = x(f_i)$. □

3.2 Persistency of ground elements

Every weighing $x : E \to \mathbb{R}$ splits the set E of ground elements into three disjoint subsets:

$E_1(x) =$ elements belonging to all optimal bases;
$E_0(x) =$ elements not belonging to any optimal basis;
$E_*(x) =$ elements that belong to some but not to all optimal bases.

Elements of $E_1(x) \cup E_0(x)$ are usually called *persistent* elements: they either belong to all optimal solutions, or to none of them. The notion of persistency of elements in combinatorial optimization was apparently first introduced by Hammer, Hansen and Simeone [9, 10], and was further considered in different settings, for example, in [5, 3, 4, 16]. In particular, Cechlárova and Lacko [4] characterized the sets $E_1(x)$ and $E_0(x)$ in terms of the rank function of the underlying matroid. On the other hand, Lemma 1 gives the following characterization of these sets in terms of min-max weights of individual elements.

Theorem 1 (Persistency)**.** *Let $x : E \to \mathbb{R}$ be a weighting, and $e \in E$ be a ground element. Then*

(1) *$e \in E_1(x)$ if and only if $x[e] > x(e)$;*

(2) *$e \in E_0(x)$ if and only if $x[e] < x(e)$;*

(3) *$e \in E_*(x)$ if and only if $x[e] = x(e)$.*

If all weights are distinct, then $B = \{e \in E : x[e] > x(e)\}$ is the unique optimal basis.

Proof. (1) To show the direction (⇒), let $e \in E_1(x)$ and take any optimal basis B; hence, $e \in B$. By Lemma 1(1), we then have $x(e) \leqslant x[e] = x(c_0)$, where c_0 is a lightest element in $\mathrm{Cut}(e, B)$. By Fact 2, the set $A = B - e + c_0$ is a basis. Held the equality $x(e) = x[e]$, then this basis would be also optimal. But $e \notin A$, a contradiction with $e \in E_1(x)$. Hence $x[e] > x(e)$ holds. The converse direction ($\Leftarrow$) follows directly from Lemma 1(2).

(2) The proof of this claim is dual. To show the direction ($\Rightarrow$), let $e \in E_0(x)$ and take any optimal basis B; hence, $e \notin B$. By Lemma 1(2), we then have $x(e) \geqslant x[e] = x(p_0)$, where $p_0 \in B$ is a heaviest element in $\mathrm{Path}(e, B)$. By Fact 2, the set $A = B - p_0 + e$ is a basis. Held the equality $x(e) = x[e]$, then this basis would be also optimal. But $e \in A$, a contradiction with $e \in E_0(x)$. Hence, $x[e] < x(e)$ holds. The converse direction ($\Leftarrow$) in (2) follows directly from Lemma 1(1).

The third clam (3) follows directly from claims (1) and (2).

Finally, assume that all weights are distinct. Then the optimal basis B is *unique*: had we two distinct optimal bases, then (by the basis exchange axiom) a heaviest element, lying in one basis but not in the other, could be replaced by a (strictly) lighter element of the other basis, contradicting the optimality of the former basis. Since the basis B is unique, we have $B = E_1(x)$ and, hence, $B = \{e \in E \colon x[e] > x(e)\}$, as desired. □

Recall that the *rank* $\mathrm{rk}(F)$ of a set $F \subseteq E$ is the maximum cardinality of an independent subset of F. That is, $\mathrm{rk}(F)$ is the maximum of $|F \cap B|$ over all bases B.

Corollary 2. *Let $x : E \to \mathbb{R}$ be a weighting. For every element $e \in E$ the following holds.*

(1) *$x[e] > x(e)$ if and only if $\mathrm{rk}(F_1) > \mathrm{rk}(F_1 - e)$, where $F_1 = \{f \in E \colon x(f) \leqslant x(e)\}$.*

(2) *$x[e] < x(e)$ if and only if $\mathrm{rk}(F_0 + e) = \mathrm{rk}(F_0)$, where $F_0 = \{f \in E \colon x(f) < x(e)\}$.*

Proof. This follows directly from Theorem 1 and the following result of Cechlárova and Lacko [4, Lemmas 2 and 3]: $e \in E_1(x)$ if and only if $\mathrm{rk}(F_1 - e) < \mathrm{rk}(F_1)$, and $e \in E_0(x)$ if and only if $\mathrm{rk}(F_0 + e) = \mathrm{rk}(F_0)$. □

3.3 Postoptimality

Let $x : E \to \mathbb{R}$ be a weighting of ground elements, and $e \in E$ a fixed ground element. For a real number $\theta \in \mathbb{R}$, let $x_\theta : E \to \mathbb{R}$ be the weighting which gives weight θ to the element e, and leaves the weights of other elements unchanged.

The following lemma gives new optimal values depending on how the (old) min-max weight of the element e is related to its (old) weight.

Lemma 2. *The difference $\tau(x_\theta) - \tau(x)$ between the new optimal value and the old one is $\min\{\theta,\ x[e]\} - x(e)$ if $x(e) \leqslant x[e]$, and is $\min\{0,\ \theta - x[e]\}$ if $x(e) \geqslant x[e]$.*

Proof. Postponed to Section 5. □

The following consequence of Lemma 2 gives the new optimal values in dependence of the new weight $\theta \in \mathbb{R}$ given to the element e.

Theorem 2 (Postoptimality)**.** *For every $\theta \in \mathbb{R}$, the difference $\tau(x_\theta) - \tau(x)$ between the new optimal value and the old one is $\max\{0, x[e] - x(e)\}$ if $\theta \geqslant x[e]$, and is $\theta - \min\{x[e], x(e)\}$ if $\theta \leqslant x[e]$.*

Note that the difference is *constant* in the first case (0 or $x[e]-x(e)$, whichever is larger), while in the second case, the difference is *negative* as long as the new weight θ given to the element e is smaller than both $x[e]$ and $x(e)$ (the optimal value then decreases).

Proof. Suppose first that $\theta \geqslant x[e]$; hence, $\theta - x[e] \geqslant 0$. By Lemma 2, the difference $\tau(x_\theta) - \tau(x)$ is $\min\{\theta,\ x[e]\} - x(e) = x[e] - x(e) \geqslant 0$ if $x[e]-x(e) \geqslant 0$, and is $\min\{0,\ \theta - x[e]\} = 0$ if $x[e]-x(e) \leqslant 0$. That is, $\tau(x_\theta) - \tau(x) = \max\{0, x[e]-x(e)\}$, as claimed.

Suppose now that $\theta \leqslant x[e]$; hence, $\theta - x[e] \leqslant 0$. By Lemma 2, the difference $\tau(x_\theta) - \tau(x)$ is $\min\{\theta,\ x[e]\} - x(e) = \theta - x(e)$ if $x(e) \leqslant x[e]$, and is $\min\{0,\ \theta - x[e]\} = \theta - x[e]$ if $x(e) \geqslant x[e]$. In both cases, the difference is $\theta - \min\{x[e], x(e)\}$, as claimed. □

Recall that the *accumulated weight* of a ground element $e \in E$ under a given weighting a weighting $x : E \to \mathbb{R}$ is the minimum $x\{e\} = \min\{x(e), x[e]\}$ of the weight and the min-max weight of this element.

Corollary 3. *If the weights are nonnegative, and if the weight of a single element is dropped down to zero, then the minimum weight of a basis decreases by exactly the accumulated weight of this element.*

Proof. Let $x : E \to \mathbb{R}_+$ be a weighting giving nonnegative weights to the ground elements, and let $e \in E$ be a ground element. We apply Theorem 2 with $\theta := 0$. Since the weights are nonnegative, $\theta \leqslant x[e]$ holds. So, Theorem 2 implies that the minimum weight of a basis decreases by exactly $\tau(x) - \tau(x_\theta) = \min\{x[e], x(e)\} - \theta = \min\{x[e], x(e)\}$. □

By Lemma 1, every basis of minimal weight determines the accumulated weights of all ground elements e (see Remark 1). The following corollary does a *converse* reduction: the minimum weight of a basis is determined by the accumulated weights of single elements.

Corollary 4. *Let $B = \{e_1, \ldots, e_r\}$ be a basis. Given a weighting $x : E \to \mathbb{R}_+$, consider the sequence of weightings $x_0, x_1, \ldots, x_r$, where $x_0 = x$, and each next weighting x_i is obtained from x by setting the weights of the elements $e_1, \ldots, e_i$ to zero. Then*

$$\tau_M(x) = x_0\{e_1\} + x_1\{e_2\} + \cdots + x_{r-1}\{e_r\}\,.$$

Let us stress that B is *any* fixed in advance basis *independent* of arriving input weightings.

Proof. Corollary 3 gives us the recursion $\tau_M(x_i) = \tau_M(x_{i+1}) + x_i\{e_{i+1}\}$ which rolls out into

$$\tau_M(x) = \tau_M(x_r) + x_{r-1}\{e_r\} + \cdots + x_1\{e_2\} + x_0\{e_1\}\,.$$

Since the weighting x_r gives zero weight to all elements $e_1, \ldots, e_r$ of the basis B, we have $x_r(B) = 0$. Since the weights are nonnegative, B is a basis of minimal x_r-weight. Hence, $\tau_M(x_r) = x_r(B) = 0$. □

3.4 Tropical Kirchhoff's effective conductance formula

Let $M = (E, \mathcal{I})$ a loopless matroid. We can obtain new matroids by "contracting" or by "deleting" a ground element $e \in E$. Namely, we can think about all of the independent sets of the matroid M being partitioned into two families: those independent sets that do not contain the element e and those that do. The sets of the former family are independent sets of the matroid $M - e$ obtained by *deleting* the element, while the sets of the latter family, with the element e removed, are independent sets of the matroid M/e obtained by *contracting* the element e. That is, the independent sets of $M - e$ are all sets $I \in \mathcal{I}$ with $e \notin I$, while those of M/e are all sets $I - e$ with $I \in \mathcal{I}$ and $e \in I$. Since the matroid M is loopless, each of these two matroids contains at least one nonempty independent set. Note that e is a ground element in none of these two matroids.

For every basis B, either $B - e$ is a basis of M/e (if $e \in B$), or B is a basis of $M - e$ (if $e \notin B$). This simple observation gives us a recursion

$$\tau_M(x) = \min\left\{\tau_{M/e}(x) + x(e),\ \tau_{M-e}(x)\right\}.$$

Using Lemma 1 and Theorem 2, we obtain the following relations between the optimal bases of the original matroid M and of the reduced matroids $M - e$ and M/e, where the first equation Eq. (1) is the tropical analogue of the Kirchhoff's effective conductance formula for electrical networks; see Section 3.5 for details. Recall that the *accumulated weight* of a ground element is the minimum $x\{e\} = \min\{x(e), x[e]\}$ of the weight and the min-max weight of this element.

Theorem 3 (Tropical Kirchhoff's formula)**.** *Let $M = (E, \mathcal{I})$ be a loopless matroid, and $e \in E$ be a ground element. Then for every weighting $x : E \to \mathbb{R}$, we have*

$$\tau_M(x) - \tau_{M/e}(x) = x\{e\}\,; \tag{1}$$

$$\tau_{M-e}(x) - \tau_M(x) = \max\{0, x[e] - x(e)\}\,; \tag{2}$$

$$\tau_{M-e}(x) - \tau_{M/e}(x) = x[e]\,. \tag{3}$$

In particular, knowing the weight $x(e)$ and the min-max weight $x[e]$ of the element e, we can compute the optimal solution $\tau_M(x)$ over the entire matroid M from the optimal solutions over any of the sub-matroids $M - e$ of M/e.

Proof. (1) By Theorem 1, we have $x\{e\} = x(e)$ if e belongs to some optimal basis B of M, and $x\{e\} = x[e]$ otherwise. In the former case, $B - e$ is an optimal basis of M/e, and its weight is $\tau_{M/e}(x) = x(B) - x(e) = \tau_M(x) - x(e)$. Hence, $\tau_M(x) - \tau_{M/e}(x) = x(e) = x\{e\}$ in this case. Suppose now that the element e belongs to none of the optimal bases of M, and let A be a lightest basis of M among all bases of M containing the element e. Hence, $\tau_{M/e}(x) = x(A - e)$. By Lemma 1(2), we have $x[e] \leqslant x(e)$ and $x(A) = x(B) - x[e] + x(e)$, and we obtain $\tau_M(x) - \tau_{M/e}(x) = x(B) - x(A - e) = x[e] = x\{e\}$ also in this case.

(2) If $x[e] \leqslant x(e)$, then Theorem 1 implies that the element e is avoided by at least one optimal basis B of M. Since this basis remains optimal also in $M - e$, we have the equality

$\tau_{M-e}(x) = \tau_M(x)$ in this case. If $x[e] > x(e)$, then (by Theorem 1) the element e is in *all* optimal bases B of M. Hence, $\tau_{M-e}(x)$ is a minimum weight $x(A)$ of a basis A of M avoiding the element e. By Lemma 1(1), we have $x(A) = x(B) - x(e) + x[e] \geqslant x(B)$. Hence, $\tau_{M-e}(x) - \tau_M(x) = x(A) - x(B) = x[e] - x(e)$ holds in this case.

(3) By Eqs. (1) and (2), we have

$$\tau_{M/e}(x) + \min\{x(e), x[e]\} = \tau_M(x) = \tau_{M-e}(x) + \min\{0, x(e) - x[e]\} .$$

This gives $\tau_{M/e}(x) + x[e] = \tau_{M-e}(x) + 0$ if $x(e) \geqslant x[e]$, and $\tau_{M/e}(x) + x(e) = \tau_{M-e}(x) + x(e) - x[e]$ if $x(e) \leqslant x[e]$. In both cases, we have $\tau_{M/e}(x) = \tau_{M-e}(x) - x[e]$, as desired. $\square$

3.5 Relation to the Kirchhoff's formula

In 1847 Kirchhoff [11] showed that the effective conductance between any pair of vertices in an electrical network can be expressed as a combinatorial formula consisting of a ratio of spanning tree polynomials. We will now show that, when applied to *graphic* matroids, Theorem 3 is exactly the tropical $(\min, +, -)$ analogue of this arithmetic $(+, \times, /)$ Kirchhoff's formula.

Let $G = (V, E)$ be an undirected connected n-vertex multi-graph (no loops, but parallel edges are allowed). Every edge $e \in E$ has its associated variable x_e, the "weight" of e. The *spanning tree polynomial* (or *Kirchhoff polynomial*) of G is the following multilinear, homogeneous polynomial

$$\kappa_G(x) = \sum_T \prod_{e \in T} x_e$$

of degree $n - 1$, where the summation is over all spanning trees T of G. In particular, for $x = \vec{1}$ (the all-1 vector), the value $\kappa_G(x)$ is the number of spanning trees of G. For an edge $e \in E$, let G/e be the graph obtained by contracting the edge e, that is, by merging the endpoints of e, and removing the resulting loop; since loops cannot contribute to a spanning tree, we can throw them away without altering $\kappa_{G/e}(x)$.

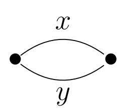
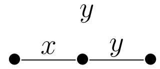
When the edges are interpreted as electrical resistors, and their weights as electrical conductances (reciprocals of electrical resistances), the Kirchhoff's effective conductance formula [11] (see also [17, Theorem 8] for a detailed exposition) states that

$$\frac{\kappa_G(x)}{\kappa_{G/e}(x)} = \text{ effective conductance between the endpoints of } e. \tag{4}$$

Over the tropical semifield $(\mathbb{R}, \min, +, -)$, min corresponds to the arithmetic addition $(+)$, addition to the arithmetic multiplication, and subtraction $(-)$ to arithmetic division (see Table 1). In particular, the spanning tree polynomial $\kappa_G(x)$ then turns into the well-known minimum weight spanning tree problem $\tau_G(x) = \min_T \sum_{e \in T} x(e)$ on the graph G. Also, the *ratio* $\kappa_G(x)/\kappa_{G/e}(x)$ in Eq. (4) turns into the *difference* $\tau_G(x) - \tau_{G/e}(x)$.

So, when applied to the graphic matroid $M = M(G)$ defined by the graph G, the first equation Eq. (1) of Theorem 3 gives us the tropical analogue $\tau_G(x) - \tau_{G/e}(x) = x\{e\}$ of the

Table 1: Correspondences between arithmetic and tropical effective conductances in electrical networks. By Ohm's and Kirchhoff's laws, the conductance is additive for resistors in parallel, and the resistance is additive for resistors in series. In the tropical case, this turns to taking the minimum for resistors in parallel, and taking the maximum for resistors in series.

	Arithmetic	Tropical
Operations	$x+y$	$\min(x,y)$
	$x \cdot y$	$x+y$
	x/y	$x-y$
Conductances of resistors:		
in parallel 	$x+y$	$\min(x,y)$
in series 	$\dfrac{1}{\frac{1}{x}+\frac{1}{y}} = \dfrac{x \cdot y}{x+y}$	$x+y-\min(x,y) = \max(x,y)$

Kirchhoff's formula Eq. (4) with the tropical "effective conductance" between the endpoints of the edge e being the accumulated weight $x\{e\} = \min\{x(e), x[e]\}$ of this edge. Theorem 3 holds as it is, but an intuitive explanation of why accumulated weights in electrical networks (at least in the series-parallel networks) play the role of effective conductances is given in Table 1.

4 Proof of Lemma 1

We will need the following consequence of Facts 1 and 2.

Lemma 3. *Let B be a basis, and $e \in B$. For every e-circuit C there is an element $f \in C - e$ such that $B - e + f$ is a basis.*

By Fact 2, this means that $(C - e) \cap \mathrm{Cut}(e, B) \neq \emptyset$ holds for all e-circuits C.

Proof. Let C be an e-circuit. Since the set $I = C - e$ is independent, it lies in some basis A, and $e \notin A$ holds since $I + e = C$ is already dependent. By Fact 1(1), there is an $f \in A$ such that $B - e + f$ and $A - f + e$ are both bases. In view of Fact 2, this is equivalent to $f \in \mathrm{Cut}(e, B)$ and $f \in \mathrm{Path}(e, A)$. Since A is a basis, and both circuits C and $\mathrm{Path}(e, A) + e$ lie in $A + e$, the uniqueness of fundamental circuits yields $\mathrm{Path}(e, A) = C - e$. Hence, $\mathrm{Cut}(e, B) \cap (C - e) \neq \emptyset$, as claimed. □

Let $x : E \to \mathbb{R}$ be a weighting of the ground elements. Recall that the accumulated weight $x[e]$ of e is the minimum, over all e-circuits C, of the maximum weight of an element in the independent set $C-e$:

$$x[e] = \min_{C \text{ } e\text{-circuit}} \max_{f \in C-e} x(f)\,,$$

The min-max weight of e is $x\{e\} = \min\{x(e), x[e]\}$. By an e-circuit *witnessing* the min-max weight $x[e]$ of an element e we will mean an e-circuit C such that $\max\{x(f)\colon f \in C-e\} = x[e]$. Recall that a basis B is *optimal* if its weight $x(B) = \sum_{f\in B} x(f)$ is smallest among *all* bases.

Proof of Lemma 1. Fix a ground element $e \in E$, and let B be an optimal basis. Lemma 1 determines the min-max weight $x[e]$ of the element e depending on whether this element belongs to the basis B or not.

Case 1: $e \in B$. Let c_0 be a lightest element in $\mathrm{Cut}(e,B)$. Our goal is to show that then

$$x(e) \leqslant x[e] = x(c_0)\,,$$

and that $B-e+c_0$ is a lightest basis among all bases avoiding the element e.

To show the inequality $x(e) \leqslant x(c_0)$, suppose contrariwise that $x(c_0) < x(e)$ holds. Since $c_0 \in \mathrm{Cut}(e,B)$, Fact 2 implies that the set $B-e+c_0$ is a basis. But its weight is then smaller than that of B, contradicting the optimality of B.

To show the inequality $x[e] \geqslant x(c_0)$, let C be an e-circuit witnessing the min-max weight $x[e]$ of the element e. By Lemma 3, there is an element g in the intersection $(C-e)\cap \mathrm{Cut}(e,B)$. Then $x(g) \leqslant x[e]$ because $g \in C-e$, and $x(g) \geqslant x(c_0)$ because $g \in \mathrm{Cut}(e,B)$. Hence, $x[e] \geqslant x(c_0)$.

To show the converse inequality $x[e] \leqslant x(c_0)$, consider the fundamental circuit $C = \mathrm{Path}(c_0,B)+c_0$ of the element c_0 relative to the basis B. Since $c_0 \in \mathrm{Cut}(e,B)$, we have $e \in \mathrm{Path}(c_0,B)$. Thus, both e and c_0 belong to the same circuit C. Let p_0 be a heaviest element in $C-e = \mathrm{Path}(c_0,B)+c_0-e$. Since in the definition of the min-max weight $x[e]$ we take the *minimum* over all circuits containing e, we have $x[e] \leqslant x(p_0)$. So, it remains to show that $x(p_0) \leqslant x(c_0)$ holds. Suppose contrariwise that $x(p_0) > x(c_0)$. Then $p_0 \neq c_0$ and, hence, $p_0 \in \mathrm{Path}(c_0,B)$. By Fact 2, the set $A = B-p_0+c_0$ is a basis. But the weight of this basis is $x(A) = x(B)-x(p_0)+x(c_0) < x(B)$, contradicting the optimality of the basis B. Thus, $x[e] \leqslant x(c_0)$, as desired.

It remains to show that $B-e+c_0$ is a lightest basis among all bases avoiding the element e. To show this, let B' be a lightest basis avoiding the element e; hence, $e \notin B'$. Take a bijection $\phi : B \to B'$ ensured by Fact 1(2). Hence, $B-g+\phi(g)$ is a basis for every element $g \in B$. Let $c := \phi(e) \in B'$.

Since $B-e+c$ is a basis, Fact 2 implies that $c \in \mathrm{Cut}(e,B)$. Since B is optimal, we have $x(g) \leqslant x(\phi(g))$ for all $g \in B$. In particular, $x(B-e) \leqslant x(B'-\phi(e)) = x(B'-c)$. The basis $A = B-e+c$ avoids the element e, and its weight is $x(A) = x(B-e)+x(c) \leqslant x(B'-c)+x(c) = x(B')$. Hence, the basis A is a lightest basis avoiding the element e. Since c_0 is a *lightest* element of $\mathrm{Cut}(e,B)$, we have $x(c_0) \leqslant x(c)$. Thus, the set $B-e+c_0$ is also a lightest basis avoiding the element e. □

Case 2: $e \notin B$. Let p_0 be a heaviest element in $\mathrm{Path}(e, B)$. Our goal is to show that then

$$x(e) \geqslant x[e] = x(p_0)\,,$$

and that $B - p_0 + e$ is a lightest basis among all bases containing the element e. The proof in this case is dual to that in Case 1.

To show the inequality $x(e) \geqslant x(p_0)$, suppose contrariwise that $x(e) < x(p_0)$. Since $p_0 \in \mathrm{Path}(e, B)$, Fact 2 implies that the set $B - p_0 + e$ is a basis. But its weight is then smaller than that of B, contradicting the optimality of B.

The inequality $x[e] \leqslant x(p_0)$ holds just because $\mathrm{Path}(e, B) + e$ is an e-circuit, and $x[e]$ takes the *minimum* (of the maximum weights) over all e-circuits.

To show the converse inequality $x[e] \geqslant x(p_0)$, Suppose contrariwise that we have a strict inequality $x[e] < x(p_0)$, and let C be an e-circuit witnessing $x[e]$. Hence, $x(f) < x(p_0)$ holds for all $f \in P := C - e$. Since $p_0 \in \mathrm{Path}(e, B)$, Fact 2 implies that $A = B - p_0 + e$ is also a basis. Since $e \in A$, Lemma 3 implies that some element $f \in P \cap \mathrm{Cut}(e, A)$ can replace e in A, that is, $A' = A - e + f = B - p_0 + f$ is a basis. But since $x(f) < x(p_0)$, we have $x(A') < x(B)$, contradicting the optimality of B.

So, it remains to show that $B - p_0 + e$ is a lightest basis among all bases containing the element e. To show this, let B' be a lightest basis containing the element e; hence, $e \in B'$. Take a bijection $\phi : B \to B'$ ensured by Fact 1(2). Hence, $B - \phi^{-1}(g) + g$ is a basis for every element $g \in B'$. Let $p := \phi^{-1}(e) \in B$.

Since $B - p + e$ is a basis, Fact 2 implies that $p \in \mathrm{Path}(e, B)$. Since the basis B is optimal, we have $x(\phi^{-1}(g)) \leqslant x(g)$ for all $g \in B'$. In particular, $x(B - p) = x(B - \phi^{-1}(e)) \leqslant x(B' - e)$. The basis $A = B - p + e$ contains the element e, and its weight is $x(A) = x(B - p) + x(e) \leqslant x(B' - e) + x(e) = x(B')$. Hence, the basis A is also a lightest basis containing the element e. Since p_0 is a *heaviest* element of $\mathrm{Path}(e, B)$, the set $B - p_0 + e$ is also a lightest basis containing the element e. □

5 Proof of Lemma 2

Let $x : E \to \mathbb{R}$ be a weighting, and $e \in E$ be an arbitrary ground element. For a real number $\theta \in \mathbb{R}$, let $x_\theta : E \to \mathbb{R}$ be the weighting which gives weight $\theta \in \mathbb{R}$ to the element e and leaves other weights unchanged. Our goal is to prove the following two assertions.

(1) If $x(e) \leqslant x[e]$, then $\tau(x_\theta) = \tau(x) + \min\{\theta,\ x[e]\} - x(e)\,.$

(2) If $x(e) \geqslant x[e]$, then $\tau(x_\theta) = \tau(x) + \min\{0,\ \theta - x[e]\}\,.$

Proof. (1) Suppose that $x(e) \leqslant x[e]$. Theorem 1 implies that then e belongs to some x-optimal basis B. There are two possibilities: either B is x_θ-optimal or not.

Claim 1. *If B is not x_θ-optimal, then the element e is avoided by all x_θ-optimal bases.*

Proof. Assume that B is not x_θ-optimal, and take an arbitrary x_θ-optimal basis B'. Suppose contrariwise that $e \in B'$. Since B is not x_θ-optimal, $x_\theta(B) > x_\theta(B')$ holds. Since the element e belongs to both bases B and B', we have $x_\theta(B') = x_\theta(e) + x(B' - e)$ and $x_\theta(B) = x_\theta(e) + x(B - e)$. But $x(B' - e) \geqslant x(B - e)$, since B is x-optimal. We thus have $x_\theta(B') \geqslant x_\theta(B)$, a contradiction with $x_\theta(B) > x_\theta(B')$. Thus, $e \notin B'$, as desired. □

Now, if the basis B is x_θ-optimal, then (since $e \in B$) we have

$$\tau(x_\theta) = x_\theta(B) = x(B - e) + x_\theta(e) = x(B) + \theta - x(e) = \tau(x) + \theta - x(e)\,.$$

If B is not x_θ-optimal, then Claim 1 tells us that the element e is avoided by all x_θ-optimal bases. Since $x_\theta(B') = x(B')$ holds for every such basis B', we have that $\tau(x_\theta)$ is the minimum x-weight of a basis avoiding the element e, and Lemma 1(1) yields

$$\tau(x_\theta) = x(B) + x[e] - x(e) = \tau(x) + x[e] - x(e)\,.$$

So, regardless of whether the basis B is x_θ-optimal or not, we have that $\tau(x_\theta)$ is either $\tau(x) + \theta - x(e)$ or $\tau(x) + x[e] - x(e)$, whichever of these two numbers is smaller.

(2) Suppose that $x(e) \geqslant x[e]$. Then, by Theorem 1, the element e is avoided by some x-optimal basis B. We again have two possibilities: either B is x_θ-optimal or not.

Claim 2. *If B is not x_θ-optimal, then the element e is contained in all x_θ-optimal bases.*

Proof. Assume that B is not x_θ-optimal, and take an arbitrary x_θ-optimal basis B'. Suppose contrariwise that $e \notin B'$. Since B is not x_θ-optimal, $x_\theta(B) > x_\theta(B')$ holds. Since $e \notin B'$, we have $x_\theta(B') = x(B')$, and since $e \notin B$, we also have $x_\theta(B) = x(B)$. So, the inequality $x_\theta(B') < x_\theta(B)$ yields $x(B') < x(B)$, contradicting the x-optimality of B. □

Now, if B is x_θ-optimal, then (since $e \notin B$) we have $\tau(x_\theta) = x_\theta(B) = x(B) = \tau(x)$. If B is not x_θ-optimal, then Claim 2 tells us that the element e is contained in all x_θ-optimal bases. Thus, $\tau(x_\theta)$ in this case is the minimum x_θ-weight of a basis containing e. By Lemma 1(2), if p_0 is an element of $\mathrm{Path}(e, B)$ of smallest x-weight, then the basis $A = B - p_0 + e$ has the smallest x-weight among all bases containing the element e. By Lemma 1(2), we also have that $x(p_0) = x[e]$. Since $e \notin B$ and $p_0 \neq e$, we have

$$\tau(x_\theta) = x_\theta(A) = x(B) - x(p_0) + \theta = \tau(x) - x[e] + \theta\,.$$

So, regardless of whether the basis B is x_θ-optimal or not, we have that $\tau(x_\theta)$ is either $\tau(x)$ or $\tau(x) + \theta - x[e]$, whichever of these two numbers is smaller. □

References

[1] R. A. Brualdi. Comments on bases in dependence structures. *Bull. Australian Math. Soc.*, 1:1, 1969. doi:10.1017/S000497270004140X.

[2] T. H. Brylawski. Some properties of basic families of subsets. *Discrete Math.*, 6:333–341, 1973. doi:10.1016/0012-365X(73)90064-2.

[3] K. Cechlárova. Persistency in the assignment and transportation problems. *Math. Methods Oper. Res.*, 47:243–254, 1998. doi:10.1007/BF01194399.

[4] K. Cechlárova and V. Lacko. Persistency in combinatorial optimization problems on matroids. *Discrete Appl. Math.*, 110:121–132, 2. doi:10.1016/S0166-218X(00)00279-1.

[5] M. C. Costa. Persistency in maximum cardinality bipartite matchings. *Oper. Res. Letters*, 15:143–149, 1994. doi:10.1016/0167-6377(94)90049-3.

[6] J. Edmonds. Matroids and the greedy algorithm. *Math. Programming*, 1:127–136, 1971. doi:10.1007/BF01584082.

[7] D. Gale. Optimal assignments in an ordered set: an application of matroid theory. *J. Combin. Theory*, 4:176–180, 1968. doi:10.1016/S0021-9800(68)80039-0.

[8] A. M. Geoffrion and R. Nauss. Parametric and postoptimality analysis in integer programming. *Management Science*, 23:453–466, 1977. doi:10.1287/mnsc.23.5.453.

[9] P. L. Hammer, P. Hansen, and B. Simeone. Vertices belonging to all or to no maximum stable sets of a graph. *SIAM J. Alg. Disc. Math.*, 4:511–522, 1982. doi:10.1137/0603052.

[10] P. L. Hammer, P. Hansen, and B. Simeone. Roof duality, complementation and persistency in quadratic 0-1 optimization. *Math. Programming*, 28(2):121–155, 1984. doi:10.1007/BF02612354.

[11] G. Kirchhoff. Über die Auflösung der Gleichungen, auf welche man bei der Untersuchungen der linearen Verteilung galvanischer Ströme geführt wird. *Ann. Phys. Chem.*, 72:497–508, 1847. Engl. transl. by J.B. O'Tolle, IRE Trans. on Circuit Theory, CT-5, March 1958.

[12] M. Libura. Sensitivity analysis for minimum weight base of a matroid. *Control and Cybernetics*, 20(3):7–24, 1991.

[13] J. G. Oxley. *Matroid theory*. Oxford University Press, 1992.

[14] R. Rado. A theorem on independence relations. *Quart. J. Math.*, 13(1):83–89, 1942. doi:10.1093/qmath/os-13.1.83.

[15] R. E. Tarjan. Sensitivity analysis of minimum spanning trees and shortest path trees. *Inf. Process. Letters*, 14(1):30–33, 1981. doi:10.1016/0020-0190(82)90137-5.

[16] T. Tassa. Finding all maximally-matchable edges in a bipartite graph. *Theoret. Comput. Sci.*, 423:59–58, 2011. doi:10.1016/j.tcs.2011.12.071.

[17] D. G. Wagner. Combinatorics of electrical networks. Lecture Notes, Dept. of Combinatorics and Optimization, University of Waterloo, 2009.