VILNIUS UNIVERSITY

Santa Račkauskienė

JOINT UNIVERSALITY OF ZETA-FUNCTIONS WITH PERIODIC COEFFICIENTS

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Contents

Introduction

In the thesis, the joint universality of periodic Hurwitz zeta-functions as well as that jointly with the Riemann zeta-function or zeta-functions of normalized cusp forms is obtained.

Actuality

Universality is a very important and useful property of zeta and L-functions, it has a series of theoretical and practical applications. Universality is the main ingredient in the proof of the functional independence of zeta and L-functions, is applied in the investigation of zero-distribution and moment problem, allows to prove various value denseness theorems, and, of course, plays a crucial role in approximation of analytic functions. One of possible practical applications is estimation of integrals over complicated analytic curves in quantum mechanics [4]. Thus, this is a motivation to extend the class of universal functions.

In practice, often approximation and estimation of systems of analytic functions is needed. This problem can be successfully solved using the joint universality of zeta-functions. The majority of zeta and L-functions have approximate functional equations, therefore, due to joint universality, simultaneous estimation of analytic functions reduces to that of rather simple Dirichlet polynomials. This is a singnificant impact of the universality of zeta-functions to the theory of analytic functions.

After Voronin's remarkable work [42], a series of famous number theorists continued their investigations on the universality of zeta-functions. The names of B. Bagchi, H. Bauer, R. Garunkštis, P. Gauthier, S. M. Gonek, J. Kaczorowski, A. Laurinčikas, K. Matsumoto, A. Reich, J. Steuding, the works of young Lithuanian, Japanese, German and Polish mathematicians clearly show the actuality of the universality problem in the theory of zeta and L-functions.

Aims and problems

The aim of the thesis is to extend the joint universality to new classes of zeta-functions. The concrete problems are the following.

1. To remove a rank condition in a joint universality theorem for periodic Hurwitz zeta-functions.

2. To weaken a rank condition in an extended joint universality theorem (a collection of periodic sequences corresponds each shift parameter) for periodic Hurwitz zeta-functions.

3. To prove a mixed joint universality theorem for the Riemann zeta-function and periodic Hurwitz zeta-functions.

4. To prove a mixed joint universality theorem for a zeta-function of normalized Hecke eigen cusp forms and periodic Hurwitz zeta-functions.

Methods

In the thesis, for the proof of joint universality theorems for zeta-functions an analytic method based on probabilistic limit theorems on the weak convergence of probability measures in the space of analytic functions is applied. This method also involves elements of the measure theory and the approximation theory of analytic functions.

Novelty

All results of thesis are new. They improve or extend joint universality results for periodic Hurwitz zeta-functions.

History of the problem

In 1975, S. M. Voronin discovered [42] the universality of the Riemann zeta-function $\zeta(s)$, $s = \sigma + it$. Roughly speaking, he proved that any non-vanishing analytic function can be approximated uniformly on some sets of the strip $D = \{s \in \mathbb{C} : \frac{1}{2} < \sigma < 1\}$ by shifts $\zeta(s + i\tau)$, $\tau \in \mathbb{R}$. We state a modern version of the Voronin theorem which proof is given in [19]. meas$\{A\}$ denotes the Lebesgue measure of a measurable set $A \subset \mathbb{R}$.

Theorem A. *Suppose that $K \subset D$ is a compact subset with connected complement, and that $f(s)$ is a continuous non-vanishing function on K which is analytic in the interior of K. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T} \text{meas} \left\{ \tau \in [0, T] : \sup_{s\in K} |\zeta(s + i\tau) - f(s)| < \varepsilon \right\} > 0.$$

Theorem A shows that the set of shifts $\zeta(s+i\tau)$ approximating a given analytic function is infinite: it has a positive lower density. A proof of Theorem A is different from the initial Voronin proof, and is based on a limit theorem on the weak convergence of probability measures in the space of analytic functions. The latter method was proposed by B. Bagchi in this thesis [1], and was developed in the monographs [19], [27] and [41].

It turned out that some other zeta-functions also have the universality property. The zeta-functions of cusp forms are among universal in the Voronin sense functions. We remind that the function $F(s)$ is callied a cusp form of weight κ with respect to the full modular group

$$SL(2,\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \ a,b,c,d \in \mathbb{Z}, \quad ad-bc=1 \right\}$$

if is holomorphic in the upper half-plane $\mathrm{Im}z > 0$, with some $\kappa \in 2\mathbb{N}$ satisfies, for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$, the functional equation

$$F\left(\frac{az+b}{cz+d}\right) = (cz+d)^{\kappa} F(z),$$

and at infinity has the Fourier series expansion

$$F(z) = \sum_{m=1}^{\infty} c(m) e^{2\pi imz}.$$

Moreover, we assume that the cusp form $F(s)$ is a simultaneous eigen function of all Hecke operators

$$(T_n f)(z) = n^{\kappa-1} \sum_{d|n} d^{-\kappa} \sum_{b=0}^{d-1} f\left(\frac{nz+bd}{dz}\right), \quad n \in \mathbb{N}.$$

It is known that, in this case, $c(1) \neq 0$. Thus, we can normalize the function $F(s)$ by taking $c(1) = 1$.

To a normalized Hecke eigen cusp form $F(z)$, we can attach the zeta-function $\zeta(s,F)$ defined, for $\sigma > \frac{\kappa+1}{2}$, by

$$\zeta(s,F) = \sum_{m=1}^{\infty} \frac{c(m)}{m^s} = \prod_p \left(1 - \frac{\alpha(p)}{p^s}\right)^{-1} \left(1 - \frac{\beta(p)}{p^s}\right)^{-1},$$

where, for primes p, $\alpha(p)$ and $\beta(p)$ are conjugate complex numbers such that $\alpha(p) + \beta(p) = c(p)$. It is well known that the function $\zeta(s,F)$ has analytic continuation to an entire function.

The theory of modular forms is given, for example, in [11] and [7].

The universality of the function $\zeta(s,F)$ was began to study in [15] and completely proved in [29]. Let $D_\kappa = \{s \in \mathbb{C} : \ \frac{\kappa}{2} < \sigma < \frac{\kappa+1}{2}\}$. Then the following analogue of Theorem A is true.

Theorem B. *Suppose that $K \subset D_\kappa$ is a compact subset with connected complement, and that $f(s)$ is a continuous non-vanishing function on K which is analytic in the interior of K. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T} \mathrm{meas} \left\{ \tau \in [0,T] : \ \sup_{s \in K} |\zeta(s+i\tau, F) - f(s)| < \varepsilon \right\} > 0.$$

A more interesting and complicated property of zeta-functions than the universality is their joint universality. The first result on joint universality also belongs to S. M. Voronin. In [43], see also [18], he obtained a joint universality theorem for Dirichlet L-functions. We remind that two Dirichlet characters χ_1 and χ_2 are equivalent if they are generated by the same primitive character. The theory of Dirichlet L-functions can be found, for example, in [38], [17]. We state a modified version of the Voronin theorem.

Theorem C. *Suppose that $\chi_1, \dots, \chi_r$ are pairwise non-equivalent Dirichlet characters, and $L(s, \chi_1)$, $\dots, L(s, \chi_r)$ are the corresponding Dirichlet L-functions. For $j = 1, \dots, r$, let $K_j \subset D$ be a compact subset with connected complement, and let $f_j(s)$ be a continuous non-vanishing function on K_j which is analytic in the interior of K_j. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T} \mathrm{meas} \left\{ \tau \in [0, T] : \sup_{1\le j\le r} \sup_{s\in K_j} |L(s + i\tau, \chi_j) - f_j(s)| < \varepsilon \right\} > 0.$$

Other versions of Theorem C were independently obtained by S. M. Gonek [8] and B. Bagchi [1], [2]. The Voronin theorem in the form of Theorem C is given in [26].

In Theorem C, a collection of analytic functions are simultaneously approximated by shifts of Dirichlet L-functions. This procedure, of course, requires a certain independence of a collection of L-functions, and this independence is expressed by the non-equivalence of Dirichlet characters. The known joint universality theorems for other zeta-functions also involve some independence hypotheses. This is clearly reflected in a joint universality theorem for Hurwitz zeta-functions. However, first we remind the definition and universality of the Hurwitz zeta function.

Let α, $0 < \alpha \le 1$, be a fixed parameter. The Hurwitz zeta-function $\zeta(s, \alpha)$ is defined, for $\sigma > 1$, by the series

$$\zeta(s, \alpha) = \sum_{m=0}^{\infty} \frac{1}{(m + \alpha)^s},$$

and has meromorphic continuation to the whole complex plane with unique simple pole at the point $s = 1$ with residue 1. The function $\zeta(s, \alpha)$ is an interesting analytical object depending on a parameter α whose arithmetical nature influences the properties of $\zeta(s, \alpha)$. The universality of $\zeta(s, \alpha)$ is contained in the following theorem.

Theorem D. *Suppose that the number α is transcendental or rational $\ne 1, \frac{1}{2}$. Let $K \subset D$ be a compact subset with connected complement, and let $f(s)$ be a continuous function on K which is analytic in the interior of K. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T} \mathrm{meas} \left\{ \tau \in [0, T] : \sup_{s\in K} |\zeta(s + i\tau, \alpha) - f(s)| < \varepsilon \right\} > 0.$$

First Theorem D by different methods has been obtained in [8] and [1], see also [40]. We see that the approximated function $f(s)$, differently from Theorem A, is not necessarily non-vanishing on K, and this is conditioned by non-existence of the Euler product over primes for the function $\zeta(s,\alpha)$ in the case of Theorem D. We have that $\zeta(s,1)=\zeta(s)$ and

$$\zeta\left(s,\frac{1}{2}\right)=(2^s-1)\zeta(s),$$

therefore, the functions $\zeta(s,1)$ and $\zeta\left(s,\frac{1}{2}\right)$ are also universal, however, the approximated function $f(s)$ must be non-vanishing on K.

The case of algebraic irrational parameter α remains an open problem.

Now we state a joint universality theorem for Hurwitz zeta-functions. Let, for $0<\alpha_j\leq 1$, $j=1,\dots,r$,

$$L(\alpha_1,\dots,\alpha_r)=\{\log(m+\alpha_j):\ m\in\mathbb{N}_0,\quad j=1,\dots,r\}.$$

Theorem E. *Suppose that the set $L(\alpha_1,\dots,\alpha_r)$ is linearly independent over the field of rational numbers $\mathbb{Q}$. For $j=1,\dots,r$, let $K_j\subset D$ be a compact subset with connected complement, and let $f_j(s)$ be a continuous function on K_j which is analytic in the interior of K_j. Then, for every $\varepsilon>0$,*

$$\liminf_{T\to\infty}\frac{1}{T}\text{meas}\left\{\tau\in[0,T]:\ \sup_{1\leq j\leq r}\sup_{s\in K_j}|\zeta(s+i\tau,\alpha_j)-f_j(s)|<\varepsilon\right\}>0.$$

A proof of Theorem E is given in [23]. For algebraically independent over $\mathbb{Q}$ numbers $\alpha_1,\dots,\alpha_r$ ($\alpha_1,\dots,\alpha_r$ are not roots of any polynomial $p(x_1,\dots,x_r)\not\equiv 0$ with rational coefficients), Theorem E by a different method has been obtained in [36].

A generalization of the Hurwitz zeta-function is the periodic Hurwitz zeta- function introduced in [12]. Let $\mathfrak{a}=\{a_m: m\in\mathbb{N}_0\}$ be a periodic sequence of complex numbers with minimal period $k\in\mathbb{N}$, and $0<\alpha\leq 1$. Then the periodic Hurwitz zeta-function $\zeta(s,\alpha;\mathfrak{a})$ is defined, for $\sigma>1$, by

$$\zeta(s,\alpha;\mathfrak{a})=\sum_{m=0}^{\infty}\frac{a_m}{(m+\alpha)^s}.$$

The periodicity of the sequence $\mathfrak{a}$ implies, for $\sigma>1$, the equality

$$\zeta(s,\alpha;\mathfrak{a})=\frac{1}{k^s}\sum_{l=0}^{k-1}a_l\zeta\left(s,\frac{l+\alpha}{k}\right).$$

This shows that the function $\zeta(s,\alpha;\mathfrak{a})$ also admits a meromorphic continuation with a simple pole at $s=1$ with residue

$$a\overset{def}{=}\frac{1}{k}\sum_{l=0}^{k-1}a_l.$$

In the case $a=0$, the function $\zeta(s,\alpha;\mathfrak{a})$ is entire.

In [12] and [13], the universality of the function $\zeta(s, \alpha; \mathfrak{a})$ with transcendental parameter α was investigated, and the following statement was proved.

Theorem F. *Suppose that the number α is transcendental. Let $K \subset D$ be a compact subset with connected complement, and let $f(s)$ be a continuous function on K which is analytic in the interior of K. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T}\mathrm{meas}\left\{\tau \in [0, T] : \sup_{s\in K} |\zeta(s + i\tau, \alpha; \mathfrak{a}) - f(s)| < \varepsilon\right\} > 0.$$

Thus, Theorem F is an analogue of Theorem D in the case of transcendental α.

The joint universality for periodic Hurwitz zeta-functions was began to study in [21]. For $j = 1 \dots, r$, let $\mathfrak{a}_j = \{a_{mj} : m \in \mathbb{N}\}$ be a periodic sequence of complex numbers with minimal period $k_j \in \mathbb{N}$, α_j, $0 < \alpha_j \leq 1$, be a fixed parameter, and $\zeta(s, \alpha_j; \mathfrak{a}_j)$ denote the corresponding periodic Hurwitz zeta-function. Denote by k the least common multiple of the periods $k_1, \dots, k_r$, and define the matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1r} \\ a_{21} & a_{22} & \dots & a_{2r} \\ \dots & \dots & \dots & \dots \\ a_{k1} & a_{k2} & \dots & a_{kr} \end{pmatrix}.$$

In [21], it was proved that if $k_j = k$, $\alpha_j = \alpha$ for $j = 1, \dots, r$, α is transcendental, and $\mathrm{rank}(A) = r$, then the functions $\zeta(s, \alpha, \mathfrak{a}_1), \dots, \zeta(s, \alpha; \mathfrak{a}_r)$ are jointly universal. In [22], the requirement that $k_j = k$ for $j = 1, \dots, r$ was removed. Finally, in [14] the following joint universality theorem was proved.

Theorem G. *Suppose that the numbers $\alpha_1, \dots, \alpha_r$ are algebraically independent over $\mathbb{Q}$, and that* $\mathrm{rank}(A) = r$. *For $j = 1, \dots, r$, let K_j and $f_j(s)$ be the same as in Theorem E. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T}\mathrm{meas}\left\{\tau \in [0, T] : \sup_{1\leq j\leq r} \sup_{s\in K_j} |\zeta(s + i\tau, \alpha_j; \mathfrak{a}_j) - f_j(s)| < \varepsilon\right\} > 0.$$

It turned out that the rank hypothesis in Theorem G can be removed, and a joint universality theorem for periodic Hurwitz zeta-functions, without using the matrix A, forms Chapter 1 of the thesis. Let $L(\alpha_1, \dots, \alpha_r)$ be the same set as in Theorem E. We give a shortered statement of Theorem 1.1.

Theorem 1.1. *Suppose that the set $L(\alpha_1, \dots, \alpha_r)$, is linearly independent over $\mathbb{Q}$. For $j = 1, \dots, r$, let K_j and $f_j(s)$ be the same as in Theorem E. Then the assertion of Theorem G is true.*

The joint universality for periodic Hurwitz zeta-functions has a more general form when a collection of periodic sequences is attached to each parameter α_j. First such an extension of the joint universality has been proposed in [31] for Lerch zeta-functions. The above idea for periodic Hurwitz zeta-functions has been applied in [24]. Let l_j, $j = 1, \dots, r$, be positive integers. For every $l = 1, \dots, l_j$, let

$\mathfrak{a}_{jl} = \{a_{mjl} : m \in \mathbb{N}_0\}$ be a periodic sequence of complex numbers with minimal period $k_{jl} \in \mathbb{N}$. Suppose that, for $j = 1, \dots, r$, α_j is a fixed parameter, $0 < \alpha_j \le 1$, and, for $\sigma > 1$,

$$\zeta(s, \alpha_j; \mathfrak{a}_{jl}) = \sum_{m=0}^{\infty} \frac{a_{mjl}}{(m + \alpha_j)^s}.$$

Denote by k the least common multiple of the periods $k_{11}, \dots, k_{1l_1}, \dots, k_{r1}, \dots, k_{rl_r}$, and define the matrix

$$B = \begin{pmatrix} a_{111} & a_{112} & \dots & a_{11l_1} & a_{122} & \dots & a_{12l_2} & \dots & a_{1r1} & a_{1r2} & \dots & a_{1rl_r} \\ a_{211} & a_{212} & \dots & a_{21l_1} & a_{222} & \dots & a_{22l_2} & \dots & a_{2r1} & a_{2r2} & \dots & a_{2rl_r} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ a_{k11} & a_{k12} & \dots & a_{k1l_1} & a_{k22} & \dots & a_{k2l_2} & \dots & a_{kr1} & a_{kr2} & \dots & a_{krl_r} \end{pmatrix}.$$

Moreover, let

$$\kappa = \sum_{j=1}^{r} l_j.$$

Then in [24], the following result has been obtained.

Theorem H. *Suppose that the system $L(\alpha_1, \dots, \alpha_r)$ is linearly independent over $\mathbb{Q}$, and that* $\operatorname{rank}(B) = \kappa$. *For every $j = 1, \dots, r$, and $l = 1, \dots, l_j$, let K_{jl} be a compact subset of the strip D with connected complement, and let $f_{jl}(s)$ be a continuous function on K_{jl} which is analytic in the interior of K_{jl}. Then, for every $\varepsilon > 0$,*

$$\liminf_{T \to \infty} \frac{1}{T} \text{meas} \left\{ \tau \in [0, T] : \sup_{1 \le j \le r} \sup_{1 \le l \le l_j} \sup_{s \in K_{jl}} |\zeta(s + i\tau, \alpha_j; \mathfrak{a}_{jl}) - f_{jl}(s)| < \varepsilon \right\} > 0.$$

In Chapter 2 of the thesis, the rank condition in Theorem H is made weaker. Let k_j be the least common multiple of the periods $k_{j1}, k_{j2}, \dots, k_{jl_j}$, $j = 1, \dots, r$. Define

$$B_j = \begin{pmatrix} a_{1j1} & a_{1j2} & \dots & a_{1jl_j} \\ a_{2j1} & a_{2j2} & \dots & a_{2jl_j} \\ \dots & \dots & \dots & \dots \\ a_{k_j j1} & a_{k_j j2} & \dots & a_{k_j jl_j} \end{pmatrix}, \quad j = 1, \dots, r.$$

Then the main result of Chapter 2 is the following theorem.

Theorem 2.1. *Suppose that the set $L(\alpha_1, \dots, \alpha_r)$ is linearly independent over $\mathbb{Q}$, and that* $\operatorname{rank}(B_j) = l_j$, $j = 1, \dots, r$. *Let K_{jl} and f_{jl} be the same as is Theorem H. Then the assertion of Theorem H is true.*

In Theorem 2.1, differently from Theorem H, we use the information related only to α_j, $j = 1, \dots, r$.

All above joint universality theorems for zeta or L-functions are of the same type. Theorem C is an example of the joint universality for functions having the Euler product over primes, while all joint theorems for periodic Hurwitz zeta-functions form a group of results for zeta-functions having no the Euler product. The paper of H. Mishou [35] is the first work on the joint universality for zeta-functions of different types: having and having no the Euler product. We call this universality a mixed joint universality. In [35], a joint universality theorem for the Riemann and Hurwitz zeta-functions has been proved.

Theorem I. *Suppose that the number α is transcendental. Let $K_1 \subset D, K_2 \subset D$ be compact subsets with connected complements, $f_1(s)$ be a continuous non-vanishing function on K_1 which is analytic in the interior of K_1, and let $f_2(s)$ be a continuous function on K_2 which is analytic in the interior of K_2. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T}\text{meas}\Big\{\tau \in [0,T] : \sup_{s\in K_1} |\zeta(s+i\tau) - f_1(s)| < \varepsilon,$$
$$\sup_{s\in K_2} |\zeta(s+i\tau,\alpha) - f_2(s)| < \varepsilon\Big\} > 0.$$

A generalization of Theorem I has been given in [16]. Let $\mathfrak{b} = \{b_m : m \in \mathbb{N}\}$ be a periodic sequence of complex numbers with minimal period $l \in \mathbb{N}$. Then the periodic zeta-function $\zeta(s;\mathfrak{b})$ is defined, for $\sigma > 1$, by

$$\zeta(s;\mathfrak{b}) = \sum_{m=1}^{\infty} \frac{b_m}{m^s}.$$

In view of periodicity of the sequence $\mathfrak{b}$, it follows that, for $\sigma > 1$,

$$\zeta(s;\mathfrak{b}) = \frac{1}{l^s}\sum_{j=1}^{l} b_j \zeta\Big(s, \frac{j}{l}\Big),$$

and this gives meromorphic continuation for $\zeta(s;\mathfrak{b})$ to the whole complex plane with possible pole at $s = 1$ with residue

$$b \stackrel{def}{=} \frac{1}{l}\sum_{j=1}^{l} b_j.$$

If $b = 0$, then the function $\zeta(s;\mathfrak{b})$ is entire.

We recall that the sequence $\mathfrak{b}$ is multiplicative if $b_1 = 1$, and $b_{mn} = b_m b_n$ for all colprimes $m, n \in \mathbb{N}$. The universality of the function $\zeta(s;\mathfrak{b})$ with multiplicative sequence $\mathfrak{b}$ has been obtained in [32]. In this case, the theorem is similar to Theorem A.

In [16], the joint universality for the functions $\zeta(s;\mathfrak{b})$ and $\zeta(s,\alpha;\mathfrak{a})$ has been obtained.

Theorem J. *Suppose that the sequence $\mathfrak{b}$ is multiplicative such that, for every prime p,*

$$\sum_{l=1}^{\infty} \frac{|b_{p^l}|}{p^{\frac{l}{2}}} \le c < 1,$$

and that the number α is transcendental. Let $K_1, K_2, f_1(s)$ and $f_2(s)$ be the same as in Theorem I. Then, for every $\varepsilon > 0$,

$$\liminf_{T\to\infty} \frac{1}{T}\text{meas}\Big\{\tau \in [0,T] : \sup_{s\in K_1} |\zeta(s+i\tau;\mathfrak{b}) - f_1(s)| < \varepsilon,$$
$$\sup_{s\in K_2} |\zeta(s+i\tau,\alpha;\mathfrak{a}) - f_2(s)| < \varepsilon\Big\} > 0.$$

A multidimensional version of Theorem J is presented in [25]. In this case, the joint universality is obtained for the collection of zeta-functions $\zeta(s;\mathfrak{b}_1), \dots, \zeta(s;\mathfrak{b}_{r_1})$ and $\zeta(s,\alpha;\mathfrak{a}_1), \dots, \zeta(s,\alpha_{r_2};\mathfrak{a}_{r_2})$.

Theorem K [25]. *Suppose that, for $j = 1, \dots, r_1$, the sequence $\mathfrak{b}_j$ is multiplicative such that, for every prime p,*

$$\sum_{l=1}^{\infty} \frac{|b_{p^l j}|}{p^{\frac{l}{2}}} \le c_j < 1,$$

and that the numbers $\alpha_1, \dots, \alpha_2$ are algebraically independent over $\mathbb{Q}$. For $j = 1, \dots, r_1$, let $K_j \subset D$ be a compact subset with connected complement, and let $f_j(s)$ be a continuous non-vanishing function on K_j which is analytic in the interior of K_j. For $j = 1, \dots, r_2$, let $\widehat{K_j} \subset D$ be a compact subset with connected complement, and let $\widehat{f_j}(s)$ be a continuous function on $\widehat{K_j}$ which is analytic in the interior of $\widehat{K_j}$. Then, for every $\varepsilon > 0$,

$$\liminf_{T\to\infty} \frac{1}{T}\text{meas}\Big\{\tau \in [0,T] : \sup_{1\le j\le r_1} \sup_{s\in K_j} |\zeta(s+i\tau;\mathfrak{b}_j) - f_j(s)| < \varepsilon,$$
$$\sup_{1\le j\le r_2} \sup_{s\in \widehat{K_j}} |\zeta(s+i\tau,\alpha_j;\mathfrak{a}_j) - \widehat{f_2}(s)| < \varepsilon\Big\} > 0.$$

In Chapter 3 of the thesis, we generalize Theorem 2.1 adding to the functions $\zeta(s,\alpha_1;\mathfrak{a}_{11}), \dots,$ $\zeta(s,\alpha_1;\mathfrak{a}_{1l_1}), \dots, \zeta(s,\alpha_r;\mathfrak{a}_{r1}), \dots, \zeta(s,\alpha_r;\mathfrak{a}_{rl_r})$ the Riemann zeta-function $\zeta(s)$. Thus, we have the following statement.

Theorem 3.1. *Suppose that the numbers $\alpha_1, \dots, \alpha_r$ are algebraically independent over $\mathbb{Q}$,* rank(B_j) *$= l_j$, $j = 1, \dots, r$, and that all hypotheses on the sets K_{jl} and functions $f_{jl}(s)$ of Theorem 2.1 hold. Moreover, let $K \subset D$ be a compact subset with connected complement, and let $f(s)$ be a continuous non-vanishing function on K which is analytic in the interior of K. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T}\text{meas}\Big\{\tau \in [0,T] : \sup_{s\in K} |\zeta(s+i\tau) - f(s)| < \varepsilon,$$
$$\sup_{1\le j\le r} \sup_{1\le l\le l_j} \sup_{s\in K_{jl}} |\zeta(s+i\tau,\alpha_j;\mathfrak{a}_{jl}) - f_{jl}(s)| < \varepsilon\Big\} > 0.$$

Chapter 4 of the thesis is devoted to a an analogue of Theorem 3.1 with the function $\zeta(s,F)$ in place of the function $\zeta(s)$. Note that, in this case, we have a more complicated situation because the functions $\zeta(s, F)$ and $\zeta(s, \alpha_j; \mathfrak{a}_{jl})$ are universal in different strips D_κ and D, respectively. We remind that here $\zeta(s, F)$ denotes the zeta-function attached to a normalized Hecke eigen cusp form F of weight κ.

Theorem 4.1. *Suppose that F is a normalized Hecke eigen cusp form of weight κ for the full modular group, the numbers $\alpha_1, \dots, \alpha_r$ are algebraically independent over $\mathbb{Q}$, and that* $\text{rank}(B_j) = l_j$, *$j = 1, \dots, r$. Let $K \subset D_\kappa$ be a compact subset with connected complement, $f(s)$ be a continuous non-vanishing function on K which is analytic in the interior of K, and that all hypothesis on the sets K_{jl} and functions f_{jl} of Theorem 2.1 hold. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T} \text{meas}\Big\{\tau \in [0, T] : \sup_{s\in K} |\zeta(s + i\tau, F) - f(s)| < \varepsilon,$$
$$\sup_{1\le j\le r} \sup_{1\le l\le l_j} \sup_{s\in K_{jl}} |\zeta(s + i\tau, \alpha_j; \mathfrak{a}_{jl}) - f_{jl}(s)| < \varepsilon\Big\} > 0.$$

History of universality in analysis can be found in a very informative paper [9], see also [41], [20], [33].

The proofs of all joint universality theorems of the thesis are based on the probabilistic approach involving limit theorems for weakly convergent probability measures on the space of analytic functions, and on explicitly given supports of the limit measures.

Approbation

The results of the thesis were presented at the Conferences of Lithuanian Mathematical Society (2009-2011), at 10^{th} International Vilnius Conference on Probability Theory and Mathematical Statistics (Vilnius, Lithuania, 28^{th} June -2^{nd} July, 2010), at the 16^{th} International Conference Mathematical Modeling and Analysis (Sigulda, Latvia, May 25-28, 2011), at the 8^{th} International Algebraic Conference in Ukraine (Lugansk, Ukraine, July 5-12, 2011), at the International Conference 27^{th} Journées Arithmétiques (Vilnius, Lithuania, 27^{th} June -1^{st} July, 2011), as well as at the doctorant conferences of Institute of Mathematics and Informatics, and at the seminars of Number theory of Vilnius University and the seminar of the Faculty of Mathematics and Informatics of Šiauliai University.

Principal publications

The main results of the thesis are published in the following papers:

1. A. Laurinčikas, S. Skerstonaitė, A joint universality theorem for periodic Hurvitz zeta-functions. II, *Lith. Math. J.* Vol. 49, No 3, 287-296 (2009).

2. A. Laurinčikas, S. Skerstonaitė, Joint universality for periodic Hurwitz zeta-functions. II, in: *New Directions in Value-Distribution Theory of Zeta and L-Functions*, Würzburg Conference, 6-10 October (2008), R. Steuding and J. Steuding (Eds.), Shaker Verlag, Aachen, 161-169 (2009).

3. S. Račkauskienė, D. Šiaučiūnas, Joint universality of some zeta functions. I, *Lietuvos Matematikos Rinkinys.* T. 51, 45-50 (2010).

4. J. Genys, R. Macaitienė, S. Račkauskienė, D. Šiaučiūnas,A mixed joint universality theorem for zeta-functions. *Mathematical Modelling and Analysis*, Vol. 15, No 4, 431-446 (2010).

5. S. Račkauskienė, D. Šiaučiūnas, A mixed joint universality theorem for zeta-functions. II, (*Submited*).

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Chapter 1

Joint universality for periodic Hurwitz zeta-functions

In this chapter, we prove a joint universality theorem for periodic Hurwitz zeta-functions $\zeta(s,\alpha_1;\mathfrak{a}_1)$, $\dots,\zeta(s,\alpha_r;\mathfrak{a}_r)$. Here, for $j=1,\dots,r$, α_j, $0<\alpha_j\leq 1$, is a fixed parameter, $\mathfrak{a}_j=\{a_{mj}:\ m\in\mathbb{N}_0\}$ is a periodic sequence of complex numbers with minimal period $k\in\mathbb{N}$, and for $\sigma>1$,

$$\zeta(s,\alpha_j;\mathfrak{a}_j)=\sum_{m=0}^{\infty}\frac{a_{mj}}{(m+\alpha_j)^s}.$$

If

$$a_j=\frac{1}{k_j}\sum_{l=0}^{k-1}a_{lj}\neq 0,$$

then the function $\zeta(s,\alpha_j;\mathfrak{a}_j)$ is entire, while if $a_j\neq 0$, the function $\zeta(s,\alpha_j;\mathfrak{a}_j)$ has a unique simple pole at $s=1$ with residue a_j.

1.1. Statement of the main theorem

We recall that

$$L(\alpha_1,\dots,\alpha_r)=\big\{\log(m+\alpha_j):\ m\in\mathbb{N}_0,\quad j=1,\dots,r\big\}.$$

For brevity, denote the elements of the set $L(\alpha_1,\dots,\alpha_r)$ by $e_{m,j}=\log(m+\alpha_j)$. The set $L(\alpha_1,\dots,\alpha_r)$ is linearly independent over the field of rational numbers $\mathbb{Q}$ if, for every finite collection $e_{m_1,j_1},\dots,e_{m_n,j_l}$, $\{m_1,\dots,m_n\}\subset\mathbb{N}_0$, $\{j_1,\dots,j_l\}\subset\{1,\dots,r\}$, the equality

$$q_{m_1,j_1}e_{m_1,j_1}+\dots+q_{m_n,j_l}e_{m_n,j_l}=0$$

with rationals $q_{m_1,j_1},\dots,q_{m_n,j_l}$ holds only in the case $q_{m_1,j_1}=\dots=q_{m_n,j_l}=0$. Obviously, in place of rationals $q_{m_1,j_1},\dots,q_{m_n,j_l}$ we may use rational integers.

We remind that $D=\{s\in\mathbb{C}:\ \frac{1}{2}<\sigma<1\}$.

Theorem 1.1. *Suppose that the set $L(\alpha_1,\dots,\alpha_r)$ is linearly independent over $\mathbb{Q}$. For $j=1,\dots,r$, let $K_j\subset D$ be a compact subset with connected complement, and let $f_j(s)$ be a continuous function on K_j which is analytic in the interior of K_j. Then, for every $\varepsilon>0$,*

$$\liminf_{T\to\infty}\frac{1}{T}\mathrm{meas}\left\{\tau\in[0,T]:\ \sup_{1\le j\le r}\sup_{s\in K_j}|\zeta(s+i\tau,\alpha_j;\mathfrak{a}_j)-f_j(s)|<\varepsilon\right\}>0.$$

We note that Theorem 1.1 removes a certain rank condition on the coefficients a_{mj} which was used in [14].

A joint limit theorem on the weak convergence of probability measures in the space of analytic functions for periodic Hurwitz zeta-functions is the main ingredient in the proof of Theorem 1.1.

1.2. Joint limit theorem

We denote by $H(D)$ the space of analytic functions on D equipped with the topology of uniform convergence on compacta. In this topology, a sequence $\{g_n(s):\ n\in\mathbb{N}\}\subset H(D)$ converges to the function $g(s)\in H(D)$ if, for every compact subset $K\subset D$,

$$\lim_{n\to\infty}\sup_{s\in K}|g_n(s)-g(s)|=0.$$

Define

$$H^r(D)=\underbrace{H(D)\times\dots\times H(D)}_{r}.$$

Let, as usual, $\mathfrak{B}(S)$ denote the class of Borel sets of a space S. Moreover, let

$$\Omega=\prod_{m=0}^{\infty}\gamma_m,$$

where $\gamma_m=\{s\in\mathbb{C}:\ |s|=1\}$ for all $m\in\mathbb{N}_0$. Since the unit circle γ is a compact, by the Tikhonov theorem, see, for example, [37], the infinite-dimensional torus Ω with the product topology and point wise multiplication is a compact topological Abelian group. Let

$$\Omega^r=\Omega_1\times\dots\times\Omega_r,$$

where $\Omega_j=\Omega$ for $j=1,\dots,r$. Then Ω^r also, by the Tikhonov theorem, is a compact topological Abelian group. Therefore [39], on $(\Omega^r,\mathfrak{B}(\Omega^r))$, the probability Haar measure m_H^r can be defined, and we obtain the probability space $(\Omega^r, \mathfrak{B}(\Omega^r), m_H^r)$. We remind that the measure m_H^r is invariant with respect to shifts by points from Ω^r, i.e.,

$$m_H^r(A) = m_H^r(\omega A) = m_H^r(A\omega)$$

for every $A \in \mathfrak{B}(\Omega^r)$ and all $\omega \in \Omega$. It is important to note that the Haar measure m_H^r is the product of the Haar measures m_{jH} on the coordinate spaces $(\Omega_j, \mathfrak{B}(\Omega_j))$, $j = 1, \dots, r$. Denote by $\omega_j(m)$ the projection of an element $\omega_j \in \Omega_j$ to the coordinate space γ_m, $m \in \mathbb{N}_0$, $j = 1, \dots, r$. Let $\underline{\omega} = (\omega_1, \dots, \omega_r) \in \Omega^r$, where $\omega_j \in \Omega_j$, $\quad j = 1, \dots, r$, and let, for brevity,

$$\underline{\alpha} = (\alpha_1, \dots, \alpha_r), \quad \underline{\mathfrak{a}} = (\mathfrak{a}_1, \dots, \mathfrak{a}_r).$$

On the probability space $(\Omega^r, \mathfrak{B}(\Omega^r), m_H^r)$, define the $H^r(D)$-valued random element $\underline{\zeta}(s, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{a}})$ by

$$\underline{\zeta}(s, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{a}}) = \big(\zeta(s, \alpha_1, \omega_1; \mathfrak{a}_1), \dots, \zeta(s, \alpha_r, \omega_r; \mathfrak{a}_r)\big),$$

where

$$\zeta(s, \alpha_j, \omega_j; \mathfrak{a}_j) = \sum_{m=0}^{\infty} \frac{a_{mj}\omega_j(m)}{(m+\alpha_j)^s}, \quad j = 1, \dots, r.$$

We note that the latter series converges uniformly on compact subsets $K \subset D$ for almost all $\omega_j \in \Omega_j$, thus it defines an $H(D)$-value random element, $j = 1, \dots, r$. Denote by $P_{\underline{\zeta}}$ the distribution of the random element $\underline{\zeta}(s, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{a}})$, i.e.,

$$P_{\underline{\zeta}}(A) = m_H^r\big(\omega \in \Omega^r : \underline{\zeta}(s, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{a}}) \in A\big), \quad A \in \mathfrak{B}(H^r(D)).$$

Let, for $A \in \mathfrak{B}(H^r(D))$,

$$P_T(A) = \frac{1}{T}\text{meas}\Big\{\tau \in [0, T] : \ \underline{\zeta}(s + i\tau, \underline{\alpha}; \underline{\mathfrak{a}}) \in A\Big\},$$

where

$$\underline{\zeta}(s, \underline{\alpha}; \underline{\mathfrak{a}}) = (\zeta(s, \alpha_1; \mathfrak{a}_1), \dots, \zeta(s, \alpha_r; \mathfrak{a}_r)).$$

This section of the chapter is devoted to the following probabilistic limit theorem.

Theorem 1.2. *Suppose that the set $L(\alpha_1, \dots, \alpha_r)$ is linearly independent over $\mathbb{Q}$. Then P_T converges weakly to the measure $P_{\underline{\zeta}}$ as $T \to \infty$.*

We divide the proof of Theorem 1.2 into severe lemmas. The first of them is a limit theorem on the torus Ω^r. Let, for $A \in \mathfrak{B}(\Omega^r)$,

$$Q_T(A) = \frac{1}{T}\text{meas}\Big\{\tau \in [0, T] : \ (((m+\alpha_1)^{-i\tau} : m \in \mathbb{N}_0), \dots, ((m+\alpha_r)^{-i\tau} : \ m \in \mathbb{N}_0)) \in A\Big\}.$$

Lemma 1.3. *Suppose that the set $L(\alpha_1, \dots, \alpha_r)$ is linearly independent over $\mathbb{Q}$. Then Q_T converges weakly to the Haar measure m_H^r as $T \to \infty$.*

A proof of Lemma 1.3 is based on the Fourier transform method on compact topological group, and is given in [24].

Now let $\sigma_1 > \frac{1}{2}$ be a fixed number, and let, for $m, n \in \mathbb{N}_0$,

$$v_n(m, \alpha_j) = \exp\left\{ -\left(\frac{m+\alpha_j}{n+\alpha_j} \right)^{\sigma_1} \right\}, \quad j = 1, \dots, r.$$

Define

$$\zeta_n(s, \alpha_j; \mathfrak{a}_j) = \sum_{m=0}^{\infty} \frac{a_{mj} v_n(m, \alpha_j)}{(m+\alpha_j)^s}, \quad j = 1, \dots, r,$$

and

$$\zeta_n(s, \alpha_j, \omega_j; \mathfrak{a}_j) = \sum_{m=0}^{\infty} \frac{a_{mj} \omega_j(m) v_n(m, \alpha_j)}{(m+\alpha_j)^s}, \quad j = 1, \dots, r,$$

It was proved in [12] that the latter series are absolutely convergent for $\sigma > \frac{1}{2}$. The next important step in the proof of Theorem 1.2 are limit theorems in the space $H^r(D)$ for the vectors

$$\underline{\zeta}_n(s, \underline{\alpha}; \underline{\mathfrak{a}}) = (\zeta_n(s, \alpha_1; \mathfrak{a}_1), \dots, \zeta_n(s, \alpha_r; \mathfrak{a}_r))$$

and

$$\underline{\zeta}_n(s, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{a}}) = (\zeta_n(s, \alpha_1, \omega_1; \mathfrak{a}_1), \dots, \zeta_n(s, \alpha_r, \omega_r; \mathfrak{a}_r)).$$

Let, for $A \in \mathfrak{B}(H^r(D))$,

$$P_{T,n}(A) = \frac{1}{T} \text{meas} \left\{ \tau \in [0, T] : \underline{\zeta}_n(s + i\tau, \underline{\alpha}; \underline{\mathfrak{a}}) \in A \right\},$$

and, for fixed $\underline{\omega}_0 \in \Omega^r$,

$$Q_{T,n}(A) = \frac{1}{T} \text{meas} \left\{ \tau \in [0, T] : \underline{\zeta}_n(s + i\tau, \underline{\alpha}, \underline{\omega}_0; \underline{\mathfrak{a}}) \in A \right\}.$$

Lemma 1.4. *Suppose that the set $L(\alpha_1, \dots, \alpha_r)$ is linearly independent over $\mathbb{Q}$. Then $P_{T,n}$ and $Q_{T,n}$ both converge weakly to the same probability measure P_n on $(H^r(D), \mathfrak{B}(H^r(D)))$ as $T \to \infty$.*

Before the proof of Lemma 1.4, we remind the well-known fact from the theory of weak convergence of probability measures. Let $(S_1, \mathfrak{B}(S_1))$ and $(S_2, \mathfrak{B}(S_2))$ be two measurable spaces, and $h : S_1 \to S_2$ be a $(\mathfrak{B}(S_1), \mathfrak{B}(S_2))$-measurable function, i.e., for every $A \in \mathfrak{B}(S_2)$,

$$h^{-1}A \in \mathfrak{B}(S_1).$$

Then every probability measure P on $(S_1, \mathfrak{B}(S_1))$ induces the unique probability measure Ph^{-1} on $(S_2, \mathfrak{B}(S_2))$ defined by

$$Ph^{-1}(A) = P(h^{-1}A), \quad A \in \mathfrak{B}(S_2).$$

The following simple lemma which proof can be found in [3], Section 5, often is very useful.

Lemma 1.5. *Suppose that P_n, $n \in \mathbb{N}$, and P be probability measures on $(S_1, \mathfrak{B}(S_1))$, $h : S_1 \to S_2$ be a continuous function, and let P_n converges weakly to P as $n \to \infty$. Then $P_n h^{-1}$ also converges weakly to Ph^{-1} as $n \to \infty$.*

Proof of Lemma 1.4. Since the series for $\zeta_n(s, \alpha_j; \mathfrak{a}_j)$ and $\zeta_n(s, \alpha_j \omega_j; \mathfrak{a}_j)$, $j = 1, \dots, r$, converges absolutely for $\sigma > \frac{1}{2}$, the functions $h_n : \Omega^r \to H^r(D)$ and $g_n : \Omega^r \to H^r(D)$ given by $h_n(\omega) = \underline{\zeta}_n(s, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{a}})$ and $g_n(\omega) = \underline{\zeta}_n(s, \underline{\alpha}, \underline{\omega\omega_0}; \underline{\mathfrak{a}})$ are continuous. Moreover, we have that $P_{T,n} = Q_T h_n^{-1}$ and $Q_{T,n} = Q_T g_n^{-1}$. Therefore, from Lemmas 1.3 and 1.5 we obtain that $P_{T,n}$ and $Q_{T,n}$ converge weakly to $m_H^r h_n^{-1}$ and $m_H^r g_n^{-1}$ respectively, as $T \to \infty$. Moreover, the invariance of the Haar measure m_H^r with respect to shifts by points from Ω^r shows that

$$m_H^r g_n^{-1} = m_H^r (f_n(f_0))^{-1} = m_H^r (f_0^{-1} f_n^{-1}) = (m_H^r f_0^{-1}) f_n^{-1} = m_H^r f_n^{-1},$$

where $f_0 : \Omega^r \to \Omega^r$ is given by $f(\omega) = \underline{\omega\omega_0}$, $\underline{\omega} \in \Omega^r$.

In order to pass from $\underline{\zeta}_n(s, \underline{\alpha}; \underline{\mathfrak{a}})$ to $\underline{\zeta}(s, \underline{\alpha}; \underline{\mathfrak{a}})$, we need an approximation of $\underline{\zeta}(s, \underline{\alpha}; \underline{\mathfrak{a}})$ and $\underline{\zeta}(s, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{a}})$ by $\underline{\zeta}_n(s, \underline{\alpha}; \underline{\mathfrak{a}})$ and $\underline{\zeta}_n(s, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{a}})$, respectively. For this, we will use a metric on $H^r(D)$ which induces its topology of uniform convergence on compacta. First, we define such a metric on $H(D)$. For $g_1, g_2 \in H(D)$, we set

$$\rho(g_1, g_2) = \sum_{l=1}^{\infty} 2^{-l} \frac{\sup_{s \in K_l} |g_1(s) - g_2(s)|}{1 + \sup_{s \in K_l} |g_1(s) - g_2(s)|},$$

where $\{K_l : \ l \in \mathbb{N}\}$ is a sequence of compact subsets of D such that

$$D = \bigcup_{l=1}^{\infty} K_l,$$

$K_l \subset K_{l+1}$ for all $l \in \mathbb{N}$, and if $K \subset D$ is a compact subset, then $K \subset K_l$ for some l. The existence of such a sequence is given in [5]. Clearly, the metric ρ induces on $H(D)$ the topology of uniform convergence on compacta.

Now, for $\underline{g}_1 = (g_{11}, \dots, g_{1r})$, $\underline{g}_2 = (g_{21}, \dots, g_{2r}) \in H^r(D)$, putting

$$\underline{\rho}(\underline{g}_1, \underline{g}_2) = \max_{1 \le j \le r} \rho(g_{1j}, g_{2j}),$$

we obtain a desired metric on $H^r(D)$.

Lemma 1.6 *The equality*

$$\lim_{n \to \infty} \limsup_{T \to \infty} \frac{1}{T} \int_0^T \underline{\rho}(\underline{\zeta}(s + i\tau, \underline{\alpha}; \underline{\mathfrak{a}}), \underline{\zeta}_n(s + i\tau, \underline{\alpha}; \underline{\mathfrak{a}})) \mathrm{d}\tau = 0$$

holds.

The proof of the lemma does not depend on arithmetical nature of the numbers $\alpha_1, \dots, \alpha_r$, and is given in [13].

Lemma 1.7 *Suppose that the set $L(\alpha_1,\ldots,\alpha_r)$ is linearly independent over $\mathbb{Q}$. Then, for almost all $\underline{\omega}\in\Omega^r$, we have the equality*

$$\lim_{n\to\infty}\limsup_{T\to\infty}\frac{1}{T}\int_0^T \underline{\rho}(\underline{\zeta}(s+i\tau,\underline{\alpha},\underline{\omega};\underline{\mathfrak{a}}),\underline{\zeta}_n(s+i\tau,\underline{\alpha},\underline{\omega};\underline{\mathfrak{a}}))\mathrm{d}\tau=0.$$

Proof. Let $a_\tau=\{(m+\alpha)^{-i\tau}:\ m\in\mathbb{N}_0\}$, $\tau\in\mathbb{R}$, $0<\alpha\le 1$, and define $\varphi_\tau:\ \Omega\to\Omega$ by $\varphi_\tau(\omega)=a_\tau\omega$, $\omega\in\Omega$. Then $\{\varphi_\tau:\ \tau\in\mathbb{R}\}$ is a one-parameter group of measurable measure - preserving transformations on Ω. A set $A\in\mathfrak{B}(\Omega)$ is invariant with respect to the group $\{\varphi_\tau:\ \tau\in\mathbb{R}\}$ if, for every $\tau\in\mathbb{R}$, the sets A and $A_\tau=\varphi_\tau(A)$ coincide up to a set of m_H-measure zero, where m_H is the probability Haar measure on $(\Omega,\mathfrak{B}(\Omega))$. The invariant sets form a σ-field which is a σ-subfield of $\mathfrak{B}(\Omega)$. A one-parameter groups $\{\varphi_\tau:\ \tau\in\mathbb{R}\}$ is ergodic if its σ-field of invariant sets consists only of sets of m_H-measure zero or one. If the set $L(\alpha)=\{\log(m+\alpha):m\in\mathbb{N}_0\}$ is linearly independent over $\mathbb{Q}$, then it is proved in [24] that the group $\{\varphi_\tau:\ \tau\in\mathbb{R}\}$ is ergodic. Since the set $L(\alpha_1,\ldots,\alpha_r)$ is linearly independent over $\mathbb{Q}$, so is each set $L(\alpha_j)$, $j=1,\ldots,r$. Combining this with the classical Birkhoff-Khintchine ergodic theorem, it is proved in [24] that, for every compact subset $K\subset D$,

$$\lim_{n\to\infty}\limsup_{T\to\infty}\frac{1}{T}\int_0^T\sup_{s\in K}\big|\zeta(s+i\tau,\alpha_j,\omega_j;\mathfrak{a}_j)-\zeta_n(s+i\tau,\alpha_j,\omega_j;\mathfrak{a}_j)\big|\mathrm{d}\tau=0$$

for almost all $\omega_j\in\Omega_j$, $j=1,\ldots,r$. This and the definition of the metric ρ imply the equality

$$\lim_{n\to\infty}\limsup_{T\to\infty}\frac{1}{T}\int_0^T\rho\big(\zeta(s+i\tau,\alpha_j,\omega_j;\mathfrak{a}_j),\zeta_n(s+i\tau,\alpha_j,\omega_j;\partial_j)\big)\mathrm{d}\tau=0$$

for almost all $\omega_j\in\Omega_j$, $j=1,\ldots,r$, which, together with the definition of the metric $\underline{\rho}$, yields the assertion of the lemma.

For the proof of Theorem 1.2, we need one more lemma on a common limit measure. Let, for $A\in\mathfrak{B}(H^r(D))$,

$$\widehat{P}_T(A)=\frac{1}{T}\mathrm{meas}\Big\{\tau\in[0,T]:\ \underline{\zeta}(s+i\tau,\underline{\alpha},\underline{\omega};\underline{\mathfrak{a}})\Big\}\in A.$$

Lemma 1.8. *Suppose that the set $L(\alpha_1,\ldots,\alpha_r)$ is linearly independent over $\mathbb{Q}$. Then P_T and $\widehat{P}_T$, both converge weakly to the same probability measure P on $(H^r(D),\mathfrak{B}(H^r(D)))$ as $T\to\infty$.*

The proof of the lemma is based on the Prokhorov theory of weak convergence of probability measures, therefore, first we will remind some fact of that theory.

Let $\{P\}$ be a family of probability measures on $(S,\mathfrak{B}(S))$. The family $\{P\}$ is called relatively compact if every sequence $\{P_n\}\subset\{P\}$ contains a weakly convergent subsequence, and the family $\{P\}$ is tight if, for every $\varepsilon>0$, there exists a compact subset $K\subset S$ such that

$$P(K)>1-\varepsilon$$

for all $P\in\{P\}$. The Prokhorov theorems connect the notions of the relative compactness and tightness.

Lemma 1.9. *If the family of probability measures is tight, then it is relatively compact.*

Lemma 1.10. *Suppose that the space S is complete and separable. If the family $\{P\}$ is relatively compact, then it is tight.*

We also need one lemma from the theory of weak convergence of probability measures. Denote by $\xrightarrow{\mathfrak{D}}$ the convergence in distribution.

Lemma 1.11. *Suppose that the space (S, d) is separable, and Y_n, X_{kn}, $k \in \mathbb{N}$, $n \in \mathbb{N}$ are S-valued random elements defined on the probability space $(\widetilde{\Omega}, \mathfrak{B}(\widetilde{\Omega}), \mathbb{P})$. Let $X_{kn} \xrightarrow{\mathfrak{D}} X_n$ as $k \to \infty$, $X_n \xrightarrow{\mathfrak{D}} X$ as $n \to \infty$ and, for every $\varepsilon > 0$,*

$$\lim_{n\to\infty} \limsup_{k\to\infty} \mathbb{P}\big(d(X_{kn}, Y_n) \geq \varepsilon\big) = 0.$$

Then $Y_n \xrightarrow{\mathfrak{D}} X$ as $n \to \infty$.

The proofs of Lemmas 1.9-1.11 can be found in [3].

Proof of Lemma 1.8. We take a random variable θ defined on a certain probability space $(\widetilde{\Omega}, \mathfrak{B}(\widetilde{\Omega}), \mathbb{P})$ and uniformly distributed on $[0, 1]$. On $(\widetilde{\Omega}, \mathfrak{B}(\widetilde{\Omega}), \mathbb{P})$, define the $H^r(D)$-valued random element $\underline{X}_{T,n} = \underline{X}_{T,n}(s) = (X_{T,n,1}(s), \ldots, X_{T,n,r}(s)) = \underline{X}_{T,n}(s, \underline{\alpha}; \underline{\mathfrak{a}})$ by

$$\underline{X}_{T,n}(s, \underline{\alpha}; \underline{\mathfrak{a}}) = \underline{\zeta}_n(s + i\theta T, \underline{\alpha}; \underline{\mathfrak{a}}).$$

Then we have, by Lemma 1.4, that

$$\underline{X}_{T,n} \xrightarrow[T\to\infty]{\mathfrak{D}} \underline{X}_n, \tag{1.1}$$

where $\underline{X}_n = \underline{X}_n(s) = (X_{n,1}(s), \ldots, X_{n,r}(s))$, is an $H^r(D)$-valued random element having the distribution P_n, and P_n is the limit measure in Lemma 1.4. We will prove that the family of probability measures $\{P_n : n \in \mathbb{N}_0\}$ is tight. We have noted above that the series for $\zeta_n(s, \alpha_j; \mathfrak{a}_j)$, $j = 1, \ldots, r$, converges absolutely for $\sigma > \frac{1}{2}$. Therefore, using the properties of the mean square of absolutely convergent Dirichlet series, we have that, for $\sigma > \frac{1}{2}$,

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \big|\zeta_n(\sigma + it, \alpha_j; \mathfrak{a}_j)\big|^2 dt = \sum_{m=0}^{\infty} \frac{|a_{mj}|^2 v_n^2(m, \alpha_j)}{(m + \alpha_j)^{2\sigma}} \leq \sum_{m=0}^{\infty} \frac{|a_{mj}|^2}{(m + \alpha_j)^{2\sigma}} \tag{1.2}$$

for all $n \in \mathbb{N}_0$ and $j = 1, \ldots, r$. Let K_l be a compact subset from the definition of the metric ρ. Then the Cauchy integral formula a standard application of the contour integration and (1.2), for all $n \in \mathbb{N}_0$ and $j = 1, \ldots, r$ lead to the inequality

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \sup_{s\in K_l} \big|\zeta_n(s + i\tau, \alpha_j; \mathfrak{a}_j)\big| dt \leq C_l \bigg(\sum_{m=0}^{\infty} \frac{|a_{mj}|^2}{(m + \alpha_j)^{2\sigma_l}} \bigg)^{\frac{1}{2}} \tag{1.3}$$

with some $C_l > 0$ and $\sigma_l > \frac{1}{2}$.

Now let $\varepsilon > 0$ be an arbitrary number, and

$$R_{jl} = \bigg(\sum_{m=0}^{\infty} \frac{|a_{mj}|^2}{(m + \alpha_j)^{2\sigma_l}} \bigg)^{\frac{1}{2}}.$$

Then, taking $M_{jl} = C_l R_{jl} 2^{l+r} \varepsilon^{-1}$, we find from (1.3) that

$$\limsup_{T\to\infty} \mathbb{P}\left(\exists j : \sup_{s\in K_l} |X_{T,n,j}(s)| > M_{jl}\right) \leq \sum_{j=1}^{r} \limsup_{T\to\infty} \mathbb{P}\left(\sup_{s\in K_l} |X_{T,n,j}(s)| > M_{jl}\right) \leq$$
$$\leq \sum_{j=1}^{r} \frac{1}{M_l} \sup_{n\in\mathbb{N}_0} \limsup_{T\to\infty} \frac{1}{T} \int_0^T \sup_{s\in K_l} \left|\zeta_n(s+i\tau, \alpha_j; \mathfrak{a}_j)\right| dt \leq \sum_{j=1}^{r} \frac{C_l R_{jl}}{M_{jl}} < \frac{\varepsilon}{2^l}.$$

This and (1.1) show that, for all $n \in \mathbb{N}_0$,

$$\mathbb{P}\left(\exists j : \sup_{s\in K_l} |X_{n,j}(s)| > M_{jl}\right) \leq \frac{\varepsilon}{2^l}. \tag{1.4}$$

Define a set

$$H_\varepsilon^r = \left\{(g_1, \dots, g_r) \in H^r(D) : \sup_{s\in K_l} |g_j(s)| \leq M_{jl}, \quad j = 1, \dots, r, \quad l \in \mathbb{N}\right\}.$$

Then the set H_ε^r is uniformly bounded, thus, is a compact subset in the space $H^r(D)$. Moreover, in view of (1.4),

$$\mathbb{P}(\underline{X}_n(s) \in H_\varepsilon^r) \geq 1 - \varepsilon \sum_{l=1}^{\infty} \frac{1}{2^l} = 1 - \varepsilon$$

for all $n \in \mathbb{N}_0$. Hence, by the definition of $\underline{X}_n(s)$, we find that

$$P_n(H_\varepsilon^r) \geq 1 - \varepsilon$$

for all $n \in \mathbb{N}_0$. This means that the family of probability measures $\{P_n : n \in \mathbb{N}_0\}$ is tight. Then, by Lemma 1.9, we have that the family $\{P_n : n \in \mathbb{N}_0\}$ is relatively compact. Therefore, there exists a subsequence $\{P_{n_k}\} \subset \{P_n\}$ such that $\{P_n\}$ converges weakly to some probability measure P on $(H^r(D), \mathfrak{B}(H^r(D)))$ as $k \to \infty$, so the relation

$$\underline{X}_{n_k} \xrightarrow[k\to\infty]{\mathfrak{D}} P \tag{1.5}$$

holds.

On $(\widehat{\Omega}, \mathfrak{B}(\widehat{\Omega}, \mathbb{P})$, define one more $H^r(D)$-valued random element $\underline{X}_T = \underline{X}_T(s, \underline{\alpha}; \underline{\mathfrak{a}})$ by

$$\underline{X}_T(s, \underline{\alpha}; \underline{\mathfrak{a}}) = \underline{\zeta}(s + i\theta T, \underline{\alpha}; \underline{\mathfrak{a}}).$$

Then, for every $\varepsilon > 0$, Lemma 1.6 implies that

$$\begin{aligned}
&\lim_{n\to\infty} \limsup_{T\to\infty} \mathbb{P}\left(\underline{\rho}(\underline{X}_T(s, \underline{\alpha}; \underline{\mathfrak{a}}), X_{T,n}(s, \underline{\alpha}; \underline{\mathfrak{a}})) \geq \varepsilon\right) \\
&= \lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T} \text{meas}\left\{\tau \in [0, T] : \underline{\rho}(\underline{\zeta}(s+i\tau, \underline{\alpha}; \underline{\mathfrak{a}}), \zeta_n(s+i\tau, \underline{\alpha}; \underline{\mathfrak{a}})) \geq \varepsilon\right\} \\
&\leq \lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T\varepsilon} \int_0^T \underline{\rho}(\underline{\zeta}(s+i\tau, \underline{\alpha}; \underline{\mathfrak{a}}), \zeta_n(s+i\tau, \underline{\alpha}; \underline{\mathfrak{a}}) \mathrm{d}\tau = 0.
\end{aligned}$$

This, and relations (1.1) and (1.5) together with Lemma 1.11 lead to

$$\underline{X}_T \xrightarrow[T\to\infty]{\mathfrak{D}} P, \tag{1.6}$$

and this is equivalent to the weak convergence of P_T to the measure P as $T \to \infty$. Moreover, relation (1.6) shows that the probability measure P does not depend on the subsequence $\{P_{n_k}\}$. Hence, taking into account the relative compactness of the family $\{P_n : n \in \mathbb{N}_0\}$, we have that every subsequence of that family converges weakly to P, thus

$$\underline{X}_n \xrightarrow[n\to\infty]{\mathfrak{D}} P. \tag{1.7}$$

It remains to show that $\widehat{P}_T$ also converges weakly to the same measure P as $T \to \infty$. For this, we define the $H^r(D)$-valued random elements

$$\underline{X}_{T,n}(s, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{a}}) = \underline{\zeta}_n(s + i\theta T, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{a}})$$

and

$$\underline{X}_T(s, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{a}}) = \underline{\zeta}(s + i\theta T, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{a}}).$$

Then, using (1.7) and Lemma 1.7, and repeating the above arguments for the random elements $\underline{X}_{T,n}(s, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{a}})$ and $\underline{X}_T(s, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{a}})$, we obtain the weak convergence of P_T to the measure P as $T \to \infty$.

For the proof of Theorem 1.2, we recall an equivalent of the weak convergence of probability measures in terms of continuity sets. Let P be a probability measure on $(S, \mathfrak{B}(S))$, $A \in \mathfrak{B}(S)$, and let ∂A denote the boundary of the set A. If $P(\partial A) = 0$, then the set A is called a continuity set of the measure P.

Lemma 1.12. *Let P and P_n, $n \in \mathbb{N}$, be probability measures on $(S, \mathfrak{B}(S))$. Then P_n, as $n \to \infty$, converges weakly to P if and only if, for every continuity set A of the measure P,*

$$\lim_{n\to\infty} P_n(A) = P(A).$$

Proof of the lemma is given in [8], Theorem 2.1.

We also need the classical Birkhoff-Khintchin ergodic theorem. Denote by $\mathbb{E}\xi$ the expectation of the random element ξ.

Lemma 1.13. *Suppose that $X(t, \omega)$ is an ergodic process, $\mathbb{E}|X(t, \omega)| < \infty$, with sample paths integrable over every finite interval in the Riemann sense. Then, for almost all ω,*

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T X(t, \omega)dt = \mathbb{E}X(0, \omega).$$

Proof of the lemma can be found, for example, in [6].

We state one more lemma from ergodicity theory.

For $\tau \in \mathbb{R}$, define

$$\underline{a}_\tau = \{((m + \alpha_1)^{-i\tau} : \ m \in \mathbb{N}_0), \dots, ((m + \alpha_r)^{-i\tau} : \ m \in \mathbb{N}_0)\},$$

and let $\{\Phi_\tau : \tau \in \mathbb{R}\}$ be the family of transformations on the torus Ω^r given by $\Phi_\tau(\underline{\omega}) = \underline{a}_\tau \underline{\omega}$, $\underline{\omega} \in \Omega^r$. Then $\{\Phi_\tau : \tau \in \mathbb{R}\}$ is a one-parameter group of measurable measure-preserving transformations on Ω^r. The ergodicity of $\{\Phi_\tau : \tau \in \mathbb{R}\}$ is defined in the same way as that of the group $\{\varphi_\tau : \tau \in \mathbb{R}\}$ used in the proof of Lemma 1.7.

Lemma 1.14. *The group $\{\Phi_\tau : \tau \in \mathbb{R}\}$ is ergodic.*

Proof of the lemma is given in [14], Lemma 3.

Proof of Theorem 1.2. In view of Lemma 1.8, it is sufficient to show that the limit measure P in that lemma coincides with $P_{\underline{\zeta}}$.

We fix a continuity set A of the measure P in Lemma 1.8. Then, by Lemmas 1.8 and 1.12, we have that

$$\lim_{T\to\infty} \frac{1}{T} \text{meas}\left\{\tau \in [0,T] : \underline{\zeta}(s+i\tau, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{a}}) \in A\right\} = P(A). \tag{1.8}$$

Let ξ be a random variable on the probability space $(\Omega^r, \mathfrak{B}(\Omega^r), m_H^r)$ given by

$$\xi = \xi(\underline{\omega}) = \begin{cases} 1 & \text{if } \underline{\zeta}(s, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{a}}) \in A, \\ 0 & \text{if } \underline{\zeta}(s, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{a}}) \notin A. \end{cases}$$

In view of Lemma 1.14, we have that the random process $\xi(\Phi_\tau(\underline{\omega}))$ is ergodic. Therefore, by Lemma 1.13, we obtain that, for almost all $\underline{\omega} \in \Omega^r$,

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \xi(\Phi_\tau(\underline{\omega}))\mathrm{d}\tau = \mathbb{E}\xi. \tag{1.9}$$

On the other hand, the definition of ξ shows that

$$\mathbb{E}\xi = \int_{\Omega^r} \xi \mathrm{d}m_H^r = m_H^r(\underline{\omega} \in \Omega^r : \underline{\zeta}(s, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{a}}) \in A),$$

that is

$$\mathbb{E}\xi = P_{\underline{\zeta}}(A). \tag{1.10}$$

Since, by the definitions of ξ and Φ_τ,

$$\frac{1}{T} \int_0^T \xi(\Phi_\tau(\underline{\omega}))\mathrm{d}\tau = \frac{1}{T} \text{meas}\left\{\tau \in [0,T] : \underline{\zeta}(s+i\tau, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{a}}) \in A\right\},$$

we see from relations (1.9) and (1.10) that, for almost all $\underline{\omega} \in \Omega^r$,

$$\lim_{T\to\infty} \frac{1}{T} \text{meas}\left\{\tau \in [0,T] : \underline{\zeta}(s+i\tau, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{a}}) \in A\right\} = P_{\underline{\zeta}}(A).$$

This, together with (1.8), shows that $P(A) = P_{\underline{\zeta}}(A)$ for all continuity sets A of the measure P. However, all continuity sets constitute a determining class [3]. Thus, the measures P and $P_{\underline{\zeta}}$ coincide for all $A \in \mathfrak{B}(H^r(D))$, and the theorem is proved.

1.3. Support of the limit measure

Denote by $S_{P_{\underline{\zeta}}}$ the support of the measure $P_{\underline{\zeta}}$. Since the space $H^r(D)$ is separable, $S_{P_{\underline{\zeta}}}$ is a minimal closed set of the space $H^r(D)$ such that $P_{\underline{\zeta}}(S_{P_{\underline{\zeta}}}) = 1$. The support $S_{P_{\underline{\zeta}}}$ consists of all points $\underline{g} \in H^r(D)$ such that $P_{\underline{\zeta}}(G) > 0$ for every open neighbourhood G of $\underline{g}$.

Theorem 1.15. *Suppose that the set $L(\alpha_1, \dots, \alpha_r)$ is linearly independent over $\mathbb{Q}$. Then the support of the measure $P_{\underline{\zeta}}$ is the whole of $H^r(D)$.*

Proof. Let, for $A_j \in H(D), \quad j = 1, \dots, r,$

$$A = A_1 \times \dots \times A_r.$$

Since the space $H^r(D)$ is separable, the σ-field $\mathfrak{B}(H^r(D))$ coincides with that generated by sets A [3]. Moreover, the Haar measure m^r_H is the product of the Haar measures $m_{1H}, \dots, m_{rH}$. Therefore,

$$\begin{aligned} P_{\underline{\zeta}}(A) = m^r_H(\underline{\omega} \in \Omega^r :\ & \underline{\zeta}(s, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{a}}) \in A) = \\ & m^r_H(\underline{\omega} \in \Omega^r :\ \underline{\zeta}(s, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{a}}) \in A_1 \times \dots \times A_r) = \\ & m^r_H(\underline{\omega} \in \Omega^r : \zeta(s, \alpha_1, \omega_1; \mathfrak{a}_1) \in A_1, \dots, \zeta(s, \alpha_r, \omega_r; \mathfrak{a}_r) \in A_r) = \\ & m_{1H}(\omega_1 \in \Omega_1 :\ \zeta(s, \alpha_1, \omega_1; \mathfrak{a}_1) \in A_1) \dots m_{rH}(\omega_r \in \Omega : \\ & \zeta(s, \alpha_r, \omega_r; \mathfrak{a}_r) \in A_r). \end{aligned} \tag{1.11}$$

Since the set $L(\alpha_1, \dots, \alpha_r)$ is linearly independent over $\mathbb{Q}$, so is each set $L(\alpha_1), \dots, L(\alpha_r)$. Therefore, we have from [13] that, for every $j = 1, \dots, r$, the support of

$$m_{jH}(\omega_j \in \Omega_j :\ \zeta(s, \alpha_j, \omega_j; \mathfrak{a}_j) \in A)$$

is the whole of $H(D)$. Thus, the latter remark and (1.11) prove the theorem.

1.4. Proof of Theorem 1.1

We start with the famous Mergelyan theorem on approximation of analytic functions by polynomials.

Lemma 1.16. *Let $K \subset \mathbb{C}$ be a compact subset with connected complement, and let $f(s)$ be a continuous function on K which is analytic in the interior of K. Then, for every $\varepsilon > 0$, there exists a polynomial $p(s)$ such that*

$$\sup_{s \in K} |f(s) - p(s)| < \varepsilon.$$

Proof of the lemma is given in [34], see also [44].

We also remind an equivalent of the weak convergence of probability measures in terms of open sets.

Lemma 1.17. *Let P and P_n, $n \in \mathbb{N}$, be probability measures on $(S, \mathfrak{B}(S))$. Then P_n, as $n \to \infty$, converges weakly to P if and only if, for every open set G of S,*

$$\lim_{n\to\infty} P_n(G) \geq P(G).$$

Proof of the lemma can be found in [3], Theorem 2.1.

Proof of Theorem 1.1. In witue of Lemma 1.16, there exist polynomials $p_1(s), \ldots, p_r(s)$ such that

$$\sup_{1\leq j\leq r} \sup_{s\in K_j} |f_j(s) - p_j(s)| < \frac{\varepsilon}{2}. \tag{1.12}$$

Let

$$G = \left\{ (g_1, \ldots, g_r) \in H^r(D) : \sup_{1\leq j\leq r} \sup_{s\in K_j} |g_j(s) - p_j(s)| < \frac{\varepsilon}{2} \right\}.$$

Clearly, G is an open set. Moreover, in view of Theorem 1.15, $(p_1(s), \ldots, p_r(s)) \in S_{P_{\underline{\zeta}}}$. Therefore, by properties of a support mentioned in the beginning of Section 1.3, the inequality $P_{\underline{\zeta}}(G) > 0$ holds. By Theorem 1.1 and Lemma 1.17, we have that

$$\liminf_{T\to\infty} P_T(G) \geq P_{\underline{\zeta}}(G).$$

Thus, we deduce from the definitions of the set G and P_T that

$$\liminf_{T\to\infty} \frac{1}{T} \text{meas} \left\{ \tau \in [0,T] : \sup_{1\leq j\leq r} \sup_{s\in K_j} |\zeta(s+i\tau, \alpha_j; \mathfrak{a}_j) - p_j(s)| < \frac{\varepsilon}{2} \right\} > 0. \tag{1.13}$$

However, inequalities (1.12) and

$$\sup_{1\leq j\leq r} \sup_{s\in K_j} |\zeta(s+i\tau, \alpha_j; \mathfrak{a}_j) - p_j(s)| < \frac{\varepsilon}{2}$$

imply

$$\sup_{1\leq j\leq r} \sup_{s\in K_j} |\zeta(s+i\tau, \alpha_j; \mathfrak{a}_j) - f_j(s)| < \varepsilon.$$

Therefore,

$$\begin{aligned} &\left\{ \tau \in [0,T] : \sup_{1\leq j\leq r} \sup_{s\in K_j} |\zeta(s+i\tau, \alpha_j; \mathfrak{a}_j) - f_j(s)| < \varepsilon \right\} \\ &\supseteq \left\{ \tau \in [0,T] : \sup_{1\leq j\leq r} \sup_{s\in K_j} |\zeta(s+i\tau, \alpha_j; \mathfrak{a}_j) - p_j(s)| < \frac{\varepsilon}{2} \right\}. \end{aligned}$$

This, together with inequality (1.3), yields that

$$\begin{aligned} &\liminf_{T\to\infty} \frac{1}{T} \text{meas} \left\{ \tau \in [0,T] : \sup_{1\leq j\leq r} \sup_{s\in K_j} |\zeta(s+i\tau, \alpha_j; \mathfrak{a}_j) - f_j(s)| < \varepsilon \right\} \\ &\geq \liminf_{T\to\infty} \frac{1}{T} \text{meas} \left\{ \tau \in [0,T] : \sup_{1\leq j\leq r} \sup_{s\in K_j} |\zeta(s+i\tau, \alpha_j; \mathfrak{a}_j) - p_j(s)| < \frac{\varepsilon}{2} \right\} > 0. \end{aligned}$$

The theorem is proved.

Chapter 2

Extended joint universality theorem for periodic Hurwitz zeta-functions

The aim of this chapter is an extension of Theorem 1.1 for a wider collection of periodic Hurwitz zeta-functions. Let l_j, $j = 1, \dots, r$, be positive integers, and, for $l = 1, \dots, l_j$, let $\mathfrak{a}_{jl} = \{a_{mjl} : m \in \mathbb{N}_0\}$ be a periodic sequence of complex numbers with minimal period $k_{jl} \in \mathbb{N}$. Suppose that, for $j = 1, \dots, r$, α_j is a fixed parameter, $0 < \alpha_j \leq 1$, and that $\zeta(s, \alpha_j; \mathfrak{a}_{jl})$ is the corresponding periodic Hurwitz zeta-function. In this chapter, we consider the joint universality for the functions

$$\zeta(s, \alpha_1; \mathfrak{a}_{11}), \dots, \zeta(s, \alpha_1; \mathfrak{a}_{1l_1}), \dots, \zeta(s, \alpha_r; \mathfrak{a}_{r1}), \dots, \zeta(s, \alpha_r; \mathfrak{a}_{rl_r}).$$

2.1. Statement of an extended joint universality theorem

Theorem 1.1 was obtained without any hypotheses on the coefficients of the functions $\zeta(s, \alpha_1; \mathfrak{a}_1, \dots,$ $\zeta(s, \alpha_r; \mathfrak{a}_r)$. However, in the case when a collection of periodic sequences corresponds each parameter α_j, we need a certain rank condition. Let k_j be the common multiple of the periods $k_{j1}, \dots, k_{jl_j}$, $j = 1, \dots, r$. Define

$$B_j = \begin{pmatrix} a_{1j1} & a_{1j2} & \dots & a_{1jl_j} \\ a_{2j1} & a_{2j2} & \dots & a_{2jl_j} \\ \dots & \dots & \dots & \dots \\ a_{k_j j1} & a_{k_j j2} & \dots & a_{k_j jl_j} \end{pmatrix}, \quad j = 1, \dots, r.$$

Then the main theorem of the chapter is of the form.

Theorem 2.1. *Suppose that the set $L(\alpha_1, \dots, \alpha_r)$ is linearly independent over $\mathbb{Q}$, and that rank$(B_j) = l_j$, $j = 1, \dots, r$. For every $j = 1, \dots, r$ and $l = 1, \dots, l_j$, let K_{jl} be a compact subset of the strip D with connected complement, and let $f_{jl}(s)$ be a continuous function on K_{jl} which is analytic in the interior of K_{jl}. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T}\text{meas}\left\{\tau \in [0, T] : \sup_{1\le j\le r}\sup_{1\le l\le l_j}\sup_{s\in K_{jl}} |\zeta(s + i\tau, \alpha_j; \mathfrak{a}_{jl}) - f_{jl}(s)| < \varepsilon\right\} > 0.$$

2.2. Extended joint limit theorem

The proof of Theorem 2.1, as that of Theorem 1.1, is based on a probability joint limit theorems for the functions $\zeta(s, \alpha_1; \mathfrak{a}_{11}), \dots, \zeta(s, \alpha_1; \mathfrak{a}_{1l_1}), \dots, \zeta(s, \alpha_r; \mathfrak{a}_{r1}), \dots, \zeta(s, \alpha_r; \mathfrak{a}_{rl_r})$. For brevity, let $\underline{\mathfrak{a}} = (\mathfrak{a}_{11}, \dots, \mathfrak{a}_{1l_1}, \dots, \mathfrak{a}_{r1}, \dots, \mathfrak{a}_{rl_r})$, and $\kappa = \sum_{j=1}^{r} l_j$, $H^\kappa(D) = \underbrace{H(D) \times \dots \times H(D)}_{\kappa}$. We preserve the notation used in Chapter 1. On the probability space $(\Omega^r, \mathfrak{B}(\Omega^r), m_H^r)$, define the $H^\kappa(D)$-valued random element $\underline{\zeta}_\kappa(s, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{a}})$ by

$$\underline{\zeta}_\kappa(s, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{a}}) = (\zeta(s, \alpha_1, \omega_1; \mathfrak{a}_{11}), \dots, \zeta(s, \alpha_r, \omega_r; \mathfrak{a}_{r1}), \dots, \zeta(s, \alpha_r, \omega_r; \mathfrak{a}_{rl_r})),$$

where

$$\zeta(s, \alpha_l, \omega_l; \mathfrak{a}_{jl}) = \sum_{m=0}^{\infty} \frac{a_{mjl}\omega_j(m)}{(m + \alpha_j)^s}, \quad j = 1, \dots, l_j.$$

Denote by $P_{\underline{\zeta},\kappa}$ the distribution of the random element $\underline{\zeta}(s, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{a}})$, i.e.,

$$P_{\underline{\zeta},\kappa}(A) = m_H^r(\underline{\omega} \in \Omega^r : \underline{\zeta}_\kappa(s, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{a}}) \in A), \quad A \in \mathfrak{B}(H^\kappa(D)).$$

In this section, we consider the weak convergence, as $T \to \infty$, for

$$P_{T,\kappa}(A) = \frac{1}{T}\text{meas}\left\{\tau \in [0, T] : \underline{\zeta}_\kappa(s + i\tau, \underline{\alpha}; \underline{\mathfrak{a}}) \in A\right\}, \quad A \in \mathfrak{B}(H^\kappa(D)),$$

where

$$\underline{\zeta}_\kappa(s, \underline{\alpha}; \underline{\mathfrak{a}}) = (\zeta(s, \alpha_1; \mathfrak{a}_{11}), \dots, \zeta(s, \alpha_1; \mathfrak{a}_{1l_1}), \dots, \zeta(s, \alpha_r; \mathfrak{a}_{r1}), \dots, \zeta(s, \alpha_r; \mathfrak{a}_{rl_r})).$$

Theorem 2.2. *Suppose that the set $L(\alpha_1, \dots, \alpha_r)$ is linearly independent over $\mathbb{Q}$. Then $P_{T,\kappa}$ converges weakly to the measure $P_{\underline{\zeta},\kappa}$ as $T \to \infty$.*

We see that the statement of Theorem 2.2 does not contain the hypothesis on the rank of the matrices B_j, therefore, the proof of Theorem 2.2 remains similar to that of Theorem 1.2. For this reason, we will present only the principal steps of the proof.

We define

$$\zeta_n(s,\alpha_j;\mathfrak{a}_{jl})=\sum_{m=0}^{\infty}\frac{a_{mjl}v_n(m,\alpha_j)}{(m+\alpha_j)^s},\quad j=1,\dots,r, l=1,\dots,l_j,$$

and

$$\zeta_n(s,\alpha_j,\omega_j;\mathfrak{a}_{jl})=\sum_{m=0}^{\infty}\frac{a_{mjl}\omega_j(m)v_n(m,\alpha_j)}{(m+\alpha_j)^s},\quad j=1,\dots,r, l=1,\dots,l_j.$$

Since the coefficients a_{mjl} are bounded, the latter series, as those for the functions $\zeta_n(s,\alpha_j;\mathfrak{a}_j)$ and $\zeta_n(s,\alpha_j,\omega_j;\mathfrak{a}_j)$ in Section 1.2, are absolutely convergent for $\sigma>\frac{1}{2}$. We start with limit theorems in the space $H^\kappa(D)$ for

$$\underline{\zeta}_{n,\kappa}(s,\underline{\alpha};\underline{\mathfrak{a}})=(\zeta_n(s,\alpha_1;\mathfrak{a}_{11}),\dots,\zeta_n(s,\alpha_1;\mathfrak{a}_{1l_1}),\dots,\zeta_n(s,\alpha_r;\mathfrak{a}_{r1}),\dots,\zeta_n(s,\alpha_r;\mathfrak{a}_{rl_r}))$$

and

$$\begin{aligned}\underline{\zeta}_{n,\kappa}(s,\underline{\alpha},\underline{\omega};\underline{\mathfrak{a}}) &= (\zeta_n(s,\alpha_1,\omega_1;\mathfrak{a}_{11}),\dots,\zeta_n(s,\alpha_1,\omega_1;\mathfrak{a}_{1l_1}),\dots,\\ &\quad \zeta_n(s,\alpha_r,\omega_r;\mathfrak{a}_{r1}),\dots,\zeta_n(s,\alpha_r,\omega_r;\mathfrak{a}_{rl_r})).\end{aligned}$$

For $A\in\mathfrak{B}(H^\kappa(D))$, define

$$P_{T,n,\kappa}(A)=\frac{1}{T}\text{meas}\Big\{\tau\in[0,T]:\ \underline{\zeta}_{n,\kappa}(s+i\tau,\underline{\alpha};\underline{\mathfrak{a}})\in A\Big\},$$

and, for any fixed $\underline{\omega}_0\in\Omega^r$,

$$Q_{T,n,\kappa}(A)=\frac{1}{T}\text{meas}\Big\{\tau\in[0,T]:\ \underline{\zeta}_{n,\kappa}(s+i\tau,\underline{\alpha},\underline{\omega}_0;\underline{\mathfrak{a}})\in A\Big\}.$$

Lemma 2.3. *Suppose that the set $L(\alpha_1,\dots,\alpha_r)$ is linearly independent over $\mathbb{Q}$. Then $P_{T,n,\kappa}$ and $Q_{T,n,\kappa}$ both converge weakly to the same probability measure $P_{n,\kappa}$ on $(H^\kappa(D),\mathfrak{B}(H^\kappa(D)))$ as $T\to\infty$.*

Proof. The lemma uses Lemmas 1.3 and 1.5, and is obtained in the same way as Lemma 1.4.

Let ρ be the same metric on $H(D)$ as in Section 1.2. For $\underline{f}=(f_{11},\dots,f_{1l_1},\dots,f_{r1},\dots,f_{rl_r})$, $\underline{g}=(g_{11},\dots,g_{1l_1},\dots,g_{r1},\dots,g_{rl_r})\in H^\kappa(D)$, define

$$\underline{\rho}_\kappa(\underline{f},\underline{g})=\max_{1\le j\le r}\max_{1\le j\le l_j}\rho(f_{jl},g_{jl}).$$

Then $\underline{\rho}_\kappa$ is a ,metric on $H^\kappa(D)$ which induces the topology of uniform convergence on compacta.

Two next lemmas give an approximation in the mean for $\underline{\zeta}_\kappa(s,\underline{\alpha};\underline{\mathfrak{a}})$ by $\underline{\zeta}_{n,\kappa}(s,\underline{\alpha};\underline{\mathfrak{a}})$ as well as for $\underline{\zeta}_\kappa(s,\underline{\alpha},\underline{\omega};\underline{\mathfrak{a}})$ by $\underline{\zeta}_{n,\kappa}(s,\underline{\alpha},\underline{\omega};\underline{\mathfrak{a}})$.

Lemma 2.4. *The equality*

$$\lim_{n\to\infty}\limsup_{T\to\infty}\frac{1}{T}\int_0^T\underline{\rho}_\kappa(\underline{\zeta}_\kappa(s+i\tau,\underline{\alpha};\underline{\mathfrak{a}}),\underline{\zeta}_{n,\kappa}(s+i\tau,\underline{\alpha};\underline{\mathfrak{a}}))\mathrm{d}\tau=0$$

holds.

Proof of the lemma is given in [24], Lemma 2 . Using the estimate [12]

$$\int_0^T |\zeta(\sigma+it,\alpha_j;\mathfrak{a}_{jl})|^2\mathrm{d}t = O(T), \quad \sigma > \frac{1}{2},$$

$j=1,\dots,r$, $l=1,\dots,l_j$, first it is proved that, for every compact subset $K\subset D$,

$$\lim_{n\to\infty}\limsup_{T\to\infty}\frac{1}{T}\int_0^T \sup_{s\in K}\left|\zeta(s+i\tau,\alpha_j;\mathfrak{a}_{jl})-\zeta_n(s+i\tau,\alpha_j;\mathfrak{a}_{jl})\right|\mathrm{d}\tau = 0,$$

$j=1,\dots,r$, $l=1,\dots,l_j$. From this and the definition of the metric $\underline{\rho}_\kappa$, the lemma follows.

Lemma 2.5. *Suppose that the set $L(\alpha_1,\dots,\alpha_r)$ is linearly independent over $\mathbb{Q}$. Then, for almost all $\underline{\omega}\in\Omega^r$, the equality*

$$\lim_{n\to\infty}\limsup_{T\to\infty}\frac{1}{T}\int_0^T \underline{\rho}_\kappa(\underline{\zeta}_\kappa(s+i\tau,\underline{\alpha},\underline{\omega};\underline{\mathfrak{a}}),\underline{\zeta}_{n,\kappa}(s+i\tau,\underline{\alpha},\underline{\omega};\underline{\mathfrak{a}}))\mathrm{d}\tau = 0$$

holds.

Proof of the lemma is given in [24], Lemma 5 . We note that the linear independence of the set $L(\alpha_1,\dots,\alpha_r)$ is not exhausted fully, the linear independence of the sets $L(\alpha_1),\dots,L(\alpha_r)$ is sufficient. The ergodicity of group $\{\varphi_\tau:\ \tau\in\mathbb{R}\}$ defined in the proof of Lemma 1.7 leads, for almost all $\omega_j\in\Omega_j$, to the estimate

$$\int_0^T |\zeta(\sigma+it,\alpha_j,\omega_j;\mathfrak{a}_{jl})|^2\mathrm{d}t = O(T), \quad \sigma > \frac{1}{2},$$

$j=1,\dots,r$, $l=1,\dots,l_j$. From this estimate, by a standard contour integration method it is derived that, for every compact subset $K\subset D$,

$$\lim_{n\to\infty}\limsup_{T\to\infty}\frac{1}{T}\int_0^T \sup_{s\in K}|\zeta(s+i\tau,\alpha_j,\omega_j;\mathfrak{a}_{jl})-\zeta_n(s+i\tau,\alpha_j,\omega_j;\mathfrak{a}_{jl})|\mathrm{d}\tau = 0$$

for almost all $\omega_j\in\Omega_j$, $j=1,\dots,r$, $l=1,\dots,l_j$. Combining the latter relation with the definition of the metric $\underline{\rho}_\kappa$ gives the assertion of the lemma.

Define one more probability measure

$$\widehat{P}_{T,\kappa}(A) = \frac{1}{T}\mathrm{meas}\left\{\tau\in[0,T]:\ \underline{\zeta}_\kappa(s+i\tau,\underline{\alpha},\underline{\omega};\underline{\mathfrak{a}})\in A\right\}, \quad A\in\mathfrak{B}(H^\kappa(D)).$$

Lemma 2.6. *Suppose that the set $L(\alpha_1,\dots,\alpha_r)$ is linearly independent over $\mathbb{Q}$. Then $P_{T,\kappa}$ and $\widehat{P}_{T,\kappa}$ both converge weakly to the same probability measure P_κ on $(H^\kappa(D),\mathfrak{B}(H^\kappa(D)))$ as $T\to\infty$.*

Proof. Let θ be the same random variable as in the proof of Lemma 1.8. On $(\widetilde{\Omega},\mathfrak{B}(\widetilde{\Omega},\mathbb{P})$, define the $H^\kappa(D)$-valued random element $\underline{X}_{T,n,\kappa}=\underline{X}_{T,n,\kappa}(s)=\underline{X}_{T,n,\kappa}(s,\underline{\alpha};\underline{\mathfrak{a}})=(X_{T,n,1,1}(s),\dots,X_{T,n,1,l_1}(s),\dots,X_{T,n,r,1}(s),\dots,X_{T,n,r,l_r}(s))$ by

$$\underline{X}_{T,n,\kappa}(s,\underline{\alpha};\underline{\mathfrak{a}}) = \underline{\zeta}_{n,\kappa}(s+i\theta T,\underline{\alpha};\underline{\mathfrak{a}}).$$

Then, in view of Lemma 2.3, we have that

$$\underline{X}_{T,n,\kappa}\xrightarrow[T\to\infty]{\mathfrak{D}}\underline{X}_{n,\kappa}, \tag{2.1}$$

where $\underline{X}_{n,\kappa} = \underline{X}_{n,\kappa}(s) = (X_{n,1,1}(s), \dots, X_{n,1,l_1}(s), \dots, X_{n,r,1}(s), \dots, X_{n,r,l_r}(s))$ is an $H^\kappa(D)$-valued random element with the distribution $P_{n,\kappa}$, where $P_{n,\kappa}$ is the limit measure in Lemma 2.3. We have to show that the family of probability measures $\{P_{n,\kappa} : n \in \mathbb{N}_0\}$ is tight.

For $j = 1, \dots, r$, $l = 1, \dots, l_j$, the series

$$\zeta_n(s, \alpha_j; \mathfrak{a}_{jl}) = \sum_{m=0}^{\infty} \frac{a_{mjl} v_n(m, \alpha_j)}{(m+\alpha_j)^s}$$

converges absolutely for $\sigma > \frac{1}{2}$. Therefore, this, for $\sigma > \frac{1}{2}$, implies

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T |\zeta_n(\sigma+it, \alpha_j; \mathfrak{a}_{jl})|^2 \mathrm{d}t = \sum_{m=0}^{\infty} \frac{|a_{mjl}|^2 v_n^2(m, \alpha_j)}{(m+\alpha_j)^{2\sigma}} \le \sum_{m=0}^{\infty} \frac{|a_{mjl}|^2}{(m+\alpha_j)^{2\sigma}} < \infty \tag{2.2}$$

for all $n \in \mathbb{N}_0$, and $j = 1, \dots, r$, $l = 1, \dots, l_j$. Let K_k be a compact subset from the definitions of the metric ρ (we use the notation K_k in place of K_l). Then, using similar arguments to the proof of Lemma 1.8, we deduce from (2.2) that, for all $n \in \mathbb{N}_0$ and $j = 1, \dots, r$, $l = 1, \dots, l_j$,

$$\limsup_{T\to\infty} \frac{1}{T} \int_0^T \sup_{s\in K_k} |\zeta_n(s+i\tau, \alpha_j; \mathfrak{a}_{jl})| \mathrm{d}\tau \le C_k \left(\sum_{m=0}^{\infty} \frac{|a_{mjl}|^2}{(m+\alpha_j)^{2\sigma_k}} \right)^{\frac{1}{2}} \tag{2.3}$$

with some $C_k > 0$ and $\sigma_k > \frac{1}{2}$.

Now we take an arbitrary $\varepsilon > 0$, and define

$$R_{jlk} = \left(\sum_{m=0}^{\infty} \frac{|a_{mjl}|^2}{(m+\alpha_j)^{2\sigma_k}} \right)^{\frac{1}{2}}.$$

Moreover, let $M_{jlk} = C_k R_{jlk} 2^{k+\kappa} \varepsilon^{-1}$. Then, in view of (2.3), we obtain that

$$\begin{aligned}
&\limsup_{T\to\infty} \mathbb{P}\left(\sup_{s\in K_k} |X_{T,n,j,l}(s)| > M_{jlk} \quad \text{for some} \quad (j,l) \right) \le \\
&\le \sum_{j=1}^{r} \sum_{l=1}^{l_j} \limsup_{T\to\infty} \mathbb{P}\left(\sup_{s\in K_k} |X_{T,n,j,l}(s)| > M_{jlk} \right) \le \\
&\le \sum_{j=1}^{r} \sum_{l=1}^{l_j} \frac{1}{M_{jlk}} \sup_{n\in\mathbb{N}_0} \limsup_{T\to\infty} \frac{1}{T} \int_0^T \sup_{s\in K_k} |\zeta_n(s+i\tau, \alpha_j; \mathfrak{a}_{jl})| \mathrm{d}\tau \\
&\le \sum_{j=1}^{r} \sum_{l=1}^{l_j} \frac{C_k R_{jlk}}{M_{jlk}} = \frac{\varepsilon}{2^{k+\kappa}} \sum_{j=1}^{r} \sum_{l=1}^{l_j} 1 < \frac{\varepsilon}{2^k}.
\end{aligned}$$

Combining this with (2.1), we find that, for all $n \in \mathbb{N}_0$,

$$\mathbb{P}\left(\sup_{s\in K_k} |X_{n,j,l}(s)| > M_{jlk} \quad \text{for some} \quad (j,l) \right) \le \frac{\varepsilon}{2^k}. \tag{2.4}$$

Define a set

$$\begin{aligned}
H_\varepsilon^\kappa \quad = \quad &\Big\{ (g_{11}, \dots, g_{1l_1}, \dots, g_{r1}, \dots, g_{rl_r}) \in H^\kappa(D) : \sup_{s\in K_k} |g_{jl}(s)| \le M_{jlk}, \\
&j = 1, \dots, r, l = 1, \dots, l_j, k \in \mathbb{N} \Big\}.
\end{aligned}$$

Then H^κ_ε is a compact subset of the space $H^\kappa(D)$, and, in virtue if (2.4),

$$\mathbb{P}\bigg(\underline{X}_{n,\kappa}(s) \in H^\kappa_\varepsilon\bigg) \geq 1 - \varepsilon \sum_{k=1}^{\infty} \frac{1}{2^k} = 1 - \varepsilon$$

for all $n \in \mathbb{N}_0$, or, by the definition of the random element $\underline{X}_{n,\kappa}$,

$$P_{n,\kappa}(H^\kappa_\varepsilon) \geq 1 - \varepsilon$$

for all $n \in \mathbb{N}_0$. Thus, we proved that the family of probability measures $\{P_{n,\kappa} : \ n \in \mathbb{N}_0\}$ is tight. Therefore, by Lemma 1.9, the family $\{P_{n,\kappa} : \ n \in \mathbb{N}_0\}$ is relatively compact. Thus, there exists a subsequence $\{P_{n_k,\kappa}\} \subset \{P_{n,\kappa}\}$ such that $P_{n_k,\kappa}$ converges weakly to a certain probability measure P_κ on $(H^\kappa(D), \mathfrak{B}(H^\kappa(D)))$ as $k \to \infty$. This is equivalent to the relation

$$\underline{X}_{n_k,\kappa} \xrightarrow[k\to\infty]{\mathfrak{D}} P_\kappa. \tag{2.5}$$

Now let

$$\underline{X}_{T,\kappa} = \underline{X}_{T,\kappa}(s, \underline{\alpha}; \underline{\mathfrak{a}}) = \underline{\zeta}_\kappa(s + i\theta T, \underline{\alpha}; \underline{\mathfrak{a}}).$$

Then Lemma 2.4, for $\varepsilon > 0$, implies

$$\begin{aligned}
&\lim_{n\to\infty} \limsup_{T\to\infty} \mathbb{P}(\underline{\rho}_\kappa(\underline{X}_{T,\kappa}(s, \underline{\alpha}; \underline{\mathfrak{a}}), \underline{X}_{T,n,\kappa}(s, \underline{\alpha}; \underline{\mathfrak{a}})) \geq \varepsilon) \\
&= \lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T} \mathrm{meas}\{\tau \in [0, T] : \underline{\rho}_\kappa(\underline{\zeta}_\kappa(s + i\tau, \underline{\alpha}; \underline{\mathfrak{a}}), \underline{\zeta}_{n,\kappa}(s + i\tau, \underline{\alpha}; \underline{\mathfrak{a}})) \geq \varepsilon\} \\
&\leq \lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T\varepsilon} \int_0^T \underline{\rho}_\kappa(\underline{\zeta}_\kappa(s + i\tau, \underline{\alpha}; \underline{\mathfrak{a}}), \underline{\zeta}_{n,\kappa}(s + i\tau, \underline{\alpha}; \underline{\mathfrak{a}}))\mathrm{d}\tau = 0.
\end{aligned}$$

This, (2.1) and (2.5) show that Lemma 1.11 can be applied, and we obtain that

$$\underline{X}_{T,\kappa} \xrightarrow[T\to\infty]{\mathfrak{D}} P_\kappa. \tag{2.6}$$

Thus, we have that $P_{T,\kappa}$ converges weakly to the measure P_κ as $T \to \infty$. Moreover, (2.6), together with relative compactness of the family $\{P_{n,\kappa} : \ n \in \mathbb{N}_0\}$, shows that

$$\underline{X}_{n,\kappa} \xrightarrow[n\to\infty]{\mathfrak{D}} P_\kappa. \tag{2.7}$$

Now we consider the weak convergence of the measure $\widehat{P}_{T,\kappa}$. Define the $H^\kappa(D)$-valued random elements

$$\underline{X}_{T,n,\kappa}(s, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{a}}) = \underline{\zeta}_{n,\kappa}(s + i\theta T, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{a}})$$

and

$$\widehat{\underline{X}}_{T,\kappa}(s, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{a}}) = \underline{\zeta}_\kappa(s + i\theta T, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{a}}).$$

Then, dealing with the latter random elements and using (2.7) and Lemma 2.5, we obtain, similarly to the case of the measure $P_{T,\kappa}$, that $\widehat{P}_{T,\kappa}$ also converges weakly to P_κ as $T \to \infty$. The lemma is proved.

Proof of Theorem 2.2. We use the same arguments as in the proof of Theorem 1.2. Let A be an arbitrary continuity set of the limit measure P_κ in Lemma 2.6. Then the weak convergence of the measure $\widehat{P}_{T,\kappa}$ to P_κ as $T \to \infty$ together with Lemma 1.12 yields the equality

$$\lim_{T\to\infty} \frac{1}{T}\text{meas}\Big\{\tau \in [0,T]:\ \underline{\zeta}_\kappa(s+i\tau, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{a}}) \in A\Big\} = P_\kappa(A). \tag{2.8}$$

On the probability space $(\Omega^r, \mathfrak{B}(\Omega^r), m_H^r)$, define the random variable ξ_κ by

$$\xi_\kappa = \xi_\kappa(\underline{\omega}) = \begin{cases} 1 & \text{if } \ \underline{\zeta}_\kappa(s, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{a}}) \in A, \\ 0 & \text{if } \ \underline{\zeta}_\kappa(s, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{a}}) \notin A. \end{cases}$$

Let $\{\Phi_\tau:\ \tau \in \mathbb{R}\}$ be the same ergodic group as in the proof of Theorem 1.2. Then we have that the random process $\xi_\kappa(\Phi_\tau(\omega))$ is ergodic, and Lemma 1.13, for almost all $\underline{\omega} \in \Omega^r$, implies the equality

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \xi_\kappa(\Phi_\tau(\omega))\mathrm{d}\tau = \mathbb{E}\xi_\kappa. \tag{2.9}$$

However, by the definition of ξ_κ,

$$\mathbb{E}\xi_\kappa = \int_{\Omega^r} \xi_\kappa \mathrm{d}m_H^r = m_H^r(\underline{\omega} \in \Omega^r:\ \zeta_\kappa(s, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{a}}) \in A) = P_{\underline{\zeta}_\kappa}(A). \tag{2.10}$$

On the other hand,

$$\frac{1}{T}\int_0^T \xi_\kappa(\Phi_\tau(\underline{\omega}))\mathrm{d}\tau = \frac{1}{T}\text{meas}\Big\{\tau \in [0,T]:\ \underline{\zeta}_\kappa(s+i\tau, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{a}}) \in A\Big\}.$$

Therefore, in view of (2.9) and (2.10),

$$\lim_{T\to\infty} \frac{1}{T}\text{meas}\Big\{\tau \in [0,T]:\ \underline{\zeta}_\kappa(s+i\tau, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{a}}) \in A\Big\} = P_{\underline{\zeta}_\kappa}(A),$$

and we have by (2.8) that $P_\kappa(A) = P_{\underline{\zeta}_\kappa}(A)$ for all continuity sets A of the measure P_κ. Hence, $P_\kappa = P_{\underline{\zeta}_\kappa}$. The theorem is proved.

2.3. Support of the limit measure

In this section, we will prove the following theorem.

Theorem 2.7. *Suppose that the set $L(\alpha_1, \ldots, \alpha_r)$ is linearly independent over $\mathbb{Q}$. Then the support of the measure $P_{\underline{\zeta}_\kappa}$ is the whole of $H^\kappa(D)$.*

For $j = 1, \ldots, r$, define

$$P_{j\underline{\zeta}}(A) = m_{jH}(\omega_j \in \Omega_j:\ (\zeta(s, \alpha_j, \omega_j; \underline{\mathfrak{a}}_{j1}), \ldots, \zeta(s, \alpha_j, \omega_j; \underline{\mathfrak{a}}_{jl_j}) \in A), \quad A \in \mathfrak{B}(H^{l_j}(D)).$$

Lemma 2.8. *Suppose that rank$B_j = l_j$. Then the support of the measure $P_{j\zeta}$ is the whole of $H^{l_j}(D)$, $j = 1, \ldots, r$.*

For the proof of Lemma 2.8, we need some auxiliary results, and we state them as separate lemmas.

Lemma 2.9. *Let the sequence* $\{\underline{g}_m = (g_{m1}, \ldots, g_{mh}) \in H^n(D) : m \in \mathbb{N}_0\}$ *satisfy the hypotheses:*

1^0 *Suppose that* $\mu_1, \ldots, \mu_h$ *be complex-valued measures on* $(\mathbb{C}, \mathfrak{B}(\mathbb{C}))$ *with compact supports contained in* D *such that*

$$\sum_{m=0}^{\infty} \left| \sum_{j=1}^{n} \int_{\mathbb{C}} g_{mj}(s) \mathrm{d}\mu_j(s) \right| < \infty.$$

Then

$$\int_{\mathbb{C}} s^k \mathrm{d}\mu_j(s) = 0, \quad k \in \mathbb{N}_0, \quad j = 1, \ldots, n;$$

2^0 *The series*

$$\sum_{m=0}^{\infty} \underline{g}_m(s)$$

is convergent in $H^n(D)$*;*

3^0 *For every compact subset* $K \subset D$*,*

$$\sum_{m=0}^{\infty} \sum_{j=1}^{n} \sup_{s \in K} |g_{mj}(s)| < \infty.$$

Then the set of all convergent series with $a(m) \in \gamma$, $m \in \mathbb{N}_0$

$$\sum_{m=0}^{\infty} a(m) \underline{g}_m(s)$$

is dense in $H^n(D)$.

The lemma is a particular case of Lemma 6 from [30].

Lemma 2.10. *Suppose that* μ *is a complex-valued measure on* $(\mathbb{C}, \mathfrak{B}(\mathbb{C}))$ *with compact support contained in the half-plane* $\{s \in \mathbb{C} : \ \sigma > \sigma_0\}$*,*

$$g(z) = \int_{\mathbb{C}} e^{sr} \mathrm{d}\mu(s), \quad z \in \mathbb{C},$$

and $g(z) \not\equiv 0$*. Then*

$$\limsup_{r \to \infty} \frac{\log |g(r)|}{r} > \sigma_0.$$

Proof of the lemma is given in [19], Lemma 6. 4. 10.

Now let $A \subset \mathbb{N}$ be a set having a positive density, i.e.,

$$\lim_{x \to \infty} \frac{1}{x} \sharp \{m \leq x : \ m \in A\} > 0.$$

Let $0 < \theta \leq \pi$. We remind that a function $g(s)$ analytic in the closed angular region $|\arg s| \leq \theta_0$ is said to be of exponential type if

$$\limsup_{r \to \infty} \frac{\log |g(re^{i\theta})|}{r} < \infty$$

uniformly in θ, $|\theta| \le \theta_0$.

Lemma 2.11. *Let $g(s)$ be a function of exponential type such that*

$$\limsup_{r\to\infty} \frac{\log|g(r)|}{r} > -1,$$

and the set $A \subset \mathbb{N}$ has a positive density. Then

$$\sum_{m\in A} |g(\log m)| = +\infty.$$

The lemma is proved in [28], Lemma 5.

Let

$$\underline{\zeta}_j(s, \alpha_j, \omega_j; \mathfrak{a}_j) = (\zeta(s, \alpha_j, \omega_j; \mathfrak{a}_j), \dots, \zeta(s, \alpha_j, \omega_j; \mathfrak{a}_{jl_j})),$$

where $\underline{\mathfrak{a}}_j = (\mathfrak{a}_{j1}, \dots, \mathfrak{a}_{jl_j})$. For $s \in D$ and $a(m) \in \gamma$, consider the series

$$\sum_{m=0}^{\infty} a(m)\underline{g}_{mj}(s), \tag{2.11}$$

where

$$\underline{g}_{mj}(s) = (g_{mj1}(s), \dots, g_{mjl_j}(s)) = \left(\frac{a_{mj1}}{(m+\alpha_j)^s}, \dots, \frac{a_{mjl_j}}{(m+\alpha_j)^s}\right).$$

Lemma 2.12. *The set of all convergent series (2.11) is dense in $H^{l_j}(D)$.*

Proof. Since $\zeta(s, \alpha_j, \omega_j; \mathfrak{a}_{jl})$ is an $H(D)$-valued random element, the series

$$\sum_{m=0}^{\infty} \frac{a_{mjl}\omega_j(m)}{(m+\alpha_j)^s}$$

converges uniformly on compact subsets of the strip D for almost all $\omega_j \in \Omega_j$. Therefore, there exists a sequence $\{b_m : b_m \in \gamma, m \in \mathbb{N}_0\}$ such that the series

$$\sum_{m=0}^{\infty} b_m \underline{g}_{mj}(s)$$

converges in $H^{l_j}(D)$. Since $a(m)b_m \in \gamma$, $m \in \mathbb{N}_0$, for the proof of the lemma it suffices to show that the set of all convergent series

$$\sum_{m=0}^{\infty} a(m)b_m\underline{g}_{mj}(s) \tag{2.12}$$

with $a(m) \in \gamma$ is dense in $H^{l_j}(D)$. For this aim, we will check the hypotheses of Lemma 2.9 for the sequence $\{b_m\underline{g}_{mj}(s) :\ m \in \mathbb{N}_0\}$. Hypothesis 2^0 of Lemma 2.9 is satisfied by the choice of the sequence

$\{b_m:\ m \in \mathbb{N}_0\}$. Let $K \subset D$ be an arbitrary compact subset. Since, for $s \in D$, the inequality $\sigma > \frac{1}{2}$ is true, we have that

$$\sum_{m=0}^{\infty} \sum_{l=1}^{l_j} \sup_{s \in K} |g_{mjl}(s) b_m| < \infty. \tag{2.13}$$

Thus, it remains to check hypothesis 1^0 of Lemma 2.9.

Let $\mu_1, \dots, \mu_{l_j}$ be complex-valued measures on $(\mathbb{C}, \mathfrak{B}(\mathbb{C}))$ with compact supports contained in D such that

$$\sum_{m=0}^{\infty} \left| \sum_{l=1}^{l_j} \int_{\mathbb{C}} g_{mjl}(s) b_m \mathrm{d}\mu_l(s) \right| < \infty. \tag{2.14}$$

Since k_j is the least common multiple of the periods $k_{j1}, \dots, k_{jl_j}$, it is the period of all sequences $\mathfrak{a}_{j1}, \dots, \mathfrak{a}_{jl_j}$. Therefore, in view of the periodicity of the coefficients a_{mjl} and the definition of $g_{mjl}(s)$, (2.14) shows that, for $k = 1, \dots, k_j$,

$$\sum_{\substack{m=0 \\ m \equiv k (\bmod k_j)}}^{\infty} \left| \sum_{l=1}^{l_j} a_{kjl} \int_{\mathbb{C}} \frac{\mathrm{d}\mu_l(s)}{(m+\alpha_j)^s} \right| < \infty. \tag{2.15}$$

We put

$$\nu_k(s) = \sum_{l=1}^{l_j} a_{kjl} \mu_l(s).$$

Then $\nu_k(s)$ also is a complex-valued measure on $(\mathbb{C}, \mathfrak{B}(\mathbb{C}))$ with compact support contained in D, $k = 1, \dots, k_j$. This together with (2.15) implies, for $k = 1, \dots, k_j$, that

$$\sum_{\substack{m=0 \\ m \equiv k (\bmod k_j)}}^{\infty} \left| \int_{\mathbb{C}} \frac{\mathrm{d}\nu_k(s)}{(m+\alpha_j)^s} \right| < \infty. \tag{2.16}$$

Now let

$$A_k = \{m \in \mathbb{N}:\ m \equiv k (\mathrm{mod}\ k_j)\}$$

and

$$\rho_k(z) = \int_{\mathbb{C}} e^{-sz} \mathrm{d}\nu_k(s), \quad z \in \mathbb{C}, \quad k = 1, \dots, k_j.$$

Then, obviously, the set A_k has a positive density, moreover, $\rho_k(z)$ is an entire function of exponential type, $k = 1, \dots, k_j$. Therefore, by Lemma 2.10, either $\rho_k(z) \equiv 0$, or

$$\limsup_{x \to \infty} \frac{\log |\rho_k(x)|}{x} > -1, \quad k = 1, \dots, k_j. \tag{2.17}$$

Here we have used the definition of the strip D, and the sign minus in the definition of $\rho_k(z)$. If inequality (2.17) takes place, then lemma 2.11 gives

$$\sum_{m \in A_k} |\rho_k(\log m)| = +\infty, \quad k = 1, \dots, k_j. \tag{2.18}$$

It is well known that, for all $s \in \mathbb{C}$,

$$e^s = 1 + O(|s|e^{|s|}).$$

Hence, for $m \geq 2$,

$$(m+\alpha_j)^{-s} = m^{-s}\left(1+\frac{\alpha_j}{m}\right)^{-s} = m^{-s}\exp\{-s\log\left(1+\frac{\alpha_j}{m}\right)\} =$$
$$= m^{-s}\exp\{\frac{O(|s|)}{m}\} = m^{-s}\left(1+\frac{O(|s|)}{m}e^{O(|s|)}\right) = m^{-s} + m^{-1-\sigma}O\left(|s|e^{O(|s|)}\right).$$

Since the compact support of the measures $\nu_1, \ldots, \nu_{k_j}$ are contained in D, this and (2.16) show that

$$\sum_{m \in A_k} |\rho_k(\log m)| < \infty, \quad k = 1, \ldots, k_j,$$

and this contradicts (2.18). Therefore, $\rho_k(z) \equiv 0$ for $k = 1, \ldots, k_j$, and, by the definitions of $\rho_k(z)$ and ν_k, we obtain the system of equations

$$\sum_{l=1}^{l_j} a_{k_j l} \int_{\mathbb{C}} e^{-sz} \mathrm{d}\mu_l(s) \equiv 0, \quad k = 1, \ldots, k_j.$$

Since $\operatorname{rank}(B_j) = l_j$, the latter system has only the solution

$$\int_{\mathbb{C}} e^{-sz} \mathrm{d}\mu_l(s) \equiv 0, \quad l = 1, \ldots, l_j.$$

Hence, by differentiation, are find that

$$\int_{\mathbb{C}} s^n \mathrm{d}\mu_l(s) = 0$$

for $n \in \mathbb{N}_0$ and $l = 1, \ldots, l_j$. Thus, hypothesis 1^0 of Lemma 2.9 is also satisfied by the sequence $\{b_m \underline{g}_{mj}(s) : m \in \mathbb{N}_0\}$. Therefore, by Lemma 2.9, the set of all convergent series (2.12) with $a(m) \in \gamma$ is dense in $H^{l_j}(D)$. The lemma is proved.

For the proof of Lemma 2.8, we will apply one more lemma. Denote by S_X the support of the random element X.

Lemma 2.13. *Let $\{X_m : m \in \mathbb{N}_0\}$ be sequence of independent $H^n(D)$-valued random elements such that the series*

$$\sum_{m=0}^{\infty} X_m$$

converges almost surely. then the support of the sum of this series is the closure of the set of $\underline{g} \in H^n(D)$ which can be written as a convergent series

$$\underline{g} = \sum_{m=0}^{\infty} \underline{g}_m, \quad \underline{g}_m \in S_{X_m}.$$

Proof of the lemma is given in [30], Lemma 5.

Proof of Lemma 2.8. By the definition of Ω_j, we have that $\{\omega_j(m): \ m \in \mathbb{N}_0\}$ is a sequence independent complex-valued random variables defined on the probability space $(\Omega_j, \mathfrak{B}(\Omega_j), m_{jH})$, $j = 1, \ldots, r$. The support of each random variable $\omega_j(m)$ is the unit circle γ. From this we have that

$$\left\{ \left(\frac{a_{mj1}\omega_j(m)}{(m+\alpha_j)^s}, \ldots, \frac{a_{mjl_j}\omega_j(m)}{(m+\alpha_j)^s} \right) : \ m \in \mathbb{N}_0 \right\}$$

is a sequences of independent $H^{l_j}(D)$-valued random elements on the probability space $(\Omega_j, \mathfrak{B}(\Omega_j), m_{jH})$, and the support of each element

$$\left(\frac{a_{mj1}\omega_j(m)}{(m+\alpha_j)^s}, \ldots, \frac{a_{mjl_j}\omega_j(m)}{(m+\alpha_j)^s} \right)$$

is the set

$$\left\{ \underline{g} \in H^{l_j}(D): \ \underline{g}(s) = \left(\frac{a_{mj1}a}{(m+\alpha_j)^s}, \ldots, \frac{a_{mjl_j}a}{(m+\alpha_j)^s} \right), a \in \gamma \right\},$$

$m \in \mathbb{N}_0$, $j = 1, \ldots, r$. Therefore, by Lemma 2.13, the support of the random element $\underline{\zeta}_j(s, \alpha_j, \omega_j; \underline{\mathfrak{a}}_j)$ is the closure of the set of all convergent series (2.11), $j = 1, \ldots, r$. This and Lemma 2.12 prove the lemma because the support of the element $\underline{\zeta}_j(s, \alpha_j, \omega_j; \underline{\mathfrak{a}}_j)$ coincides with the support of the measure $P_{j\zeta}$, $j = 1, \ldots, r$.

Proof of Theorem 2.7. For $A_j \in \mathfrak{B}(H^{l_j}(D))$, $j = 1, \ldots, r$, let

$$A = A_1 \times \cdots \times A_r. \tag{2.19}$$

Since the space $H^\kappa(D)$ is separable, we have [3] that the σ-field $\mathfrak{B}(H^\kappa(D))$ coincides with

$$\mathfrak{B}(H^{l_1}(D)) \times \cdots \times \mathfrak{B}(H^{l_r}(D)),$$

that is, it coincides with a σ-field generated by sets (2.19). We also remind that the measure m_H^r is the product of the measures m_{jH} on $(\Omega_j, \mathfrak{B}(\Omega_j))$, $j = 1, \ldots, r$. Therefore, we have

$$\begin{aligned} P_{\underline{\zeta}_\kappa}(A) \ &= \ m_H^r(\underline{\omega} \in \Omega^r: \ \underline{\zeta}_\kappa(s, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{a}}) \in A) = \\ & \quad\ m_{1H}(\omega_1 \in \Omega_1: \ \underline{\zeta}_1(s, \alpha_1, \omega_1; \underline{\mathfrak{a}}_1) \in A_1 \times \cdots \times \\ & \quad\ m_{rH}(\omega_r \in \Omega_r: \ \underline{\zeta}_r(s, \alpha_r, \omega_r; \underline{\mathfrak{a}}_r) \in A_r). \end{aligned} \tag{2.20}$$

By Lemma 2.8, the support of the measure

$$m_{jH}(\omega_j \in \Omega_j: \ \underline{\zeta}_j(s, \alpha_j, \omega_j; \underline{\mathfrak{a}}_j) \in A_j)$$

is the whole of $H^{l_j}(D)$, $j = 1, \ldots, r$. Therefore, the theorem follows from (2.20).

2.4. Proof of Theorem 2.1

The proof of Theorem 2.1 is similar to that of Theorem 1.1.

Proof of Theorem 2.1. By Lemma 1.16, there exist polynomials $p_{jl}(s)$, $j = 1, \dots, r$, $l = 1, \dots, l_j$, such that

$$\sup_{1 \le j \le r} \sup_{1 \le l \le l_j} \sup_{s \in K_{jl}} |f_{jl}(s) - p_{jl}(s)| < \frac{\varepsilon}{2}. \tag{2.21}$$

Define

$$G = \left\{ (g_{11}, \dots, g_{1l_1}, \dots, g_{r1}, \dots, g_{rl_r}) \in H^{\kappa}(D) : \sup_{1 \le j \le r} \sup_{1 \le l \le l_j} \sup_{s \in K_{jl}} |g_{jl}(s) - p_{jl}(s)| < \frac{\varepsilon}{2} \right\}.$$

The set G is open in the space $H^{\kappa}(D)$, and, by Theorem 2.7, $(p_{11}, \dots, p_{1l_1}, \dots, p_{r1}, \dots, p_{rl_r})$ is an element of the support of the measure $P_{\underline{\zeta}\kappa}$. Therefore, Theorem 2.2 and Lemma 1.17 show that

$$\begin{aligned} \liminf_{T \to \infty} \frac{1}{T} \operatorname{meas}\left\{ \tau \in [0, T] : \sup_{1 \le j \le r} \sup_{1 \le l \le l_j} \sup_{s \in K_{jl}} |\zeta(s + i\tau, \alpha_j; \mathfrak{a}_{jl}) - p_{jl}(s)| < \frac{\varepsilon}{2} \right\} \\ \ge P_{\underline{\zeta}\kappa}(G) > 0. \end{aligned}$$

This and inequality (2.21) complete the proof of Theorem 2.1.

Chapter 3

Mixed joint universality for periodic Hurwitz zeta-functions and the Riemana zeta-function.

We preserve the notation of Chapter 3, and consider the joint universality for the periodic Hurwitz zeta-functions $\zeta(s,\alpha_1;\mathfrak{a}_{11}),\dots,\zeta(s,\alpha_1;\mathfrak{a}_{1l_1}),\dots,\zeta(s,\alpha_r;\mathfrak{a}_{r1}),\dots,\zeta(s,\alpha_r;\mathfrak{a}_{rl_r})$ and the Riemann zeta-function $\zeta(s)$ which is defined, for $\sigma>1$, by the series

$$\zeta(s)=\sum_{m=1}^{\infty}\frac{1}{m^s},$$

and can be analytically continued to the whole complex plane, except for a simple pole at the point $s=1$ with residue 1. The Euler product over primes

$$\zeta(s)=\prod_p\left(1-\frac{1}{p^s}\right)^{-1},\quad \sigma>1, \tag{3.1}$$

is a very important object in the theory of the function $\zeta(s)$.

3.1. Statement of a mixed joint universality theorem

The periodic Hurwitz zeta-functions $\zeta(s,\alpha_j;\mathfrak{a}_{jl})$, $j=1,\dots,r$, $l=1,\dots,l_j$, with transcendental parameter α_j have not the Euler product over primes while the function $\zeta(s)$ can be defined by equality (3.1). Thus, the collection $\zeta(s)$, $\zeta(s,\alpha_1;\mathfrak{a}_{11}),\dots,\zeta(s,\alpha_1;\mathfrak{a}_{1l_1}),\dots,\zeta(s,\alpha_r;\mathfrak{a}_{r1}),\dots,\zeta(s,\alpha_r;\mathfrak{a}_{rl_r})$ consists of zeta-functions of different types, and this is reflected in their joint universality - in so called a mixed joint universality theorem. We remind that the numbers $\alpha_1,\dots,\alpha_r$ are algebraically independent over $\mathbb{Q}$ if there is no polynomials $p(x_1,\dots,x_r)\not\equiv 0$ with rational coefficients such that

$$p(\alpha_1,\dots,\alpha_r)=0.$$

Theorem 3.1. *Suppose that the numbers $\alpha_1, \dots, \alpha_r$ are algebraically independent over $\mathbb{Q}$, and that* $\operatorname{rank}(B_j) = l_j$, *$j = 1, \dots, r$. For every $j = 1, \dots, r$ and $l = 1, \dots, l_j$, let K_{jl} be a compact subset of the strip D with connected complement, and let $f_{jl}(s)$ be a continuous function on K_{jl} which is analytic in the interior of K_{jl}. Moreover, let $K \subset D$ be a compact subset with connected complement, and let $f(s)$ be a continuous non-vanishing function on K which is analytic in the interior of K. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T}\mathrm{meas}\Bigg\{\tau \in [0,T] :\ \sup_{s\in K} |\zeta(s+i\tau) - f(s)| < \varepsilon,$$
$$\sup_{1\le j\le r}\ \sup_{1\le l\le l_j}\ \sup_{s\in K_{jl}} |\zeta(s+i\tau, \alpha_j; \mathfrak{a}_{jl}) - f_{jl}(s)| < \varepsilon\Bigg\} > 0.$$

3.2. Joint limit theorem for periodic Hurwitz zeta-functions and the Riemann zeta-function

We start the proof of Theorem 3.1 with a joint limit theorem in the space of analytic functions for the functions $\zeta(s)$, $\zeta(s, \alpha_1; \mathfrak{a}_{11}), \dots, \zeta(s, \alpha_1; \mathfrak{a}_{1l_1}), \dots, \zeta(s, \alpha_r; \mathfrak{a}_{r1}), \dots, \zeta(s, \alpha_r;$
$\mathfrak{a}_{rl_r})$. In this chapter, we use the notation

$$\kappa = \sum_{j=1}^{r} l_j + 1,$$

and $H^\kappa(D) = \underbrace{H(D) \times \cdots \times H(D)}_{\kappa}$. Moreover, we introduce a torus

$$\widehat{\Omega} = \prod_p \gamma_p,$$

where $\gamma_p = \{s \in \mathbb{C} :\ |s| = 1\}$ for all primes p. By the Tikhonov theorem, with the product topology and pointwise multiplication, the torus Ω is a compact topological Abelian group. Therefore, on $(\widehat{\Omega}, \mathfrak{B}(\widehat{\Omega}))$, the probability Haar measure $\widehat{m}_H$ can be defined, and we have the probability space $(\widehat{\Omega}, \mathfrak{B}(\widehat{\Omega}), \widehat{m}_H)$. Denote by $\widehat{\omega}(p)$ the projection of $\widehat{\omega} \in \widehat{\Omega}$ to γ_p. We also use the probability space $(\Omega, \mathfrak{B}(\Omega), m_H)$ defined in Section 1.2.

Now let

$$\underline{\Omega} = \widehat{\Omega} \times \Omega_1 \times \cdots \times \Omega_r,$$

where $\Omega_j = \Omega$ for $j = 1, \dots, r$. Then, by the Tikhonov theorem again, $\underline{\Omega}$ is a compact topological Abelian group, and we obtain a new probability space $(\underline{\Omega}, \mathfrak{B}(\underline{\Omega}), \underline{m}_H)$, where $\underline{m}_H$ is the probability Haar measure on $(\underline{\Omega}, \mathfrak{B}(\underline{\Omega}))$. We preserve the notation of previous sections for $\underline{\alpha}$ and $\underline{\mathfrak{a}}$, and denote by $\underline{\omega} = (\widehat{\omega}, \omega_1, \ldots, \omega_r)$ the elements of $\widehat{\Omega}$. On the probability space $(\underline{\Omega}, \mathfrak{B}(\underline{\Omega}), \underline{m}_H)$, define the $H^\kappa(D)$-valued random element $\underline{\zeta}_\kappa(s, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{a}})$ by the formula

$$\underline{\zeta}_\kappa(s, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{a}}) = \big(\zeta(s, \widehat{\omega}), \zeta(s, \alpha_1, \omega_1; \mathfrak{a}_{11}), \ldots, \zeta(s, \alpha_1, \omega_1; \mathfrak{a}_{1l_1}), \ldots,$$
$$\zeta(s, \alpha_r, \omega_r; \mathfrak{a}_{r1}), \ldots, \zeta(s, \alpha_r, \omega_r; \mathfrak{a}_{rl_r})\big),$$

where

$$\zeta(s, \widehat{\omega}) = \prod_p \left(1 - \frac{\widehat{\omega}(p)}{p^s}\right)^{-1},$$

and the $H(D)$-valued random elements $\zeta(s, \alpha_j, \omega_j; \mathfrak{a}_{jl})$ are the same as in Chapter 2, $j = 1, \ldots, r$, $l = 1, \ldots, l_j$. Denote by $P_{\underline{\zeta}_\kappa}$ the distribution of the random element $\underline{\zeta}_\kappa$. Moreover, let

$$\underline{\zeta}_\kappa(s, \underline{\alpha}; \underline{\mathfrak{a}}) = \big(\zeta(s), \zeta(s, \alpha_1; \mathfrak{a}_{11}), \ldots, \zeta(s, \alpha_1; \mathfrak{a}_{1l_1}), \ldots,$$
$$\zeta(s, \alpha_r; \mathfrak{a}_{r1}), \ldots, \zeta(s, \alpha_r; \mathfrak{a}_{rl_r})\big),$$

and

$$P_{T,\kappa}(A) = \frac{1}{T}\text{meas}\Big\{\tau \in [0, T] : \ \underline{\zeta}_\kappa(s + i\tau, \underline{\alpha}; \underline{\mathfrak{a}}) \in A\Big\}, \quad A \in \mathfrak{B}(H^\kappa(D)).$$

The main result of this section is the following statement.

Theorem 3.2. *Suppose that the numbers $\alpha_1, \ldots, \alpha_r$ are algebraically independent over $\mathbb{Q}$. Then $P_{T,\kappa}$ converges weakly to the measure $P_{\underline{\zeta}_\kappa}$ as $T \to \infty$.*

We start the proof of Theorem 3.2 with a limit theorem on the torus $\underline{\Omega}$. Denote by $\mathcal{P}$ the set of all primes, and define

$$Q_T(A) = \frac{1}{T}\text{meas}\Big\{\tau \in [0, T] : \ \big((p^{-i\tau} : p \in \mathcal{P}), ((m + \alpha_1)^{-i\tau} : \ m \in \mathbb{N}_0), \ldots,$$
$$((m + \alpha_r)^{-i\tau} : \ m \in \mathbb{N}_0)\big) \in A\Big\}, \quad A \in \mathfrak{B}(\underline{\Omega}).$$

Lemma 3.3. *Suppose that the numbers $\alpha_1, \ldots, \alpha_r$ are algebraically independent over $\mathbb{Q}$. Then the measure $\mathbb{Q}_T$ converges weakly to the Haar measure $\underline{m}_H$ as $T \to \infty$.*

Proof. The dual group of $\underline{\Omega}$ is isomorphic to

$$\mathcal{D} = \left(\bigoplus_{p \in \mathcal{P}} \mathbb{Z}_p\right) \bigoplus_{j=1} \left(\bigoplus_{m \in \mathbb{N}_0} \mathbb{Z}_{jm}\right),$$

where $\mathbb{Z}_p = \mathbb{Z}$ and $\mathbb{Z}_{jm} = \mathbb{Z}$ for all $p \in \mathcal{P}$ and $m \in \mathbb{N}_0$, $j = 1, \ldots, r$, respectively. An element $\underline{k} = (\underline{k}_{\mathcal{P}}, \underline{k}_{r\mathbb{N}_0}) \in \mathcal{D}$, $\underline{k}_{\mathcal{P}} = (k_p : p \in \mathcal{P})$, $\underline{k}_{r\mathbb{N}_0} = (k_{lm} : m \in \mathbb{N}_0, j = 1, \ldots, r)$, where only a finite number of integers k_p and k_{jm} are distinct from zero, acts on $\underline{\Omega}$ by

$$\underline{\omega} \to \underline{\omega}^{\underline{k}} = \prod_{p \in \mathcal{P}} \widehat{\omega}^{k_p}(p) \prod_{j=1}^{r} \prod_{m \in \mathbb{N}_0} \omega_j^{k_{jm}}(m).$$

Therefore, the Fourier transform $g_T(\underline{k})$ of the measure $\mathbb{Q}_T$ is

$$\begin{aligned} g_T(\underline{k}) &= \int_{\Omega}\Big(\prod_{p\in\mathcal{P}}\widehat{\omega}^{k_p}(p)\prod_{j=1}^{r}\prod_{m\in\mathbb{N}_0}\omega_j^{k_{jm}}(m)\Big)\mathrm{d}\mathbb{Q}_T = \\ & \frac{1}{T}\int_0^T\prod_{p\in\mathcal{P}}p^{-ik_p\tau}\prod_{j=1}^{r}\prod_{m\in\mathbb{N}_0}(m+\alpha_j)^{-ik_{jm}\tau}\mathrm{d}\tau, \end{aligned} \tag{3.2}$$

where, as above, only a finite number of integers k_p and k_{jm} are distinct from zero. It is well known that the set $\{\log p: \ p\in\mathcal{P}\}$ is linearly independent over $\mathbb{Q}$, and this follows from the unique factorization of positive integers. Since the numbers $\alpha_1,\dots,\alpha_r$ are algebraically independent over $\mathbb{Q}$, hence the set

$$L = \stackrel{def}{=}\big\{\big(\log p: p\in\mathcal{P}\big),\big(\log(m+\alpha_j): \ m\in\mathbb{N}_0, j=1,\dots,r\big)\big\}$$

is linearly independent over $\mathbb{Q}$. Really, if there exist integers k_p and k_{jm}, not all zeros, such that

$$\begin{aligned} k_1\log p_1+\dots+k_n\log p_n+k_{1m_1}\log(m_1+\alpha_1)+\dots+k_{n_1m_{n_1}}(m_{n_1}+\alpha_1)+\dots \\ +k_{rm_r}\log(m_r+\alpha_r)+\dots+k_{n_rm_{n_r}}\log(m_{n_r}+\alpha_r)=0, \end{aligned}$$

then we obtain that

$$\begin{aligned} p_1^{k_1}\dots p_n^{k_n}(m_1+\alpha_1)^{k_{1m_1}}\dots(m_{n_1}+\alpha_1)^{k_{n_1m_{n_1}}}\dots \\ (m_r+\alpha_r)^{k_{rm_r}}\dots(m_{n_r}+\alpha_r)^{k_{n_rm_{n_r}}}=1, \end{aligned}$$

and this contradicts the algebraic independence of the numbers $\alpha_1,\dots,\alpha_r$. Here p_j denotes a certain prime number not the j th in the set $\mathcal{P}$.

We find by (3.2) that

$$g_T(\underline{k}) = \begin{cases} 1 & \text{if } \ \underline{k}=\underline{0}, \\ \frac{1-\exp\left\{-iT\left(\sum_{p\in\mathcal{P}}k_p\log p+\sum_{j=1}^{r}\sum_{m\in\mathbb{N}_0}k_{jm}\log(m+\alpha_j)\right)\right\}}{T\left(\sum_{p\in\mathcal{P}}k_p\log p+\sum_{j=1}^{r}\sum_{m\in\mathbb{N}_0}k_{jm}\log(m+\alpha_j)\right)} & \text{if } \ \underline{k}\neq\underline{0}. \end{cases}$$

Thus,

$$\lim_{T\to\infty}g_T(\underline{k}) = \begin{cases} 1 & \text{if } \ \underline{k}=\underline{0}, \\ 0 & \text{if } \ \underline{k}\neq\underline{0}. \end{cases}$$

This and a continuity theorem for probability measures on compact topological groups [10], Theorem 1.4.2, prove the lemma.

We use the same notation as in previous chapters for $v_n(m,\alpha_j)$ and $\zeta_n(s,\alpha_j;\mathfrak{a}_{jl})$, and define additionally

$$v_n(m)=\exp\left\{-\left(\frac{m}{n}\right)^{\sigma_1}\right\}, \quad m,n\in\mathbb{N}.$$

Then we have that the series

$$\zeta_n(s)=\sum_{m=1}^{\infty}\frac{v_n(m)}{m^s}$$

is absolutely convergent for $\sigma > \frac{1}{2}$. For $m \in \mathbb{N}$, define

$$\widehat{\omega}(m) = \prod_{p^l \| m} \widehat{\omega}^l(p),$$

where $p^l \parallel m$ means that $p^l | m$ but $p^{l+1} \nmid m$, and let

$$\zeta_n(s, \widehat{\omega}) = \sum_{m=1}^{\infty} \frac{v_n(m)\widehat{\omega}(m)}{m^s}.$$

Since $|\widehat{\omega}(m)| = 1$, the latter series also is absolutely convergent for $\sigma > \frac{1}{2}$. For brevity, let

$$\begin{aligned}\underline{\zeta}_{n,\kappa}(s, \underline{\alpha}; \underline{\mathfrak{a}}) = \big(&\zeta_n(s), \zeta_n(s, \alpha_1; \mathfrak{a}_{11}), \dots, \zeta_n(s, \alpha_1; \mathfrak{a}_{1l_1}), \dots, \\ &\zeta_n(s, \alpha_r; \mathfrak{a}_{r1}), \dots, \zeta_n(s, \alpha_r; \mathfrak{a}_{rl_r})\big)\end{aligned}$$

and

$$\begin{aligned}\underline{\zeta}_{n,\kappa}(s, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{a}}) = \big(&\zeta_n(s, \widehat{\omega}), \zeta_n(s, \alpha_1, \underline{\omega}_1; \mathfrak{a}_{11}), \dots, \zeta_n(s, \alpha_1, \underline{\omega}_1; \mathfrak{a}_{1l_1}), \dots, \\ &\zeta_n(s, \alpha_r, \underline{\omega}_r; \mathfrak{a}_{r1}), \dots, \zeta_n(s, \alpha_r, \underline{\omega}_r; \mathfrak{a}_{rl_r})\big).\end{aligned}$$

For $A \in \mathfrak{B}(h^\kappa(D))$, now define

$$P_{T,n,\kappa}(A) = \frac{1}{T}\text{meas}\Big\{\tau \in [0, T] : \ \underline{\zeta}_{n,\kappa}(s + i\tau, \underline{\alpha}; \underline{\mathfrak{a}}) \in A\Big\},$$

and, for any fixed $\underline{\omega}_0 = (\widehat{\omega}_0, \omega_{10}, \dots, \omega_{r0}) \in \underline{\Omega}$,

$$Q_{T,n,\kappa}(A) = \frac{1}{T}\text{meas}\Big\{\tau \in [0, T] : \ \underline{\zeta}_{n,\kappa}(s + i\tau, \underline{\alpha}, \underline{\omega}_0; \underline{\mathfrak{a}}) \in A\Big\}.$$

We note that $P_{T,n,\kappa}$ and $Q_{T,n,\kappa}$ are different from those of Section 2.2 because $\underline{\zeta}_{n,\kappa}(s + i\tau, \underline{\alpha}; \underline{\mathfrak{a}})$ and $\underline{\zeta}_{n,\kappa}(s + i\tau, \underline{\alpha}, \underline{\omega}_0; \underline{\mathfrak{a}})$ are different from similar collections of Section 2.2.

Lemma 3.4. *Suppose that the numbers $\alpha_1, \dots, \alpha_r$ are algebraically independent over $\mathbb{Q}$. Then $P_{T,n,\kappa}$ and $Q_{T,n,\kappa}$ both converge weakly to the same probability measure $P_{n,\kappa}$ on $(H^\kappa(D), \mathfrak{B}(H^\kappa(D)))$ as $T \to \infty$.*

Proof. Since the series $\zeta_n(s)$ and $\zeta_n(s, \alpha_j; \mathfrak{a}_{jl})$, $j = 1, \dots, r$, $l = 1, \dots, l_j$, converge absolutely for $\sigma > \frac{1}{2}$, the function $h_{n,\kappa} : \ \underline{\Omega} \to H^\kappa(D)$ given by the formula

$$h_{n,\kappa}(\underline{\omega}) = \underline{\zeta}_{n,\kappa}(s, \alpha, \underline{\omega}; \underline{\mathfrak{a}})$$

is continuous. Moreover, we have that

$$\begin{aligned}h_{n,\kappa}\big(&(p^{-i\tau} : p \in \mathcal{P}), ((m + \alpha_1)^{-i\tau} : m \in \mathbb{N}_0), \dots, \\ &((m + \alpha_r)^{-i\tau} : \ m \in \mathbb{N}_0)\big) = \underline{\zeta}_n(s + i\tau, \underline{\alpha}; \underline{\mathfrak{a}}).\end{aligned}$$

Therefore, we have that $P_{t,n,\kappa} = Q_T h_{n,\kappa}^{-1}$. This, the continuity of $h_{n,\kappa}$, Lemmas 3.3 and 1.5 show that $P_{T,n,\kappa}$ converges weakly to $P_{n,\kappa} = \underline{m}_H h_{n,\kappa}^{-1}$ as $T \to \infty$.

Similarly, we obtain that $Q_{T,n,\kappa}$ converges weakly to $\underline{m}_H g_{n,\kappa}^{-1}$ as $T \to \infty$, where $g_{n,\kappa} : \underline{\Omega} \to H^\kappa(D)$ is related to $h_{n,\kappa}$ by $g_{n,\kappa}(\underline{\omega}) = h_{n\kappa}(\underline{\omega}\underline{\omega}_0)$. Since the Haar measure $\underline{m}_H$ is invariant with respect to the translations by points from $\underline{\Omega}$, this implies the equality $\underline{m}_H g_{n,\kappa}^{-1} = \underline{m}_H h_{n,\kappa}^{-1}$, and the lemma is proved.

Now we define a metric on $h^\kappa(D)$. For

$$\underline{f} = (f_0, f_{11}, \dots, f_{1l_1}, \dots, f_{r1}, \dots, f_{rl_r}) \in H^\kappa(D)$$

and

$$\underline{g} = (g_0, g_{11}, \dots, g_{1l_1}, \dots, g_{r1}, \dots, g_{rl_r}) \in H^\kappa(D),$$

define

$$\underline{\rho}_\kappa(\underline{f}, \underline{g}) = \max\left(\rho(f_0, g_0), \max_{1\le j\le r} \max_{1\le l\le l_j} \rho(f_{jl}, g_{jl})\right),$$

where ρ is the same metric on $H(D)$ as in previous chapters. Then $\underline{\rho}_\kappa$ is a metric on $H^\kappa(D)$ inducing its topology.

Now we will approximate the vectors $\underline{\zeta}_\kappa(s, \underline{\alpha}; \underline{\mathfrak{a}})$ and $\underline{\zeta}_\kappa(s, \alpha, \underline{\omega}; \underline{\mathfrak{a}})$ by $\underline{\zeta}_{n,\kappa}(s, \underline{\alpha}; \underline{\mathfrak{a}})$ and $\underline{\zeta}_{n,\kappa}(s, \alpha, \underline{\omega}; \underline{\mathfrak{a}})$, respectively.

Lemma 3.5. *The equality*

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T} \int_0^T \underline{\rho}_\kappa\big(\underline{\zeta}_\kappa(s+i\tau, \underline{\alpha}; \underline{\mathfrak{a}}), \underline{\zeta}_{n,\kappa}(s+i\tau, \underline{\alpha}; \underline{\mathfrak{a}})\big)\mathrm{d}\tau = 0$$

holds.

Proof. It is known [19] that

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T} \int_0^T \rho\big(\zeta(s+i\tau), \zeta_n(s+i\tau)\big)\mathrm{d}\tau = 0.$$

This, Lemma 2.4 with remark on the notation by $\underline{\rho}_\kappa$ of a different metric in Chapter 2, and the definition of the metric $\underline{\rho}_\kappa$ prove the lemma.

Lemma 3.6. *Suppose that the numbers $\alpha_1, \dots, \alpha_r$ are algebraically independent over $\mathbb{Q}$. Then, for almost all $\underline{\omega} \in \underline{\Omega}$, the equality*

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T} \int_0^T \underline{\rho}_\kappa\big(\underline{\zeta}_\kappa(s+i\tau, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{a}}), \underline{\zeta}_{n,\kappa}(s+i\tau, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{a}})\big)\mathrm{d}\tau = 0$$

holds.

Proof. In [19], it is obtained that, for almost all $\widehat{\omega} \in \widehat{\Omega}$,

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T} \int_0^T \rho\big(\zeta(s+i\tau, \widehat{\omega}), \zeta_n(s+i\tau, \widehat{\omega})\big)\mathrm{d}\tau = 0.$$

From this, Lemma 2.5 and the definition of the metric $\underline{\rho}_\kappa$, the lemma follows because the algebraic independence over $\mathbb{Q}$ of the numbers $\alpha_1, \dots, \alpha_r$ implies the linear independence over $\mathbb{Q}$ of the set $L(\alpha_1, \dots, \alpha_r)$.

Define one more probability measure

$$P_{T,\kappa}(A) = \frac{1}{T}\text{meas}\left\{\tau \in [0,T] : \underline{\zeta}_{\kappa}(s+i\tau, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{a}}) \in A\right\}, \quad A \in \mathfrak{B}(H^{\kappa}(D)).$$

Lemma 3.7. *Suppose that the numbers $\alpha_1, \dots, \alpha_r$ are algebraically independent over $\mathbb{Q}$. Then $P_{T,\kappa}$ and $\widehat{P}_{T,\kappa}$ both converge weakly to the same probability measure P_κ on $(H^{\kappa}(D), \mathfrak{B}(H^{\kappa}(D)))$ as $T \to \infty$.*

Proof. We generalize the proof of Lemma 2.6. On the probability space $(\widetilde{\Omega}, \mathfrak{B}(\widetilde{\Omega}), \mathbb{P})$, define the $H^{\kappa}(D)$-valued random element $\underline{X}_{T,n,\kappa}$ by

$$\begin{aligned}\underline{X}_{T,n,\kappa} = \underline{X}_{T,n,\kappa}(s) = \underline{X}_{T,n,\kappa}(s,\underline{\alpha};\underline{\mathfrak{a}}) = \big(&X_{T,n}(s), X_{T,n,1,1}(s), \dots, X_{T,n,1,l_1}(s), \dots, \\ &X_{T,n,r,1}(s), \dots, X_{T,n,r,l_r}(s)\big) = \underline{\zeta}_{n,\kappa}(s+i\theta T, \underline{\alpha}; \underline{\mathfrak{a}}).\end{aligned}$$

Then, by Lemma 3.4,

$$\underline{X}_{T,n,\kappa} \xrightarrow[T\to\infty]{\mathfrak{D}} \underline{X}_{n,\kappa}, \tag{3.3}$$

where

$$\underline{X}_{n,\kappa} = X_{n,\kappa}(s) = \big(X_n(s), X_{n,1,1}(s), \dots, X_{n,1,l_1}(s), \dots, X_{n,r,1}(s), \dots, X_{n,r,l_r}(s)\big),$$

is an $H^{\kappa}(D)$-valued random element with the distribution $P_{n,\kappa}$ ($P_{n,\kappa}$ is the limit measure in Lemma 3.4). Since the series for $\zeta_n(s)$ and $\zeta_n(s,\alpha_j;\mathfrak{a}_{jl})$, $j=1,\dots,r$, $l=1,\dots,l_j$, converge absolutely for $\sigma > \frac{1}{2}$, we have that, for $\sigma > \frac{1}{2}$,

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \big|\zeta_n(\sigma+it)\big|^2 \mathrm{d}t = \sum_{m=1}^{\infty} \frac{v_n^2(m)}{m^{2\sigma}} \le \sum_{m=1}^{\infty} \frac{1}{m^{2\sigma}} \tag{3.4}$$

for all $n \in \mathbb{N}$, and (2.2) is true for all $n \in \mathbb{N}$, and $j=1,\dots,r$, $l=1,\dots,l_j$. Then, using the Caushy integral formula, contour integration and (3.4), we find that, for $n \in \mathbb{N}$,

$$\limsup_{T\to\infty} \frac{1}{T}\int_0^T \sup_{s\in K_k} \big|\zeta_n(s+i\tau)\big| \mathrm{d}\tau \le \widetilde{C}_k \left(\sum_{m=1}^{\infty} \frac{1}{m^{2\widetilde{\sigma}_k}}\right)^{\frac{1}{2}} \tag{3.5}$$

with some $\widetilde{C}_k > 0$ and $\widetilde{\sigma}_k > \frac{1}{2}$. Here K_k is a compact subset from the definition of the metric ρ. Let

$$\widetilde{R}_k = \left(\sum_{m=1}^{\infty} \frac{1}{m^{2\widetilde{\sigma}_k}}\right)^{\frac{1}{2}},$$

and let other notation remain the same as in the proof of Lemma 2.6. Then, taking $\widetilde{M}_k = \widetilde{C}_k \widetilde{R}_k 2^{k+1}\varepsilon^{-1}$

and $M_{jlk} = C_k R_{jlk} 2^{k+\kappa}\varepsilon^{-1}$, we deduce from (3.5) and (2.3) that

$$\limsup_{T\to\infty} \mathbb{P}\big(\sup_{s\in K_k} |X_{T,n}(s)| > \widetilde{M}_k \text{ or } \sup_{s\in K_k} |X_{T,n,j,l}(s)| > M_{jlk}$$
$$\text{for some } (j,l)\big) \le \limsup_{T\to\infty} \mathbb{P}\big(\sup_{s\in K_k} |X_{T,n}(s)| > \widetilde{M}_k\big)$$
$$+ \sum_{j=1}^{r}\sum_{l=1}^{l_j} \limsup_{T\to\infty} \mathbb{P}\big(\sup_{s\in K_k} |X_{T,n,j,l}(s)| > M_{jlk}\big)$$
$$\le \frac{1}{\widetilde{M}_k} \sup_{n\in\mathbb{N}} \limsup_{T\to\infty} \frac{1}{T}\int_0^T \sup_{s\in K_k} |\zeta_n(s+i\tau)|\mathrm{d}\tau$$
$$+ \sum_{j=1}^{r}\sum_{l=1}^{l_j} \frac{1}{M_{jlk}} \sup_{n\in\mathbb{N}_0} \limsup_{T\to\infty} \frac{1}{T}\int_0^T \sup_{s\in K_k} |\zeta_n(s+i\tau,\alpha_j;\mathfrak{a}_{jl})|\mathrm{d}\tau$$
$$\le \frac{\widetilde{C}_R \widetilde{R}_k}{\widetilde{M}_k} + \sum_{j=1}^{r}\sum_{l=1}^{l_j} \frac{C_k R_{jlk}}{M_{jlk}} = \frac{\varepsilon}{2^{k+1}} + \frac{\varepsilon}{2^{k+\kappa}} \sum_{j=1}^{\varepsilon}\sum_{l=1}^{l_j} 1 \le$$
$$\frac{\varepsilon}{2^{k+1}} + \frac{\varepsilon}{2^{k+1}} = \frac{\varepsilon}{2^k}.$$

This together with (3.3) leads, for all $n \in \mathbb{N}$, to the inequality

$$\mathbb{P}\big(\sup_{s\in K_k} |X_n(s)| > \widetilde{M}_k \text{ or } \sup_{s\in K_k} |X_{n,j,l}(s)| > M_{jlk} \text{ for some } (j,l)\big) \le \frac{\varepsilon}{2^k}, \quad k\in\mathbb{N}. \tag{3.6}$$

Now we take a set

$$H_\varepsilon^\kappa = \big\{(g_0, g_{11},\dots,g_{1l_1},\dots,g_{r1},\dots,g_{rl_r}) \in H^\kappa(D):\ \sup_{s\in K_k} |g_0(s)| < \widetilde{M}_k,$$
$$\sup_{s\in K_k} |g_{jl}(s)| \le M_{jlk},\ j=1,\dots,r,\ l=1,\dots,l_j,\ k\in\mathbb{N}\big\}.$$

Then the set H_ε^κ is uniformly bounded, thus it is compact on the space $H^\kappa(D)$. Moreover, in view of (3.6),

$$\mathbb{P}\big(\underline{X}_{n,\kappa}(s) \in H_\varepsilon^\kappa\big) \ge 1 - \varepsilon \sum_{k=1}^{\infty} \frac{1}{2^k} = 1-\varepsilon$$

for all $n\in\mathbb{N}$. Hence,

$$P_{n,\kappa}(H_\varepsilon^\kappa) \ge 1-\varepsilon$$

for all $n \in \mathbb{N}$. Thus, we obtained that the family of probability measures $\{P_{n,\kappa}:\ n\in\mathbb{N}\}$ is tight. Therefore, by Lemma 1.9, the latter family is relatively compact, and there exists a sequence $\{P_{n_k,\kappa}:\ k\in\mathbb{N}\} \subset \{P_{n,\kappa}:\ n\in\mathbb{N}\}$ weakly convergent to some probability measure P_κ on $(H^\kappa(D), \mathfrak{B}(H^\kappa(D))$ as $k\to\infty$. Hence

$$X_{n_k,\kappa} \xrightarrow[k\to\infty]{\mathfrak{D}} P_\kappa. \tag{3.7}$$

Let

$$\underline{X}_{T,\kappa} = X_{T,\kappa}(s) = \underline{\zeta}_\kappa(s+i\theta T, \underline{\alpha}; \underline{\mathfrak{a}})$$

be one more $H^\kappa(D)$-valued random element defined on the probability space $(\widetilde{\Omega}, \mathfrak{B}(\widetilde{\Omega}), \mathbb{P})$. Then, by Lemma 3.5, we have that, for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \limsup_{T\to\infty} \mathbb{P}\big(\underline{\rho}_\kappa(\underline{X}_{T,n}(s), X_{T,n}(s)) \geq \varepsilon\big)$$
$$\leq \lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T\varepsilon} \int_0^T \underline{\rho}_\kappa\big(\underline{\zeta}_\kappa(s+i\tau, \underline{\alpha}; \underline{\mathfrak{a}}), \underline{\zeta}_{n,\kappa}(s+i\tau, \underline{\alpha}; \underline{\mathfrak{a}})\big)\mathrm{d}\tau = 0.$$

This, (3.3) and (3.7) together with Lemma 1.11 imply the relation

$$\underline{X}_{T,\kappa} \xrightarrow[T\to\infty]{\mathfrak{D}} P_\kappa \tag{3.8}$$

which is equivalent to the weak convergence of $P_{T,\kappa}$ to P_κ as $T \to \infty$. Moreover, it follows from (3.8) that the measure P_κ is independent of the choice of the sequence $\{P_{n_k,\kappa}\}$. Thus, we have that

$$\underline{X}_{n,\kappa} \xrightarrow[n\to\infty]{D} P_\kappa. \tag{3.9}$$

Now we consider the measure $\widehat{P}_{T,\kappa}$. For this, define

$$\widehat{\underline{X}}_{T,n,\kappa}(s) = \underline{\zeta}_{n,\kappa}(s+i\theta T, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{a}})$$

and

$$\widehat{\underline{X}}_{T,\kappa}(s) = \underline{\zeta}_\kappa(s+i\theta T, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{a}}).$$

Repeating the above arguments for the random elements $\widehat{\underline{X}}_{T,n,\kappa}(s)$ and $\widehat{\underline{X}}_{T,\kappa}(s)$, and using Lemmas 3.4 and 3.6 as well as relation (3.9), we obtain that the measure $\widehat{P}_{T,\kappa}$ also converges weakly to P_κ as $T \to \infty$. The lemma is proved.

In virtue of Lemma 3.7, for the proof of Theorem 3.2 it suffices to show that the limit measure P_κ in Lemma 3.7 coincides with $P_{\underline{\zeta}_\kappa}$. To prove this, we need some results from ergodic theory. Let, for $\tau \in \mathbb{R}$,

$$\underline{a}_\tau = \big\{(p^{-i\tau} : p \in \mathcal{P}), ((m+\alpha_1)^{-i\tau} : m \in \mathbb{N}_0), \dots, ((m+\alpha_r)^{-i\tau} : m \in \mathbb{N}_0)\big\}.$$

Define $\underline{\Phi}_\tau(\underline{\omega}) = \underline{a}_\tau \underline{\omega}$, $\underline{\omega} \in \underline{\Omega}$. Then $\{\underline{\Phi}_\tau : \tau \in \mathbb{R}\}$ is a one-parameter group of measurable measure preserving transformations on $\underline{\Omega}$. Moreover, the following statement is true.

Lemma 3.8. *Suppose that the numbers $\alpha_1, \dots, \alpha_r$ are algebraically independent over $\mathbb{Q}$. Then the group $\{\underline{\Phi}_\tau : \tau \in \mathbb{R}\}$ is ergodic.*

Proof of the lemma is given in [25], Lemma 7.

Proof of Theorem 3.2. We apply standard arguments. We fix a continuity set A of the limit measure P_κ in Lemma 3.7. Then, by Lemmas 3.7 and 1.12

$$\lim_{T\to\infty} \frac{1}{T} \mathrm{meas}\Big\{\tau \in [0,T] : \underline{\zeta}_\kappa(s+i\tau, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{a}}) \in A\Big\} = P_\kappa(A). \tag{3.10}$$

Consider a random variable ξ_κ defined on $(\underline{\Omega}, \mathfrak{B}(\underline{\Omega}), \underline{m}_H)$ by the formula

$$\xi_\kappa(\underline{\omega}) = \begin{cases} 1 & \text{if } \underline{\zeta}_\kappa(s, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{a}}) \in A, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, its expectation

$$\mathbb{E}\xi_\kappa = \underline{m}_H(\underline{\omega} \in \underline{\Omega} : \ \underline{\zeta}_\kappa(s, \alpha, \underline{\omega}; \underline{\mathfrak{a}}) \in A) = P_{\underline{\zeta}_\kappa}(A). \tag{3.11}$$

In view of Lemma 3.8, the random process $\xi_\kappa(\underline{\Phi}_\tau(\underline{\omega}))$ is ergodic. Therefore, by Lemma 1.13, we have that, for almost all $\underline{\omega} \in \underline{\Omega}$,

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \xi_\kappa(\underline{\Phi}_\tau(\underline{\omega}))\mathrm{d}\tau = \mathbb{E}\xi_\kappa. \tag{3.12}$$

On the other hand, the definitions of ξ_κ and $\underline{\Phi}_\tau$ yield

$$\frac{1}{T} \int_0^T \xi_\kappa(\underline{\Phi}_\tau(\underline{\omega}))\mathrm{d}\tau = \frac{1}{T}\mathrm{meas}\Big\{\tau \in [0,T] : \ \underline{\zeta}_\kappa(s+i\tau, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{a}}) \in A\Big\}.$$

Thus, by (3.11) and (3.12), for almost all $\underline{\omega} \in \underline{\Omega}$,

$$\lim_{T\to\infty} \frac{1}{T}\mathrm{meas}\Big\{\tau \in [0,T] : \ \underline{\zeta}_\kappa(s+i\tau, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{a}}) \in A\Big\} = P_{\underline{\zeta}_\kappa}(A).$$

Combining this with (3.10), we obtain that $P_\kappa(A) = P_{\underline{\zeta}_\kappa}(A)$ for all continuity sets A of the measure P_κ. Hence, $P_\kappa(A) = P_{\underline{\zeta}_\kappa}(A)$ for all $A \in \mathfrak{B}(H^\kappa(D))$ because the continuity sets form a determining class [3]. The theorem is proved.

3.3. Support of the limit measure

In this section, we give explicitly the support of the measure $P_{\underline{\zeta}_\kappa}$. Define

$$S = \big\{g \in H(D) : \ g(s) \neq 0 \ \text{ or } \ g(s) \equiv 0\big\}.$$

Let $\kappa_1 = \kappa - 1$.

Theorem 3.9. *Suppose that the numbers $\alpha_1, \dots, \alpha_r$ are algebraically independent over $\mathbb{Q}$, and that* $\mathrm{rank}(B_j) = l_j$, $j = 1, \dots, r$. *Then the support of $P_{\underline{\zeta}_\kappa}$ is the set $S \times H^{\kappa_1}(D)$.*

Proof. We write

$$H^\kappa(D) = H(D) \times H^{\kappa_1}(D).$$

Since the spaces $H(D)$ and $H_1^\kappa(D)$ are separable, we have [3] that

$$\mathfrak{B}(H^\kappa(D)) = \mathfrak{B}(H(D)) \times \mathfrak{B}(H^{\kappa_1}(D)).$$

Thus, it suffices to consider $P_{\underline{\zeta}_\kappa}(A)$ with $A = A_1 \times A_{\kappa_1}$, $A \in \mathfrak{B}(H(D))$, $A_{\kappa_1} \in \mathfrak{B}(H^{\kappa_1}(D))$. Let, as in Chapter 1, $\Omega^r = \Omega_1 \times \cdots \times \Omega_r$, where $\Omega_j = \Omega$ for all $j = 1, \dots, r$, and let m_H^r be the Haar measure on $(\Omega^r, \mathfrak{B}(\Omega^r))$. Then we have that the Haar measure $\underline{m}_H$ is the product of the Haar measures $\widehat{m}_H$ and m_H^r. (We recall that $\underline{\Omega} = \widehat{\Omega} \times \Omega^r$, and $\widehat{m}_H$ is the Haar measure on $(\widehat{\Omega}, \mathfrak{B}(\widehat{\Omega}))$). Hence, we find that

$$
\begin{aligned}
P_{\underline{\zeta}_\kappa}(A) \quad = \quad & \underline{m}_H\big(\underline{\omega} \in \underline{\Omega} : \underline{\zeta}_\kappa(s, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{a}}) \in A\big) \\
& = \underline{m}_H\big(\underline{\omega} \in \underline{\Omega} :\ \zeta(s, \widehat{\omega}) \in A_1, \big(\zeta(s, \alpha_1, \omega_1; \mathfrak{a}_{11}), \dots, \zeta(s, \alpha_1, \omega_1; \mathfrak{a}_{1l_1}), \dots, \\
& \zeta(s, \alpha_r, \omega_r; \mathfrak{a}_{r1}), \dots, \zeta(s, \alpha_r, \omega_r; \mathfrak{a}_{rl_r})\big) \in A_{\kappa_1}\big) \\
& = \widehat{m}_H(\widehat{\omega} \in \widehat{\Omega} :\ \zeta(s, \widehat{\omega}) \in A_1) \\
& \times m_H^r\big((\omega_1, \dots, \omega_r) \in \Omega^r : \big(\zeta(s, \alpha_1, \omega_1; \mathfrak{a}_{11}), \dots, \zeta(s, \alpha_1, \omega_1; \mathfrak{a}_{1l_1}), \dots, \\
& \zeta(s, \alpha_r, \omega_r; \mathfrak{a}_{r1}), \dots, \zeta(s, \alpha_r, \omega_r; \mathfrak{a}_{rl_r})\big) \in A_{\kappa_1}\big).
\end{aligned} \tag{3.13}
$$

In [19], it is obtained that the support of the $H(D)$-valued random element $\zeta(s, \widehat{\omega})$ is the set S, i.e., S is a minimal closed set such that

$$
\widehat{m}_H\big(\widehat{\omega} \in \widehat{\Omega} :\ \zeta(s, \widehat{\omega}) \in S\big) = 1. \tag{3.14}
$$

Moreover, by Theorem 2.7, the support of the $H^{\kappa_1}(D)$-valued random element

$$
\big(\zeta(s, \alpha_1, \omega_1; \mathfrak{a}_{11}), \dots, \zeta(s, \alpha_1, \omega_1; \mathfrak{a}_{1l_1}), \dots, \zeta(s, \alpha_r, \omega_r; \mathfrak{a}_{r1}), \dots, \zeta(s, \alpha_r, \omega_r; \mathfrak{a}_{rl_r})\big)
$$

is the whole of $H^{\kappa_1}(D)$, i.e., $H^{\kappa_1}(D)$ is a minimal closed set such that

$$
\begin{aligned}
m_H^r\big((\omega_1, \dots, \omega_r) \in \Omega^r :\ & \big(\zeta(s, \alpha_1, \omega_1; \mathfrak{a}_{11}), \dots, \zeta(s, \alpha_1, \omega_1; \mathfrak{a}_{1l_1}), \dots, \\
& \zeta(s, \alpha_r, \omega_r; \mathfrak{a}_{r1}), \dots, \zeta(s, \alpha_r, \omega_r; \mathfrak{a}_{rl_r})\big) \in H^{\kappa_1}(D)\big) = 1.
\end{aligned}
$$

This, (3.13) and (3.14) complete the proof of the theorem.

3.4. Proof of Theorem 3.1

A proof of Theorem 3.1 is based on Theorems 3.2 and 3.9 as well as on Lemma 1.16.

First suppose that the functions $f(s)$ and $f_{jl}(s)$ have analytic continuations to the whole strip D, and the analytic continuation of $f(s)$ has no zeros. Define

$$
\begin{aligned}
G \quad = \quad & \Big\{(g_0, g_{11}, \dots, g_{1l_1}, \dots, g_{r1}, \dots, g_{rl_r}) \in H^\kappa(D) : \\
& \sup_{s \in K} |g_0(s) - f(s)| \le \frac{\varepsilon}{2}, \sup_{1 \le j \le r} \sup_{1 \le l \le l_j} \sup_{s \in K_{jl}} |g_{jl}(s) - f_{jl}(s)| < \frac{\varepsilon}{2}\Big\}.
\end{aligned}
$$

The set G is open in the space $H^\kappa(D)$. Therefore, Theorem 3.2, together with Lemma 1.17, implies

$$
\liminf_{T \to \infty} \frac{1}{T} \mathrm{meas}\Big\{\tau \in [0, T] :\ \underline{\zeta}_\kappa(s + i\tau, \underline{\alpha}; \underline{\mathfrak{a}}) \in G\Big\} \ge P_{\underline{\zeta}_\kappa}(G). \tag{3.15}
$$

However, by Theorem 3.9, $(f, f_{11}, \dots, f_{1l_1}, \dots, f_{r1}, \dots, f_{rl_r})$ is an element of the support of the measure $P_{\underline{\zeta}_\kappa}$. Thus, $P_{\underline{\zeta}_\kappa}(G) > 0$, and the definition of G and (3.15) yield

$$\liminf_{T\to\infty} \frac{1}{T}\text{meas}\Big\{\tau \in [0,T] : \sup_{s\in K} |\zeta(s+i\tau) - f(s)| < \frac{\varepsilon}{2},$$
$$\sup_{1\le j\le r} \sup_{1\le l\le l_j} \sup_{s\in K_{jl}} |\zeta(s+i\tau, \alpha_j; \mathfrak{a}_{jl}) - f_{jl}(s)| < \frac{\varepsilon}{2}\Big\} > 0. \tag{3.16}$$

Now let the functions $f(s)$ and $f_{jl}(s)$ satisfy the hypotheses of the theorem. Then, by Lemma 1.16, there exist polynomials $p(s)$, $p(s) \neq 0$ on K, and $p_{jl}(s)$ such that

$$\sup_{s\in K} |f(s) - p(s)| < \frac{\varepsilon}{4} \tag{3.17}$$

and

$$\sup_{1\le j\le r} \sup_{1\le l\le l_j} \sup_{s\in K} |f(s) - p(s)| < \frac{\varepsilon}{2}. \tag{3.18}$$

Since $p(s) \neq 0$ on K, we can define a continuous branch of the function $\log p(s)$ in K which will be analytic in the interior of K. By Lemma 1.16 again, we can find a polynomial $g(s)$ such that

$$\sup_{s\in K} |p(s) - e^{q(s)}| < \frac{\varepsilon}{4}.$$

This, together with (3.17), shows that

$$\sup_{s\in K} |f(s) - e^{q(s)}| < \frac{\varepsilon}{2}. \tag{3.19}$$

However, $e^{q(s)} \neq 0$, therefore, the functions $e^{q(s)}$ and $p_{jl}(s)$ satisfy all hypotheses under which (3.16) holds. Thus, we have that

$$\liminf_{T\to\infty} \frac{1}{T}\text{meas}\Big\{\tau \in [0,T] : \sup_{s\in K} |\zeta(s+i\tau) - e^{q(s)}| < \frac{\varepsilon}{2},$$
$$\sup_{1\le j\le r} \sup_{1\le l\le l_j} \sup_{s\in K_{jl}} |\zeta(s+i\tau, \alpha_j; \mathfrak{a}_{jl}) - p_{jl}(s)| < \frac{\varepsilon}{2}\Big\} > 0. \tag{3.20}$$

It is easily seen that, in view of (3.19) and (3.18),

$$\Big\{\tau \in [0,T] : \sup_{s\in K} |\zeta(s+i\tau) - e^{q(s)}| < \frac{\varepsilon}{2},$$
$$\sup_{1\le j\le r} \sup_{1\le l\le l_j} \sup_{s\in K_{jl}} |\zeta(s+i\tau, \alpha_j; \mathfrak{a}_{jl}) - p_{jl}(s)| < \frac{\varepsilon}{2}\Big\} \subset$$
$$\Big\{\tau \in [0,T] : \sup_{s\in K} |\zeta(s+i\tau) - f(s)| < \varepsilon,$$
$$\sup_{1\le j\le r} \sup_{1\le l\le l_j} \sup_{s\in K_{jl}} |\zeta(s+i\tau, \alpha_j; \mathfrak{a}_{jl}) - f_{jl}(s)| < \varepsilon\Big\}.$$

This and (3.20) prove the theorem.

Chapter 4

Mixed joint universality for periodic Hurwitz zeta-functions and the zeta-function of cusp form

Let $F(z)$ be a normalized Hecke eigen cusp form of weight κ for the full modular group

$$SL(2,\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}, \quad ad - bc = 1 \right\}.$$

This means that $F(z)$ is a holomorphic function in the half-plane $\mathrm{Im} z > 0$, for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$ satisfies the functional equation

$$F\left(\frac{az+b}{cz+d}\right) = (cz+d)^{\kappa} F(z),$$

and is a simultaneous eigen function of all Hecke operators

$$(T_n f)(z) = n^{\kappa-1} \sum_{d|n} d^{-\kappa} \sum_{b=0}^{d-1} f\left(\frac{nz+bd}{dz}\right), \quad n \in \mathbb{N}.$$

In this case, the function $F(z)$ has at infinity the Fourier series expansion

$$F(z) = \sum_{m=1}^{\infty} c(m) e^{2\pi i m z}, \quad c(1) = 1.$$

The zeta-function $\zeta(s, F)$ attached to $F(z)$ is defined, for $\sigma > \frac{\kappa+1}{2}$, by the series

$$\zeta(s, F) = \sum_{m=1}^{\infty} \frac{c(m)}{m^s}.$$

Moreover, $\zeta(s, F)$ is analytically continuable to an entire function, and, for $\sigma > \frac{\kappa+1}{2}$, has the Euler product expansion over primes

$$\zeta(s, F) = \prod_{p} \left(1 - \frac{\alpha(p)}{p^s}\right)^{-1} \left(1 - \frac{\beta(p)}{p^s}\right)^{-1},$$

where $\alpha(p)$ and $\beta(p)$ are conjugate complex numbers related to $c(m)$ by the equality $\alpha(p)+\beta(p)=c(p)$.

In this chapter, we consider the joint universality for a collection of zeta-functions $\zeta(s,F), \zeta(s,\alpha_1;\mathfrak{a}_{11}), \dots, \zeta(s,\alpha_1;\mathfrak{a}_{1l_1}), \dots, \zeta(s,\alpha_r;\mathfrak{a}_{r1}), \dots, \zeta(s,\alpha_r;\mathfrak{a}_{1rl_r})$.

4.1 Statement of the main theorem

Let $D_\kappa = \left\{s \in \mathbb{C} : \frac{\kappa}{2} < \sigma < \frac{\kappa+1}{2}\right\}$. Other notation is the same as in Chapter 3.

Theorem 4.1. *Suppose that F is a normalized Hecke eigen cusp form of weight κ for the full modular group, the numbers $\alpha_1, \dots, \alpha_r$ are algebraically independent over $\mathbb{Q}$, and* $\operatorname{rank}(B_j) = l_j$, *$j = 1, \dots, r$. Let $K \subset D_\kappa$ be a compact subset with connected complement, and $f(s)$ be a continuous non-vanishing function on K which is analytic in the interior of K. Moreover, for $j = 1, \dots, r$, $l = 1, \dots, l_j$, let K_{jl} be a compact subset of the strip D with connected complement, and let $f_{jl}(s)$ be a continuous function on K_{jl} which is analytic in the interior of K_{jl}. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T} \operatorname{meas}\Big\{\tau \in [0,T] : \sup_{s\in K} |\zeta(s+i\tau,F) - f(s)| < \varepsilon,$$
$$\sup_{1\le j\le r}\ \sup_{1\le j\le l_j}\ \sup_{s\in K_{jl}} |\zeta(s+i\tau,\alpha_j;\mathfrak{a}_{jl}) - f_{jl}(s)| < \varepsilon\Big\} > 0.$$

We see that Theorem 4.1 is an analogue of Theorem 3.1 in which the Riemann zeta-function $\zeta(s)$ is replaced by the function $\zeta(s,F)$. This change requires some additional arguments because in Theorem 4.1 we have two different strips D_κ and D.

4.2. Joint limit theorem for periodic Hurwitz zeta-functions and the function $\zeta(s,F)$

In this chapter, let

$$v_1 = \sum_{j=1}^{r} l_j, \quad v = v_1 + 1.$$

Denote by $H(D_\kappa)$ the space of analytic functions on D_κ endowed with the topology of uniform convergence on compacta, and let

$$H^v(D_\kappa, D) = H(D_\kappa) \times \underbrace{H(D) \times \dots \times H(D)}_{v_1}.$$

Moreover, for brevity, we set

$$\underline{\zeta}_v(\hat{s}, s, \underline{\alpha}; \underline{\mathfrak{a}}, F) = \big(\zeta(s, F), \zeta(s, \alpha_1; \mathfrak{a}_{11}), \ldots,$$
$$\zeta(s, \alpha_1; \mathfrak{a}_{1l_1}), \ldots, \zeta(s, \alpha_r; \mathfrak{a}_{r1}), \ldots, \zeta(s, \alpha_r; \mathfrak{a}_{rl_r})\big)$$

On the probability space $(\underline{\Omega}, \mathfrak{B}(\underline{\Omega}), \underline{m}_H)$, define the $H^v(D_\kappa, D)$-valued random element $\underline{\zeta}_v(\widehat{s}, s, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{a}}, F)$ by the formula

$$\underline{\zeta}_v(\hat{s}, s, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{a}}, F) = \big(\zeta(\hat{s}, \hat{\omega}, F), \zeta(s, \alpha_1, \omega_1; \mathfrak{a}_{11}), \ldots,$$
$$\zeta(s, \alpha_1, \omega_1; \mathfrak{a}_{1l_1}), \ldots, \zeta(s, \alpha_r, \omega_r; \mathfrak{a}_{r1}), \ldots, \zeta(s, \alpha_r, \omega_r; \mathfrak{a}_{rl_r})\big),$$

where

$$\zeta(\hat{s}, \hat{\omega}, F) = \prod_p \left(1 - \frac{\alpha(p)\hat{\omega}(p)}{p^s}\right)^{-1} \left(1 - \frac{\beta(p)\hat{\omega}(p)}{p^s}\right)^{-1}.$$

Other notation is the same as in Chapter 3. Denote by $P_{\underline{\zeta}_v}$ the distribution of the random element $\underline{\zeta}_v(\hat{s}, s, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{a}}, F)$, i.e., for $A \in \mathfrak{B}(H^v(D_\kappa, D))$,

$$P_{\underline{\zeta}_v}(A) = \underline{m}_H \left(\underline{\omega} \in \underline{\Omega} : \ \underline{\zeta}_v(\hat{s}, s, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{a}}, F) \in A\right).$$

In this section, we consider the weak convergence of

$$P_{T,v}(A) = \frac{1}{T} \text{meas} \left\{\tau \in [0, T] : \ \underline{\zeta}_v(\hat{s} + i\tau, s + i\tau, \underline{\alpha}; \underline{\mathfrak{a}}, F) \in A\right\}, \quad A \in \mathfrak{B}(H^v(D_\kappa, D))$$

as $T \to \infty$.

Theorem 4.2. *Suppose that the numbers $\alpha_1, \ldots, \alpha_r$ are algebraically independent over $\mathbb{Q}$. Then $P_{T,v}$ converges weakly to the measure $P_{\underline{\zeta}_v}$ as $T \to \infty$.*

Proof of Theorem 4.2 is analogical to that of Theorem 3.1. Therefore, we will omit some details.

For $n \in \mathbb{N}$, define

$$\zeta_n(\hat{s}, F) = \sum_{m=1}^{\infty} \frac{c(m) v_n(m)}{m^s}$$

and

$$\zeta_n(\hat{s}, \hat{\omega} F) = \sum_{m=1}^{\infty} \frac{c(m)\hat{\omega}(m) v_n(m)}{m^s}.$$

Then we have [15] that the latter series are absolutely convergent for $\sigma > \frac{\kappa}{2}$. For brevity, we set

$$\underline{\zeta}_{n,v}(\hat{s}, s, \underline{\alpha}; \underline{\mathfrak{a}}, F) = (\zeta_n(\hat{s}, F), \zeta_n(s, \alpha_1; \mathfrak{a}_{11}), \ldots, \zeta_n(s, \alpha_1; \mathfrak{a}_{1l_1}), \ldots,$$
$$\zeta_n(s, \alpha_r; \mathfrak{a}_{r1}), \ldots, \zeta_n(s, \alpha_r; \mathfrak{a}_{rl_r}))$$

and

$$\underline{\zeta}_{n,v}(\hat{s}, s, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{a}}, F) = (\zeta_n(\hat{s}, \hat{\omega}, F), \zeta_n(s, \alpha_1, \omega_1; \mathfrak{a}_{11}), \ldots, \zeta_n(s, \alpha_1, \omega_1; \mathfrak{a}_{1l_1}), \ldots,$$
$$\zeta_n(s, \alpha_r, \omega_r; \mathfrak{a}_{r1}), \ldots, \zeta_n(s, \alpha_r, \omega_r; \mathfrak{a}_{rl_r})).$$

Now on the space $(H^v(D_\kappa, D), \mathfrak{B}(H^v(D_\kappa, D)))$, define two probability measures

$$P_{T,n,v}(A) = \frac{1}{T}\text{meas}\left\{\tau \in [0,T] : \underline{\zeta}_{n,v}(\hat{s}+i\tau, s+i\tau, \underline{\alpha}; \underline{\mathfrak{a}}, F) \in A\right\}$$

and

$$Q_{T,n,v}(A) = \frac{1}{T}\text{meas}\left\{\tau \in [0,T] : \underline{\zeta}_{n,v}(\hat{s}+i\tau, s+i\tau, \underline{\alpha}, \underline{\omega}_0; \underline{\mathfrak{a}}, F) \in A\right\},$$

where $\underline{\omega}_0 = (\hat{\omega}_0, \omega_{10}, ..., \omega_{r0})$ is a fixed element of $\underline{\Omega}$.

Lemma 4.3. *Suppose that the numbers $\alpha_1, \dots, \alpha_r$ are algebraically independent over $\mathbb{Q}$. Then $P_{T,n,v}$ and $Q_{T,n,v}$ both converge weakly to the same probability measure $P_{n,v}$ on $(H^v(D_\kappa, D), \mathfrak{B}(H^v(D_\kappa, D)))$ as $T \to \infty$.*

Proof. We repeat the arguments used in the proof of Lemma 3.4. The absolute convergence of the series for $\zeta_n(\hat{s}, F)$ and $\zeta_n(s, \alpha_j; \mathfrak{a}_{jl})$, $j = 1, \dots, r$, $l = 1, \dots, l_j$, implies the continuity of the function $h_{n,v} : \underline{\Omega} \to H^v(D_\kappa, D)$ defined by the formula

$$h_{n,v}(\underline{\omega}) = \underline{\zeta}_n(\widehat{s}, s, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{a}}, F).$$

Moreover, we have that

$$h_{n,v}\big((p^{-i\tau} : p \in \mathcal{P}), \big((m+\alpha_1)^{-i\tau} : m \in \mathbb{N}_0\big), \dots, \big((m+\alpha_r)^{-i\tau} : m \in \mathbb{N}_0\big)\big) =$$
$$= \underline{\zeta}_{n,v}(\hat{s}+i\tau, s+i\tau, \underline{\alpha}; \underline{\mathfrak{a}}, F).$$

Hence, $P_{T,n,v} = Q_T h_{n,v}^{-1}$, where Q_T is the measure from Lemma 3.3. This, the continuity of the function $h_{n,v}$ and Lemma 1.5 together with Lemma 3.3 show that the measure $P_{T,n,v}$ converges weakly to $P_{n,v} = \underline{m}_H h_{n,v}^{-1}$ as $T \to \infty$.

Now let the function $g_{n,v} : \underline{\Omega} \to H^v(D_\kappa, D)$ be given by the formula $g_{n,v}(\underline{\omega}) = h_{n,v}(\underline{\omega}\,\underline{\omega}_0)$. Then the above arguments show that the measure $Q_{T,n,v}$ converges weakly to the measure $\underline{m}_H g_{nv}^{-1}$ as $T \to \infty$. However, the invariance of the Haar measure $\underline{m}_H$ implies the equality $\underline{m}_H h_{n,v}^{-1} = \underline{m}_H g_{n,v}^{-1}$. This proves the lemma

Let $\{\hat{K}_k : k \in \mathbb{N}\}$ be a sequence of compact subsets of D_κ such that

$$D_\kappa = \bigcup_{k=1}^{\infty} \hat{K}_k,$$

$\hat{K}_l \subset \hat{K}_{l+1}$ for all $l \in \mathbb{N}$, and, for every compact $\hat{K} \subset D_\kappa$, there exist l such that $\hat{K} \subset \hat{K}_l$. For $\hat{f}, \hat{g} \in H(D_\kappa)$, define

$$\hat{\rho}(\hat{f}, \hat{g}) = \sum_{k=1}^{\infty} 2^{-k} \frac{\sup_{s \in \hat{K}_k} |\hat{f}(s) - \hat{g}(s)|}{1 + \sup_{s \in \hat{K}_k} |\hat{f}(s) - \hat{g}(s)|}.$$

Then $\hat{\rho}(\hat{f}, \hat{g})$ is a metric on $H(D_\kappa)$ which induces the topology of uniform convergence on compacta. For $\underline{f} = (\hat{f}, f_{11}, \dots, f_{1l_1}, \dots, f_{r1}, \dots, f_{rl_r})$, $\underline{g} = (\hat{g}, g_{11}, \dots, g_{1l_1}, \dots, g_{r1}, \dots, g_{rl_r}) \in H^v(D_\kappa, D)$, define

$$\rho_v(\underline{f}, \underline{g}) = \max\left(\hat{\rho}(\widehat{f}, \hat{g}), \max_{1 \le j \le r} \max_{1 \le l \le l_j} \rho(f_{jl}, g_{jl})\right).$$

Then we have that ρ_v is a metric on $H^v(D_\kappa, D)$ inducing its topology.

Having the metric on $H^v(D_\kappa, D)$, we can approximate $\underline{\zeta}_v(\hat{s}, s, \underline{\alpha}; \underline{\mathfrak{a}}, F)$ by $\underline{\zeta}_{n,v}(\hat{s}, s, \underline{\alpha}; \underline{\mathfrak{a}}, F)$, and $\underline{\zeta}_v(\hat{s}, s, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{a}}, F)$ by $\underline{\zeta}_{n,v}(\hat{s}, s, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{a}}, F)$.

Lemma 4.4. *The relation*

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T} \int_0^T \underline{\rho}_v \Big(\underline{\zeta}_v(\hat{s}+i\tau, s+i\tau, \underline{\alpha}; \underline{\mathfrak{a}}, F), \underline{\zeta}_{n,v}(\hat{s}+i\tau, s+i\tau, \underline{\alpha}; \underline{\mathfrak{a}}, F) \Big) \mathrm{d}\tau = 0.$$

holds.

Proof. In [29], it was obtained that, for every compact subset $K \subset D_\kappa$,

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T} \int_0^T \sup_{s\in K} |\zeta(s+i\tau, F) - \zeta_n(s+i\tau, F)| \mathrm{d}\tau = 0.$$

Hence, we have that

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T} \int_0^T \hat{\rho}\left(\zeta(\hat{s}+i\tau, F), \zeta_n(\hat{s}+i\tau, F)\right) \mathrm{d}\tau = 0. \tag{4.1}$$

From the assertion of type of Lemma 2.4, it follows that

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T} \int_0^T \max_{1\le j\le r} \max_{1\le l\le l_j} \rho\left(\zeta(s+i\tau, \alpha_j; \mathfrak{a}_{jl}), \zeta_n(s+i\tau, \alpha_j; \mathfrak{a}_{jl}\right) \mathrm{d}\tau = 0.$$

This, (4.1) and the definition of the metric ρ_v prove the lemma.

Lemma 4.5. *Suppose that the numbers $\alpha_1, \dots, \alpha_r$ are algebraically independent over $\mathbb{Q}$. Then, for almost all $\underline{\omega} \in \underline{\Omega}$,*

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T} \int_0^T \underline{\rho}_v \left(\underline{\zeta}_v(\hat{s}+i\tau, s+i\tau, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{a}}, F), \underline{\zeta}_v(\hat{s}+i\tau, s+i\tau, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{a}}, F) \right) \mathrm{d}\tau = 0.$$

Proof. In [15], it was proved that, for every compact subset $K \subset D_\kappa$,

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T} \int_0^T \sup_{s\in K} |\zeta(s+i\tau, \hat{\omega}, F) - \zeta_n(s+i\tau, \hat{\omega}, F)| \mathrm{d}\tau = 0$$

for almost all $\hat{\omega} \in \hat{\Omega}$. From this, we obtain that, for almost all $\hat{\omega} \in \hat{\Omega}$,

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T} \int_0^T \hat{\rho}\left(\zeta(\hat{s}+i\tau, \hat{\omega}, F), \zeta_n(\hat{s}+i\tau, \hat{\omega}, F)\right) \mathrm{d}\tau = 0. \tag{4.2}$$

The assertion of type of Lemma 2.5 yields the relation

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T} \int_0^T \max_{1\le j\le r} \max_{1\le l\le l_j} \rho\left(\zeta(s+i\tau, \alpha_j, \omega_j; \mathfrak{a}_{jl}), \zeta_n(s+i\tau, \alpha_j, \omega_j; \mathfrak{a}_{jl})\right) \mathrm{d}\tau = 0 \tag{4.3}$$

for almost all $(\omega_1, ..., \omega_r) \in \Omega_1 \times \cdots \times \Omega_r$. Since the measure $\underline{m}_H$ is the product of the Haar measures on $(\hat{\Omega}, \mathfrak{B}(\hat{\Omega}))$, and on $(\Omega_1 \times \cdots \times \Omega_r, \mathfrak{B}(\Omega_1 \times \cdots \times \Omega_r))$, relations (4.2), (4.3) and the definition of the metric ρ_v imply, for almost all $\underline{\omega} \in \underline{\Omega}$, the equality of the lemma.

For $\underline{\omega} \in \underline{\Omega}$, define one more probability measure

$$\hat{P}_{T,v}(A) = \frac{1}{T}\text{meas}\left\{\tau \in [0,T] : \underline{\zeta}_v(\hat{s}+i\tau, s+i\tau, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{a}}, F) \in A\right\}, \quad A \in \mathfrak{B}(H^v(D_\kappa, D)).$$

Lemma 4.6. *Suppose that the numbers $\alpha_1, \ldots, \alpha_r$ are algebraically independent over $\mathbb{Q}$. Then $P_{T,v}$ and $\widehat{P}_{T,v}$ both converge weakly to the same probability measure P_v on $(H^v(D_\kappa, D), \mathfrak{B}(H^v(D_\kappa, D)))$ as $T \to \infty$.*

Proof. We follow the proof of Lemma 3.7. On the probability space $(\widetilde{\Omega}, \mathfrak{B}(\widetilde{\Omega}), \mathbb{P})$, define the $H^v(D_\kappa, D)$-valued random element $\underline{X}_{T,n,v}$ by the formula

$$\begin{aligned}\underline{X}_{T,n,v} = \underline{X}_{T,n,v}(\hat{s}, s) = \big(&X_{T,n}(\hat{s}), X_{T,n,1,1}(s), \ldots, X_{T,n,1,l_1}(s), \ldots,\\ &X_{T,n,r,1}(s), \ldots, X_{T,n,r,l_r}(s)\big) = \underline{\zeta}_{n,v}(\hat{s}+i\theta T, s+i\theta T, \underline{\alpha}; \underline{\mathfrak{a}}, F).\end{aligned}$$

Then Lemma 4.3 implies the relation

$$\underline{X}_{T,n,v} \xrightarrow[T\to\infty]{\mathfrak{D}} \underline{X}_{n,v}, \tag{4.4}$$

where

$$\underline{X}_{n,v} = \underline{X}_{n,v}(\hat{s}, s) = (X_n(\hat{s}), X_{n,1,1}(s), \ldots, X_{n,1,l_1}(s), \ldots, X_{n,r,1}(s), \ldots, X_{n,r,l_r}(s))$$

is an $H^v(D_\kappa, D)$-valued random element with the distribution $P_{n,v}$ in the notation of Lemma 4.3. We have mentioned above that the series for $\zeta_n(s,F)$ converges absolutely for $\sigma > \frac{\kappa}{2}$. Therefore, for $\sigma > \frac{\kappa}{2}$

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T |\zeta_n(\sigma+it, F)|^2 \, \mathrm{d}t = \sum_{m=1}^{\infty} \frac{c^2(m)v_n(m)}{m^{2\sigma}} \leq \sum_{m=1}^{\infty} \frac{c^2(m)}{m^{2\sigma}} < \infty$$

for all $n \in \mathbb{N}$, because of the Deligne estimate [11]

$$c(m) = O\left(m^{\frac{\kappa-1}{2}}\right).$$

Thus, a simple application of the Cauchy integral formula lead to the inequality

$$\limsup_{T\to\infty} \frac{1}{T}\int_0^T \sup_{s\in\hat{K}_k} |\zeta_n(s+i\tau, F)| \, \mathrm{d}\tau \leq \hat{C}_k \left(\sum_{m=1}^{\infty} \frac{c^2(m)}{m^{2\hat{\sigma}_k}}\right)^{\frac{1}{2}}, \quad n \in \mathbb{N}, \tag{4.5}$$

with some $\hat{C}_k > 0$ and $\hat{\sigma}_k > \frac{\kappa}{2}$. We set

$$\hat{R}_k = \left(\sum_{m=1}^{\infty} \frac{c^2(m)}{m^{2\hat{\sigma}_k}}\right)^{\frac{1}{2}},$$

and take $\hat{M}_k = \hat{C}_k \hat{R}_k 2^{k+1}\varepsilon^{-1}$ and $M_{jlk} = C_k R_{jlk} 2^{k+v}\varepsilon^{-1}$, $\varepsilon > 0$, where we preserve the notation of Section 2.3. Now, from (2.3) and (4.5), we obtain that

$$\limsup_{T\to\infty} \mathbb{P}\left(\sup_{\hat{s}\in\hat{K}_k} |X_{T,n}(\hat{s})| > \hat{M}_k \text{ or } \sup_{s\in K_k} |X_{T,n,j,l}(s)| > M_{jlk} \text{ for some } (j,l)\right)$$

$$
\begin{aligned}
&\leq \limsup_{T\to\infty} \mathbb{P}\left(\sup_{\hat{s}\in\hat{K}_k} |X_{T,n}(\hat{s})| > \hat{M}_k\right) + \sum_{j=1}^{r}\sum_{l=1}^{l_j} \limsup_{T\to\infty} \mathbb{P}\left(\sup_{\hat{s}\in\hat{K}_k} |X_{T,n,j,l}(s)| > M_{jlk}\right) \\
&\leq \frac{1}{\hat{M}_k} \sup_{n\in\mathbb{N}} \limsup_{T\to\infty} \frac{1}{T}\int_0^T \sup_{\hat{s}\in\hat{K}_k} |\zeta_n(\hat{s}+i\tau, F)|\,\mathrm{d}\tau \\
&+\sum_{j=1}^{r}\sum_{l=1}^{l_j} \frac{1}{M_{jlk}} \sup_{n\in\mathbb{N}_0} \limsup_{T\to\infty} \frac{1}{T}\int_0^T \sup_{s\in K_k} |\zeta_n(s+i\tau, \alpha_j; \mathfrak{a}_{jl})|\,\mathrm{d}\tau \\
&\leq \frac{\hat{C}_k\hat{R}_k}{\hat{M}_k} + \sum_{j=1}^{r}\sum_{l=1}^{l_j} \frac{C_k R_{jlk}}{M_{jlk}} = \frac{\varepsilon}{2^{k+1}} + \frac{\varepsilon}{2^{k+v}} \sum_{j=1}^{r}\sum_{l=1}^{l_j} 1 \leq \frac{\varepsilon}{2^{k+1}} + \frac{\varepsilon}{2^{k+1}} = \frac{\varepsilon}{2^k}.
\end{aligned}
$$

Using relation (4.4), hence we deduce that, for all $n\in\mathbb{N}$,

$$
P\left(\sup_{\hat{s}\in\hat{K}_k} |X_n(\hat{s})| > \hat{M}_k \text{ or } \sup_{s\in K_k} |X_{n,j,l}(s)| > M_{jlk} \text{ for some } (j,l)\right) \leq \frac{\varepsilon}{2^k}. \tag{4.6}
$$

Define a set

$$
\begin{aligned}
H_\varepsilon^v \quad = \quad &\Bigg\{ (g, g_{11}, \ldots, g_{1l_1}, \ldots, g_{r1}, \ldots, g_{rl_r}) \in H^v(D_\kappa, D) : \sup_{\hat{s}\in\hat{K}_k} |g(\hat{s})| \leq \hat{M}_k, \\
&\sup_{s\in K_k} |g_{jl}(s)| \leq M_{jlk},\ j=1,\ldots,r,\ l=1,\ldots,l_j,\ k\in\mathbb{N}\Bigg\}.
\end{aligned}
$$

Then H_ε^v is a compact subset of the space $H^v(D_\kappa, D)$, and, by (4.6),

$$
\mathbb{P}\left(\underline{X}_{n,v}(\hat{s}, s) \in H_\varepsilon^v\right) \geq 1 - \varepsilon \sum_{k=1}^{\infty} \frac{1}{2^k} = 1 - \varepsilon
$$

for all $n\in\mathbb{N}$. Thus, by the definition of the random element $\underline{X}_n(\hat{s}, s)$,

$$
P_{n,v}\left(H_\varepsilon^v\right) \geq 1 - \varepsilon
$$

for all $n\in\mathbb{N}$. This means that the family of probability measures $\{P_{n,v} :\ n\in\mathbb{N}\}$ is tight,and, by Lemma 1.9, it is relatively compact. Therefore, there exists a subsequence $\{P_{n_k,v} :\ k\in\mathbb{N}\} \subset \{P_{n,v} : n\in\mathbb{N}\}$ such that $P_{n_k,v}$ converges weakly to a certain probability measure P_v on $(H^v(D_\kappa, D), \mathfrak{B}(H^v(D_\kappa, D)))$ as $k\to\infty$. This can be written in the form

$$
\underline{X}_{n_k,v} \xrightarrow[k\to\infty]{\mathfrak{D}} P_v. \tag{4.7}
$$

Define one more $H^v(D_\kappa, D)$-valued random element $\underline{X}_{T,v} = \underline{X}_{T,v}(\hat{s}, s)$ by the formula

$$
\underline{X}_{T,v}(\hat{s}, s) = \underline{\zeta}(\hat{s}+i\theta T, s+i\theta T, \underline{\alpha}; \underline{\mathfrak{a}}, F).
$$

Then Lemma 4.4 shows that, for every $\varepsilon > 0$,

$$
\begin{aligned}
&\lim_{n\to\infty} \limsup_{T\to\infty} \mathbb{P}\bigg(\underline{\rho}_v\left(\underline{X}_{T,n,v}(\hat{s}, s), \underline{X}_{T,v}(\hat{s}, s)\right) \geq \varepsilon\bigg) = \\
&\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T}\mathrm{meas}\bigg\{\tau\in[0,T] :\ \underline{\rho}_v\left(\underline{\zeta}(\hat{s}+i\tau, s+i\tau, \underline{\alpha}; \underline{\mathfrak{a}}, F),\right. \\
&\left.\underline{\zeta}_{n,v}(\hat{s}+i\tau, s+i\tau, \underline{\alpha}; \underline{\mathfrak{a}})\right) \geq \varepsilon\bigg\} \leq \lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T\varepsilon}\int_0^T \underline{\rho}_v\bigg(\underline{\zeta}_v(\hat{s}+i\tau,
\end{aligned}
$$

$$s+i\tau,\underline{\alpha};\underline{\mathfrak{a}},F),\underline{\zeta}_{n,v}(\hat{s}+i\tau,s+i\tau,\underline{\alpha};\underline{\mathfrak{a}},F)\Big)\mathrm{d}\tau=0.$$

The latter relation, (4.4), (4.7) and Lemma 1.11 yield

$$\underline{X}_{T,v}\xrightarrow[T\to\infty]{\mathfrak{D}}P_v, \tag{4.8}$$

thus, $P_{T,v}$ converges weakly to P_v as $T\to\infty$. The relation (4.8) also shows that the measure P_v is independent of the choice of the sequence $\{P_{n_k,v}:\ k\in\mathbb{N}\}$, and this gives the relation

$$\underline{X}_{n,v}\xrightarrow[n\to\infty]{\mathfrak{D}}P_v. \tag{4.9}$$

It remains to show that the measure $\hat{P}_{T,v}$ also converges weakly to P_v as $T\to\infty$. We set

$$\underline{\hat{X}}_{T,n,v}(\hat{s},s)=\underline{\zeta}_{n,v}(\hat{s}+i\theta T,s+i\theta T,\underline{\alpha},\underline{\omega};\underline{\mathfrak{a}},F)$$

and

$$\underline{\hat{X}}_{T,v}(\hat{s},s)=\underline{\zeta}_{v}(\hat{s}+i\theta T,s+i\theta T,\underline{\alpha},\underline{\omega};\underline{\mathfrak{a}},F).$$

Then the above arguments, together with Lemmas 4.3 and 4.5, and relation (4.9) applied for the random elements $\underline{\hat{X}}_{T,n,v}(\hat{s},s)$ and $\underline{\hat{X}}_{T,v}(\hat{s},s)$ show that the measure $\hat{P}_{T,v}$ also converges weakly to P_v as $T\to\infty$. The lemma is proved.

Proof of Theorem 4.2. In order to prove Theorem 4.2, it suffices to show that the limit measure P_v in Lemma 4.6 is the distribution of the random element $\underline{\zeta}_v(\hat{s},s,\underline{\alpha},\underline{\omega};\underline{\mathfrak{a}},F)$. For this, we repeat the proof of Theorem 3.2. Let A be a fixed continuity set of the limit measure P_v in Lemma 4.6. Then Lemmas 4.6 and 1.12 imply the relation

$$\lim_{T\to\infty}\frac{1}{T}\mathrm{meas}\Big\{\tau\in[0,T]:\ \underline{\zeta}_v(\hat{s}+i\tau,s+i\tau,\underline{\alpha},\underline{\omega};\underline{\mathfrak{a}},F)\in A\Big\}=P_v(A). \tag{4.10}$$

On the probability space $(\underline{\Omega},\mathfrak{B}(\underline{\Omega}),\underline{m}_H)$, define the random variable ξ_v by the formula

$$\xi_v(\underline{\omega})=\begin{cases}1 & \text{if}\quad \underline{\zeta}_v(\hat{s},s,\underline{\alpha},\underline{\omega};\underline{\mathfrak{a}},F)\in A,\\ 0 & \text{otherwise.}\end{cases}$$

Then we have that

$$\mathbb{E}\xi_v=\underline{m}_H\Big(\underline{\omega}\in\underline{\Omega}:\underline{\zeta}_v(\hat{s},s,\underline{\alpha},\underline{\omega};\underline{\mathfrak{a}},F)\in A\Big)=P_{\underline{\zeta}_v}(A). \tag{4.11}$$

Let $\{\underline{\Phi}_\tau:\ \tau\in\mathbb{R}\}$ be the same group as in the proof of Theorem 3.2. Then Lemma 3.8 implies the ergodicity of the random process $\xi_v(\underline{\Phi}_\tau(\underline{\omega}))$. This together with Lemma 1.13 shows that, for almost all $\underline{\omega}\in\underline{\Omega}$,

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T\xi_v\left(\underline{\Phi}_\tau(\underline{\omega})\right)\mathrm{d}\tau=\mathbb{E}\xi_v. \tag{4.12}$$

However, on the other hand, the definitions of ξ_v and $\underline{\Phi}_\tau$ imply the equality

$$\frac{1}{T}\int_0^T \xi_v\left(\underline{\Phi}_\tau(\underline{\omega})\right) \mathrm{d}\tau = \frac{1}{T}\text{meas}\left\{\tau \in [0,T]:\ \underline{\zeta}_v(\hat{s}+i\tau, s+i\tau, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{a}}, F) \in A\right\}.$$

Therefore, in view of (4.11) and (4.12), we have that, for almost all $\underline{\omega} \in \underline{\Omega}$,

$$\lim_{T\to\infty}\frac{1}{T}\text{meas}\left\{\tau \in [0,T]:\ \underline{\zeta}_v(\hat{s}+i\tau, s+i\tau, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{a}}, F) \in A\right\} = P_{\underline{\zeta}_v}(A).$$

This and (4.10) show that $P_v(A) = P_{\underline{\zeta}_v}(A)$. Since A is an arbitrary continuity set of P_v, hence $P_v(A) = P_{\underline{\zeta}_v}(A)$ for all continuity sets of P_v. Since all continuity sets form a determining class, we obtain that $P_v(A) = P_{\underline{\zeta}_v}(A)$ for all $A \in \mathfrak{B}(H^v(D_\kappa, D))$. This complete the proof of Theorem 4.2.

4.3. Support of the limit measure

Define

$$S_\kappa = \{g \in H(D_\kappa):\ g(s) \neq 0 \ \text{ or } \ g(s) \equiv 0\}.$$

We recall that $v_1 = v - 1$.

Theorem 4.7. *Suppose that the numbers $\alpha_1, \dots, \alpha_r$ are algebraically independent over $\mathbb{Q}$, and that* $\text{rank}(B_j) = l_j$, *$j = 1\dots, r$. Then the support of the measure $P_{\underline{\zeta}_v}$ is the set $S_\kappa \times H^{v_1}(D)$.*

Proof. By the definition,

$$H^v(D_\kappa, D) = H(D_\kappa) \times H^{v_1}(D).$$

In virtue of the separability of the spaces $H(D_\kappa)$ and $H^{v_1}(D)$, the equality

$$\mathfrak{B}(H^v(D_\kappa, D)) = \mathfrak{B}(H(D_\kappa)) \times \mathfrak{B}(H^{v_1}(D))$$

holds. Hence, it suffices to consider the measure $P_{\underline{\zeta}_v}$ for $A = B \times C$, where $B \in \mathfrak{B}(H(D_\kappa))$ and $C \in \mathfrak{B}(H^{v_1}(D))$. Therefore, using the same notation as in the proof of Theorem 3.9, we find that, for $A = B \times C \in \mathfrak{B}(H^v(D_\kappa, D))$,

$$\begin{aligned}
&P_{\underline{\zeta}_v}(A) = \underline{m}_H\left(\underline{\omega} \in \underline{\Omega}:\ \underline{\zeta}_v(\hat{s}, s, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{a}}, F) \in A\right)\\
&= \underline{m}_H\Big(\underline{\omega} \in \underline{\Omega}:\ \zeta(\hat{s}, \hat{\omega}, F) \in B, \big(\zeta(s, \alpha_1, \omega_1; \mathfrak{a}_{11}), \dots, \zeta(s, \alpha_1, \omega_1; \mathfrak{a}_{1l_1}), \dots,\\
&\zeta(s, \alpha_r, \omega_r; \mathfrak{a}_{r1}), \dots, \zeta(s, \alpha_r, \omega_r; \mathfrak{a}_{rl_r})\big) \in C\Big)\\
&= \hat{m}_H\left(\hat{\omega} \in \hat{\Omega}:\ \zeta(\hat{s}, \hat{\omega}, F) \in B\right) m_H^r\Big((\omega_1, \dots, \omega_r) \in \Omega^r:\ \big(\zeta(s, \alpha_1, \omega_1; \mathfrak{a}_{11}), \dots,\\
&\zeta(s, \alpha_1, \omega_1; \mathfrak{a}_{1l_1}), \dots, \zeta(s, \alpha_r, \omega_r; \mathfrak{a}_{r1}), \dots, \zeta(s, \alpha_r, \omega_r; \mathfrak{a}_{rl_r})\big) \in C\Big).
\end{aligned} \tag{4.13}$$

In [30], it was obtained that the support of the random element $\zeta(\hat{s}, \hat{\omega}, F)$ is the set S_κ, i. e., S_κ is a minimal closed subset of $H(D_\kappa)$ such that

$$\hat{m}_H\left(\hat{\omega} \in \hat{\Omega}:\ \zeta(\hat{s}, \hat{\omega}, F) \in S_\kappa\right) = 1. \tag{4.14}$$

As in the proof of Theorem 3.9, we have that the support of the random element $(\zeta(s,\alpha_1,\omega_1;\mathfrak{a}_{11})$ $,\ldots,\zeta(s,\alpha_1,\omega_1;\mathfrak{a}_{1l_1}),\ldots,\zeta(s,\alpha_r,\omega_r;\mathfrak{a}_{r1}),\ldots,\zeta(s,\alpha_r,\omega_r;\mathfrak{a}_{rl_r}))$ is the space $H^{v_1}(D)$. Thus, from this, (4.15) and (4.14), the theorem follows.

4.4. Proof of Theorem 4.1

A proof of Theorem 4.1 differs from that of Theorem 3.1 only by details of the notation.

By Lemma 1.16, there exist polynomials $p(s)$ and $p_{jl}(s)$, $j=1,\ldots,r$, $l=1,\ldots,l_j$, such that

$$\sup_{s\in K}|f(s)-p(s)|<\frac{\varepsilon}{4} \tag{4.15}$$

and

$$\sup_{1\le j\le r}\sup_{1\le l\le l_j}\sup_{s\in K_{jl}}|f_{jl}(s)-p_{jl}(s)|<\frac{\varepsilon}{2}. \tag{4.16}$$

Since $f(s)\neq 0$ on K, $p(s)\neq 0$ on K as well if ε is small enough. Thus, on K we can define a continuous branch of the logarithm $\log p(s)$ which will be an analytic function in the interior of K. Therefore, by Lemma 1.16, there exists a polynomial $q(s)$ such that

$$\sup_{s\in K}\left|p(s)-\mathrm{e}^{q(s)}\right|<\frac{\varepsilon}{4}.$$

This together with (4.16) shows that

$$\sup_{s\in K}\left|f(s)-\mathrm{e}^{q(s)}\right|<\frac{\varepsilon}{2}. \tag{4.17}$$

Define

$$\begin{aligned} G \quad = \quad & \Big\{(g,g_{11},\ldots,g_{1l_1},\ldots,g_{r1},\ldots,g_{rl_r})\in H^v(D_\kappa,D): \\ & \sup_{\hat{s}\in K}|g(\hat{s})-\mathrm{e}^{q(\hat{s})}|<\frac{\varepsilon}{2},\ \sup_{1\le j\le r}\sup_{1\le l\le l_j}\sup_{s\in K_{jl}}|g_{jl}(s)-p_{jl}(s)|<\frac{\varepsilon}{2}\Big\}. \end{aligned}$$

In view of Theorem 4.7, the collection $\big(\mathrm{e}^{q(s)},p_{11}(s),\ldots,p_{1l_1}(s),\ldots,p_{r1}(s),\ldots,p_{rl_r}(s)\big)$ is an element of the support of the measure $P_{\underline{\zeta}_v}$. Since the set G is open, hence we have that $P_{\underline{\zeta}_v}(G)>0$. Therefore, Theorem 4.2 and Lemma 1.17 yield

$$\begin{aligned} &\liminf_{T\to\infty}\frac{1}{T}\mathrm{meas}\Big\{\tau\in[0,T]:\ \underline{\zeta}_v(\hat{s}+i\tau,s+i\tau,\underline{\alpha};\underline{\mathfrak{a}},F)\in G\Big\}= \\ &\liminf_{T\to\infty}\frac{1}{T}\mathrm{meas}\Big\{\tau\in[0,T]:\ \sup_{s\in K}|\zeta(s+i\tau,F)-e^{q(s)}|<\frac{\varepsilon}{2}, \\ &\sup_{1\le j\le r}\sup_{1\le l\le l_j}\sup_{s\in K_{jl}}|\zeta(s+i\tau,\alpha_j;\mathfrak{a}_{jl})-p_{jl}(s)|<\frac{\varepsilon}{2}\Big\}\ge P_{\underline{\zeta}_v}(G)>0. \end{aligned} \tag{4.18}$$

However, in view of (4.16) and (4.17),

$$\Big\{\tau\in[0,T]:\ \sup_{s\in K}\left|\zeta(s+i\tau,F)-e^{q(s)}\right|<\frac{\varepsilon}{2},\ \sup_{1\le j\le r}\sup_{1\le l\le l_j}\sup_{s\in K_{jl}}\left|\zeta(s+i\tau,\alpha_j;\mathfrak{a}_{jl})-\right.$$

$$p_{jl}(s)\big| < \frac{\varepsilon}{2}\Bigg\} \subset \Bigg\{\tau \in [0,T]:\ \sup_{s\in K}\big|\zeta(s+i\tau, F) - f(s)\big| < \varepsilon,$$

$$\sup_{1\le j\le r}\sup_{1\le l\le l_j}\sup_{s\in K_{jl}}\big|\zeta(s+i\tau, \alpha_j; \mathfrak{a}_{jl}) - f_{jl}(s)\big| < \varepsilon\Bigg\}$$

This and (4.18) lead to the assertion of Theorem 4.1.

Conclusions

In the thesis, the following approximation properties of periodic Hurwitz zeta-functions were established:

1. The periodic Hurwitz zeta-functions $\zeta(s, \alpha_1; \mathfrak{a}_1), \dots, \zeta(s, \alpha_r; \mathfrak{a}_r)$ with parameters $\alpha_1, \dots, \alpha_r$ such that the set $L(\alpha_1, \dots, \alpha_r) = \{\log(m + \alpha_j) : \ m \in \mathbb{N}_0, \quad j = 1, \dots, r\}$ is linearly independent over field of rational numbers $\mathbb{Q}$ are jointly universal.

2. The periodic Hurwitz zeta-functions $\zeta(s, \alpha_1; \mathfrak{a}_{11}), \dots, \zeta(s, \alpha_1; \mathfrak{a}_{1l_1}), \dots, \zeta(s, \alpha_r; \mathfrak{a}_{r1}), \dots, \zeta(s, \alpha_r; \mathfrak{a}_{rl_r})$ with the set $L(\alpha_1, \dots, \alpha_r)$ linearly independent over $\mathbb{Q}$ and a rank condition related only to each fixed α_j, $j = 1, \dots, r$, are jointly universal.

3. The Riemann zeta-function $\zeta(s)$ and periodic Hurwitz zeta-functions $\zeta(s, \alpha_1; \mathfrak{a}_{11}), \dots, \zeta(s, \alpha_1; \mathfrak{a}_{1l_1}) \dots, \zeta(s, \alpha_r; \mathfrak{a}_{r1}), \dots, \zeta(s, \alpha_r; \mathfrak{a}_{rl_r})$ with algebraically independent over $\mathbb{Q}$ parameters $\alpha_1, \dots, \alpha_r$ and a rank condition related only to each fixed α_j, $j = 1, \dots, r$, are jointly universal.

4. The zeta-function $\zeta(s, F)$ attached to a normalized Hecke eigen cusp form F for the full modular group and periodic Hurwitz zeta-functions $\zeta(s, \alpha_1; \mathfrak{a}_{11}), \dots, \zeta(s, \alpha_1; \mathfrak{a}_{1l_1}), \dots, \zeta(s, \alpha_r; \mathfrak{a}_{r1}), \dots, \zeta(s, \alpha_r; \mathfrak{a}_{rl_r})$ with algebraically independent over $\mathbb{Q}$ parameters $\alpha_1, \dots, \alpha_r$ and a rank condition related only to each fixed α_j, $j = 1, \dots, r$, are jointly universal.

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Notation

j, k, l, m, n	natural numbers
p	prime number
(m, n)	greatest common divisor of natural m and n
$\mathcal{P}$	set of all prime numbers
$\mathbb{N}$	set of all natural numbers
$\mathbb{N}_0$	$\mathbb{N} \cup \{0\}$
$\mathbb{Z}$	set of all integer numbers
$\mathbb{R}$	set of all real numbers
$\mathbb{C}$	set of all complex numbers
i	imaginary unity: $i = \sqrt{-1}$
$s = \sigma + it$	complex variable
$\Re s = \sigma$	real part of s
$\Im s = t$	imaginary part of s
$\bigoplus_m A_m$	direct sum of sets A_m
$A \times B$	Cartesian product of the sets A and B
A^m	Cartesian product of m copies of the set A
$\text{meas}\{A\}$	Lebesgue measure of the set A
$H(D)$	space of analytic functions on D
$\xrightarrow{\mathcal{D}}$	convergence in distribution
$\mathcal{B}(S)$	class of Borel sets of the space S
χ	Dirichlet character
$L(s, \chi)$	Dirichlet L-function
$SL(2, \mathbb{Z})$	full modular group
$F(z)$	cusp form
$f(x) = O(g(x))$, $x \in I$	means that $\lvert f(x)\rvert \leq Cg(x)$, $x \in I$

$\zeta(s)$ Riemann zeta-function defined by

$$\zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m^s}, \text{ for } \sigma > 1,$$

and by analytic continuation elsewhere

$\zeta(s; \mathfrak{a})$ periodic zeta-function defined by

$$\zeta(s; \mathfrak{a}) = \sum_{m=1}^{\infty} \frac{a_m}{m^s}, \text{ for } \sigma > 1,$$

and by analytic continuation elsewhere

$\zeta(s, \alpha)$ Hurwitz zeta-function defined by

$$\zeta(s, \alpha) = \sum_{m=1}^{\infty} \frac{1}{m+\alpha^s}, \text{ for } \sigma > 1,$$

and by analytic continuation elsewhere

$\zeta(s, \alpha; \mathfrak{a})$ periodic Hurwitz zeta-function defined by

$$\zeta(s, \alpha; \mathfrak{a}) = \sum_{m=0}^{\infty} \frac{a_m}{(m+1)^s}, \text{ for } \sigma > 1,$$

and by analytic continuation elsewhere

$\Gamma(s)$ Euler gamma-function defined by

$$\Gamma(s) = \int_0^{\infty} \mathrm{e}^{-\mathrm{x}} \mathrm{x}^{\mathrm{s}-1} \mathrm{dx} \text{ for } \sigma > 0$$

and by analytic continuation elsewhere