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INVESTIGATIONS OF THE ACCURACY OF APPROXIMATIONS OF SEMIGROUPS

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PUSGRUPIŲ APROKSIMACIJŲ TIKSLUMO TYRIMAI

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Abstract

In this thesis we investigate the convergence of Euler's and Yosida approximations of operator semigroups. We obtain asymptotic expansions for Euler's approximations of semigroups with optimal bounds for the remainder terms. We provide various explicit formulas for the coefficients for these expansions. For Yosida approximations of semigroups we obtain two optimal error bounds with optimal constants. We also construct asymptotic expansions for Yosida approximations of semigroups and provide optimal bounds for the remainder terms of these expansions.

Reziumė

Disertacijoje tiriamas operatorių pusgrupių Eulerio ir Josidos aproksimacijų konvergavimas. Gauti Eulerio aproksimacijų asimptotiniai skleidiniai ir optimalūs liekamųjų narių įverčiai. Taip pat pateiktos įvairios šių skleidinių koeficientų analizinės išraiškos. Josidos aproksimacijoms buvo rasti du optimalūs konvergavimo greičio įverčiai su optimaliomis konstantomis. Taip pat gauti Josidos aproksimacijų asimptotiniai skleidiniai ir liekamųjų narių įverčiai.

Contents

Introduction

The semigroup theory plays an important role in many research areas, one of which is the theory of evolution equations. It is well known that a strongly continuous semigroup $S(t) = e^{tA}$ gives the solution $u(t) = S(t)x$ of the abstract Cauchy problem

$$\begin{cases} \frac{du(t)}{dt} = Au(t), \qquad t > 0, \\ u(0) = x. \end{cases}$$

Since A is usually an unbounded linear operator, it is often difficult to study the semigroup $S(t)$ or the solution of the corresponding Cauchy problem directly. For this purpose, various approximations are used, including Euler's or Yosida approximations of semigroups. The question of convergence rate of these approximations to the given semigroup arises naturally and was investigated in many articles (e.g. [14], [32], [11], [6]). In our work we applied a simple method to obtain optimal error bounds and asymptotic expansions for some approximations of semigroups.

The aim of this thesis is to investigate the asymptotic behaviour of some approximations for semigroups. In particular, we provide asymptotic expansions for Euler's approximations of bounded holomorphic semigroups and obtain optimal bounds for the remainder terms of the expansions. For Euler's approximations in general Banach algebras, we provide explicit formulas for asymptotic expansions using another approach. We also investigate the convergence of Yosida approximations. In case of bounded holomorphic semigroups of contractions we provide two optimal error bounds with optimal constants. We also provide asymptotic expansions and optimal bounds for the remainder terms of these expansions.

To obtain asymptotic expansions we use an approach which was used by Bentkus in [5] for analysis of errors in the Central Limit Theorem and in approximations by accompanying laws and applied by Bentkus and Paulauskas in [7] to derive optimal convergence rates in Chernoff–type lemmas and Euler's approximations of semigroups. We use this method to obtain optimal error bounds for Yosida approximations as well. Another interesting approach to analysis of error bounds and asymptotic expansions for Euler's approximations of semigroups in general Banach algebras was proposed by Bentkus in [6]. This method is based on applications of the Fourier–Laplace transforms and a reduction of the problem to the convergence rates and asymptotic expansions in the Law of Large Numbers.

The thesis consists of an introduction, 4 chapters and bibliography.

1. In the first chapter we give some results from the general theory of semigroups. Section 1.1 is dedicated to the basic definitions and properties of operator semigroups, their resolvents and generators. In section 1.2 we introduce classes of differentiable and holomorphic semigroups. Section 1.3 is devoted to approximations of semigroups and in Section 1.4 we describe the main method which we used to obtain optimal error bounds and asymptotic expansions for approximations of semigroups.

2. In the second chapter we obtain asymptotic expansions for Euler's approximations of semigroups using the method of multiplicative representations how it was proposed by Bentkus in [5]. We also obtain the so-called inverse expansions and provide optimal bounds for the remainder terms.

3. In the third chapter we present another approach to asymptotic expansions and convergence rates for Euler's approximations of semigroups. This approach was used by Bentkus in [6]. We provide explicit formulas for the coefficients and the remainder terms of the expansions obtained by using this alternative method.

4. In Chapter 4 we provide an optimal point-wise error bound and asymptotic expansions for Yosida approximations of semigroups. We also obtain the optimal bounds for the remainder terms. Using another variant of the same method we obtain another optimal bound for the error of approximation and provide the inverse asymptotic expansions with optimal remainder terms.

The results of this thesis were presented at the Conferences of Lithuanian Mathematical Society (2006, 2008). Moreover, the results of the thesis were presented at the seminar on Probability Theory and Mathematical Statistics of Institute of Mathematics and Informatics.

The main results of the thesis are published in the following papers:

1. M. Vilkienė. Optimal convergence rate for Yosida approximations of bounded holomorphic semigroups. Lithuanian Mathematical Journal, 49(2): 234—239, 2009.

2. M. Vilkienė. Asymptotic expansions for Yosida approximations of semigroups. Liet. Matem. Rink., 48/49(spec. nr.):78—83, 2008.

3. M. Vilkienė. Another approach to asymptotic expansions for Euler's approximations of semigroups. Lithuanian Mathematical Journal, 46(2): 217—232, 2006.

4. M. Vilkienė. Explicit formulas in asymptotic expansions for Euler's approximations of semigroups. Liet. Matem. Rink., 46(spec. nr.): 64—69, 2006.

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1 Semigroups of operators

1.1 Strongly continuous semigroups

Let X be a complex Banach space and let $L(X)$ be the space of bounded linear operators on X. We denote the norms on X and $L(X)$ by the same notation $\|\cdot\|$.

Definition 1.1. *A function* $S : \mathbb{R}_+ \mapsto L(X)$ *is called a semigroup if it satisfies the semigroup property*

$$S(t+s) = S(t)S(s) \tag{1.1}$$

for all $s, t \geq 0$ *and S(0)=I (I is identity operator on* X*) (see [33], Def. 1.1.1).*

Definition 1.2. *A semigroup* $S(t)$, $0 \leq t < \infty$ *of bounded linear operators on* X *is a strongly continuous semigroup if*

$$\lim_{t \downarrow 0} S(t)x = x \tag{1.2}$$

for every $x \in X$ *(see [33], Def. 1.1.2).*

The property (1.2) means that $S(t)$ is continuous in strong operator topology on X for all $t \geq 0$ (see p. 37 in [19]). Strongly continuous semigroups of bounded linear operators on X are often called semigroups of class C_0 or simply C_0 semigroups.

If semigroup is continuous in uniform operator topology on X for all $t \geq 0$ then it is called a *uniformly continuous semigroup.*

By Theorem 1.2.2 in [33] every strongly continuous semigroup $S(t)$ is exponentially bounded, i.e. there exist constants $\omega \geq 0$ and $M \geq 1$ such that

$$\|S(t)\| \leq Me^{\omega t} \qquad \text{for} \quad 0 \leq t < \infty.$$

If $\omega = 0$ then $S(t)$ is called *uniformly bounded* (or *bounded*) semigroup. If $\|S(t)\| \leq 1$ then it is called a *semigroup of contractions.*

Generator

One way to describe a strongly continuous semigroups is through its generator.

Definition 1.3. *The (infinitesimal) generator A of a strongly continuous semigroup $S(t)$ on a Banach space X is the operator*

$$Ax := \lim_{h \downarrow 0} \frac{1}{h}(S(h)x - Ix)$$

defined for every x in its domain $D(A) = \{x \in X : \lim_{h\downarrow 0}(S(h)x - Ix)/h \text{ exists}\}$ (see p. 49 in [19]).

In other words, the right derivatives at $t = 0$ of the orbit maps $S(\cdot)x$ are equal to Ax for all $x \in D(A)$. For uniformly continuous semigroups the generator A is equal to the derivative of $S(\cdot)$ at zero, i.e. $A = \dot{S}(0)$ (see p. 17 in [19]). The generator of a strongly continuous semigroup is a closed and densely defined operator that determines the semigroup *uniquely* (see p. 51 in [19]). Also, by Theorem 1.2.4 in [33], if A is the generator of the strongly continuous semigroup $S(t)$ then for $x \in D(A)$, $S(t)x \in D(A)$ and

$$\frac{d}{dt}S(t)x = AS(t)x = S(t)Ax.$$

By Theorem 1.1.2 in [33] a semigroup is uniformly continuous if and only if it's generator is a bounded linear operator. If A is a generator of a uniformly continuous semigroup $S(t)$ then we can write

$$S(t) = \sum_{n=0}^{\infty} \frac{(tA)^n}{n!}. \tag{1.3}$$

For this reason we often denote a uniformly continuous semigroup by $S(t) = \exp\{tA\}$. This notation is also widely used for strongly continuous semigroups though the generator of a strongly continuous semigroup is usually unbounded and the series (1.3) does not converge. We will justify this notation later in Section 1.3 using the so-called "exponential formulas". The notation $S(t) = \exp\{-tA\}$ is also used by some authors (see for example [7], [32], [5]). By Definition 1.3, the generator of such semigroup is understood to be a linear operator $-A$.

Resolvent

The *resolvent set* $\rho(A)$ of a linear operator A (not necessarily bounded) in X is the set of all $\lambda \in \mathbb{C}$ for which $\lambda I - A$ is invertible, i.e., $(\lambda I - A)^{-1}$ is a bounded linear operator in X. The family

$$R(\lambda) = (\lambda I - A)^{-1}, \qquad \lambda \in \rho(A)$$

is called the *resolvent* of A.

By Theorem 1.5.2 in [33], if A is the generator of a bounded strongly continuous semigroup $S(t)$, satisfying $\|S(t)\| \leq M$ $(M \geq 1)$, then the resolvent set $\rho(A)$ contains $\mathbb{R}_+$ and

$$\|R^n(\lambda)\| \leq \frac{M}{\lambda^n} \qquad \text{for} \quad \lambda > 0, \quad n = 1, 2, \dots .$$

1.2 Classes of semigroups

In this section we give the definitions and some properties of two interesting subclasses of strongly continuous semigroups.

Differentiable semigroups

For strongly continuous semigroup $S(t)$, $t \geq 0$ with generator A on X, the orbit maps $S(t)x$ are differentiable for $t \geq 0$ if $x \in D(A)$. Thus these orbits are differentiable for all $x \in X$ only if A is bounded (and $D(A) = X$). Here we introduce a class of strongly continuous semigroups for which the orbit maps are differentiable for all $x \in X$, but not for all t (see p. 109 in [19], p. 51 in [33]).

Definition 1.4. *Let $S(t)$ be a strongly continuous semigroups. We say that $S(t)$ is differentiable for $t > t_0$ if for every $x \in X$, the function $t \mapsto S(t)x$ is differentiable for $t > t_0$. A semigroup $S(t)$ is called differentiable if it is differentiable for $t > 0$.*

By Corollary 2.4.4 in [33], if $S(t)$ is a differentiable strongly continuous semigroup, then $S(t)$ is differentiable infinitely many times in the uniform operator topology for $t > 0$ and by Lemma 2.4.2 in [33] it's nth derivative satisfies $S^{(n)}(t) = A^n S(t)$ for $t > 0$ and $n = 1, 2, \dots$.

Holomorphic semigroups

Next we define the holomorphic semigroups. This class of semigroups plays an important role in the theory of evolution equations. See Section 3.7 in [1], Section 2.5 in [33], Section II.4 in [19] for many interesting properties of holomorphic semigroups.

Definition 1.5. *Let $\Sigma_\theta = \{z : |\arg z| < \theta\}$ be a sector in the complex plane for some $\theta > 0$ and for $z \in \Sigma_\theta$ let $S(z) \in L(X)$. The family $S(z)$, $z \in \Sigma_\theta$ is called a holomorphic semigroup in Σ_θ if :*

(i) the function $z \mapsto S(z)$ is analytic in Σ_θ,

(ii) $S(0) = I$ and $\lim_{z\to 0, z\in\Sigma_\theta} S(z)x = x$ for every $x \in X$,

(iii) $S(z_1 + z_2) = S(z_1)S(z_2)$ for $z_1, z_2 \in \Sigma_\theta$.

A semigroup $S(t)$, $t \geq 0$ is called holomorphic if it is holomorphic in some sector Σ_θ containing the nonnegative real axis.

A semigroup $S(t)$, $t \geq 0$ is called a *bounded holomorphic semigroup in the sector* Σ_θ if it has a bounded holomorphic extension to $\Sigma_{\theta'}$ for each $\theta' \in (0, \theta)$. We call $S(t)$, $t \geq 0$ a *bounded holomorphic semigroup* if it is a bounded holomorphic semigroup in some sector Σ_θ, $\theta > 0$. Note that if $S(t)$ is a bounded semigroup which is holomorphic then it is not necessarily a bounded holomorphic semigroup. For example, (see p. 153 in [1]), take $X = \mathbb{C}$ and $S(t) = e^{it}$ $(t \geq 0)$. It is obviously a bounded semigroup ($|e^{it}| = 1$ for all $t \in \mathbb{R}$) and e^{iz} is holomorphic in the entire $\mathbb{C}$. But it is not bounded in any sector Σ_θ, $\theta > 0$: take $z = -y\cot(\theta/2) + iy$ (then $|\arg z| < \theta$), then $|e^{iz}| = e^{-y} \to \infty$ as $y \to -\infty$.

1.3 Approximations of semigroups

In this section we introduce Euler's and Yosida approximations of semigroups. The need for approximations arises because in many important cases the study of the unbounded operator A and the semigroup that it generates is very complicated. Therefore it is often useful to try to build up the given operator and its semigroup from the simpler ones. There are many generation theorems which involve various approximations (see [22], [1], [19], [33], [42]). Euler's and Yosida approximations are used in the theory of partial differential equations to approximate the solutions of evolution equations (see [33], [19], [21], [25]).

In Section 1.1 we noted that any uniformly continuous semigroup $S(t)$ can be given by the converging (since its generator A is bounded) exponential series

$$S(t) = \sum_{n=0}^{\infty} \frac{(tA)^n}{n!}.$$

and therefore we often denote a uniformly continuous semigroup by $S(t) = e^{tA}$. For strongly continuous semigroups the above series does not converge in most cases since A is unbounded (see Exercise 3.12 in [19] for an example of the strongly continuous semigroups for whose generator the above series converges only when $t = 0$ or $x = 0$).

With the help of Euler's and Yosida approximations we provide two more exponential formulas which justify the notation of any strongly continuous semigroup $S(t)$ as an exponential function $S(t) = e^{tA}$ of its (unbounded) generator A. For more exponential formulas see, e.g., [22], p. 354.

Euler's approximations of semigroups

Let $S(t)$, $t \geq 0$ be a strongly continuous semigroup with a generator A and resolvent $R(\lambda)$. We consider the functions

$$t \mapsto E^n(t) := (n/t)^n R^n(n/t) = (I - tA/n)^{-n}, \qquad n \in \mathbb{N},$$

which are called the *Euler's approximations* of the semigroup $S(t)$. *Euler's exponential formula*

$$S(t)x = \lim_{n \to \infty} E^n(t)x,$$

was first proved by Hille in the strong operator topology for strongly continuous semigroups(see [22], [19]). It is known to converge in the uniform operator topology for bounded holomorphic semigroups ([14]).

Yosida approximations of semigroups

In Euler's exponential formula we approximate the semigroup $S(t)$ by the powers of bounded operators. Another method to construct a strongly continuous semigroup is to approximate its generator A by some sequence of bounded operators A_n and then hope that a strongly continuous semigroup $S(t)$ is the limit of exponentials (uniformly continuous semigroups) $e^{A_n t}$. In our work, we consider only contraction semigroups because the technical details are much easier in this case. The general case could be deduced from this one by renorming the Banach space X so that the bounded strongly continuous semigroup becomes a semigroup of contractions (see, e.g., [33] for more details).

Let A be a generator of strongly continuous semigroup $S(t)$. We define the *Yosida approximant* of A by

$$A_\lambda = \lambda A(\lambda I - A)^{-1},$$

for all $\lambda > 0$. It can be shown (see Lemma 1.3.4 in [33]) that if $S(t)$ is semigroup of contractions then A_λ is the generator of a uniformly continuous semigroup of contractions $S_\lambda(t)$. Furthermore,

$$S(t)x = \lim_{\lambda\to\infty} S_\lambda(t)x, \quad \text{for } \ x \in X.$$

We call $S_\lambda(t)$, $\lambda > 0$ *Yosida approximations* of contraction semigroup $S(t)$.

1.4 Convergence rate and asymptotic expansions

In our work we investigate the convergence of the Euler's and Yosida approximations to the corresponding semigroup $S(t)$. In particular, we are interested in estimation of errors

$$\|S_\lambda(t) - S(t)\|, \quad \text{as } \lambda \to \infty, \tag{1.4}$$

and

$$\|S_\lambda(t)x - S(t)x\|, \quad \text{as } \lambda \to \infty. \tag{1.5}$$

where $S_\lambda(t)$ are Yosida approximations of the semigroup $S(t)$.

The convergence rate of Euler's approximations $E_n(t)$ to a semigroup $S(t)$ was analysed in many research articles. For semigroups in Hilbert spaces generated by m-sectorial operators, Cachia and Zagrebnov in [14] obtained the bound $\Delta_n = O(n^{-1}\ln n)$, where $\Delta_n = \|E_n(t) - S(t)\|$. Paulauskas in [32] improved the bound to the optimal $O(n^{-1})$ using a new method based on the results and methods of probability theory related to the Central Limit Theorem. Cachia in [11] extended the bound $O(n^{-1}\ln n)$ to the case of bounded holomorphic semigroups and noticed that the Paulauskas argument can be applied to improve the bound to $O(n^{-1})$. Bentkus in [6] obtained asymptotic expansions for Euler's approximations of semigroups and optimal error bounds using an approach based on applications of the Fourier–Laplace transforms and a reduction of the problem to the convergence rates and asymptotic expansions in the Law of Large Numbers. In particular, for bounded differentiable semigroups, the error bound $\Delta_n = O(n^{-1})$ was obtained. It covers and refines all known related results obtained for semigroups in Banach algebras. New proofs for $\Delta_n = O(n^{-1})$ in case of quasi-sectorial contraction semigroups were presented in [2], [43].

Bentkus and Paulauskas in [7] obtained error bound for Euler's approximations $(I+tA)^{-n}$ of bounded holomorphic semigroup e^{-tA} generated by $-A$

$$\|(I + tA/n)^{-n} - e^{-tA}\| \leq 4K^3/n, \tag{1.6}$$

using multiplicative representations for these approximations. Such an approach was used by Bentkus in [5] for analysis of errors in the Central Limit Theorem and in approximations by accompanying laws and applied by Bentkus and Paulauskas in [7] to derive optimal convergence rates in Chernoff–type lemmas and Euler's approximations of semigroups. We use this method to obtain error bounds for (1.4) and (1.5). It is based on application of Newton-Leibniz formula along a smooth curve $\gamma(\tau)$, connecting two close objects $\gamma(0)$ and $\gamma(1)$:

$$\gamma(1) - \gamma(0) = \int_0^1 \gamma'(\tau)d\tau.$$

Here the main problem is to to choose γ so that the consequent proofs become as simple and transparent as possible. One of possible choices of γ was described in [5], Section 4 as follows.

Let $f, g : [0,1] \to \mathcal{B}$ be sufficiently smooth functions taking values in a Banach algebra $\mathcal{B}$ and such that $f(0) = g(0) = I$. Consider the union $R_{f,g} = f([0,1]) \bigcup g([0,1])$ of images of f and g. We assume that operators from the set $R_{f,g} \subseteq \mathcal{B}$ commute.

We are interested in estimation of the difference (note that $\Delta_n(0) = 0$)

$$\Delta_n = \Delta_n(1) = f^n(1) - g^n(1) \qquad \text{for} \quad n = 1, 2, \ldots$$

Write $\gamma(\tau) = f^n(\tau) g^n(1-\tau)$ and note that $\gamma(1) = f^n(1)$ and $\gamma(0) = g^n(1)$, since we assume that $f(0) = g(0) = I$. Hence $\Delta_n = \gamma(1) - \gamma(0)$. Using Newton-Leibniz formula

$$\gamma(1) - \gamma(0) = \int_0^1 \gamma'(\tau)\, d\tau,$$

we can write

$$\Delta_n = n \int_0^1 f^{n-1}(\tau)(f'(\tau)g(1-\tau) - f(\tau)g'(1-\tau))g^{n-1}(1-\tau)d\tau. \tag{1.7}$$

For example, to obtain the error bound (1.6), Bentkus and Paulauskas take $f(\tau) = (I + \tau tA/n)^{-1}$, $g(\tau) = \exp\{-\tau tA/n\}$ so that the *multiplicative representation* of the difference $\Delta_n = (I + tA/n)^{-n} - \exp\{-tA\}$ becomes

$$\Delta_n = \frac{1}{n} \int_0^1 \tau (tA)^2 (I + \tau tA/n)^{-n-1} (\tau t/n) \exp\{-t(1-\tau)A\}\, d\tau. \tag{1.8}$$

Then Bentkus and Paulauskas estimate the integrand in the left side of (1.8) to derive (1.6). In [38] we use this representation to obtain asymptotic expansions for Euler's approximations of semigroups. However, other choices of γ may be possible and may lead to even nicer results which we demonstrate with Yosida approximations in [40] and [41].

Iterative applications of (1.7) may lead to asymptotic expansions for approximations and semigroups. Using this approach, for Euler's approximations of semigroups we provide asymptotic expansions of type

$$E^n(t/n) = S(t) + \frac{a_1}{n} + \cdots + \frac{a_k}{n^k} + o\left(\frac{1}{n^k}\right) \quad \text{as} \quad n \to \infty,$$

with coefficients a_k depending on semigroup $S(t) = \exp\{tA\}$ and independent of n. We also obtain asymptotic expansions of the semigroup via Euler's approximations (the so called inverse expansions)

$$S(t) = E^n(t/n) + \frac{b_1}{n} + \cdots + \frac{b_k}{n^k} + o\left(\frac{1}{n^k}\right) \quad \text{as} \quad n \to \infty,$$

with b_k which are linear combinations of functions $t \mapsto (tA)^m E^n(t/n)$ with coefficients independent of n. We provide optimal bounds for the remainder terms of these expansions. Using this method we obtain asymptotic expansions (both direct and inverse) for Yosida approximations as well. In [6] Bentkus proposed another approach to obtain error bounds and asymptotic expansions for Euler's approximations of semigroups in Banach algebras. We describe this method briefly in Chapter 3 where we provide explicit formulas for these expansions.

2 Asymptotic expansions for Euler's approximations of semigroups

2.1 Introduction

Let X be a Banach space and $L(X)$ be a space of bounded linear operators on X. Let $S(t) = \exp\{tA\}$, $t \geq 0$ be a strongly continuous semigroup of operators from $L(X)$ with a generator A and let $R(\lambda) = (\lambda I - A)^{-1}$ be the resolvent of A. Denote

$$E(\lambda) := (1/\lambda)R(1/\lambda).$$

We define the functions

$$E^n(t/n) = (n/t)^n R^n(n/t) = (I - tA/n)^{-n}, \qquad n \in \mathbb{N},$$

which are called the Euler's approximations of the semigroup $S(t)$.

We consider asymptotic expansions of Euler's approximations

$$E^n(t/n) = S(t) + \frac{a_1}{n} + \cdots + \frac{a_k}{n^k} + o\left(\frac{1}{n^k}\right) \quad \text{as} \quad n \to \infty, \tag{2.1}$$

with coefficients a_k depending on $S(t)$ and independent of n. We also obtain asymptotic expansions of the semigroup

$$S(t) = E^n(t/n) + \frac{b_1}{n} + \cdots + \frac{b_k}{n^k} + o\left(\frac{1}{n^k}\right) \quad \text{as} \quad n \to \infty, \tag{2.2}$$

with b_k which are linear combinations of functions $t \mapsto (tA)^m E^n(t/n)$ with coefficients independent of n. We provide explicit and optimal bounds for the remainder terms in (2.1) and (2.2).

To obtain asymptotic expansions we use a decomposition of Euler's approximations which is an integro-differential identity. This approach was described in Section 1.4. That is, we express the difference $D_0 = E^n(t/n) - S(t)$ using the multiplicative representation (1.8). In our notation, we have

$$D_0 = E^n(t/n) - S(t) = \frac{1}{n}\int_0^1 \tau(tA)^2 E^{n+1}(\tau t/n) S(t(1-\tau))\, d\tau. \tag{2.3}$$

Iterative applications of this identity lead to asymptotic expansions of Euler's approximations $E^n(t/n)$ and semigroup $S(t)$.

2.2 Short asymptotic expansions

To make the exposition more comprehensible, we first formulate the results in the special case of short expansions, i.e., expansions with remainders $O(n^{-2})$. We start with a related integro-differential identity.

Theorem 2.1. *Assume that the semigroup $S(t)$ is differentiable. Then the following integro-differential identity holds*

$$E^n(t/n) = S(t) + \frac{(tA)^2}{2n} S(t) + D_1, \tag{2.4}$$

where the remainder term D_1 is given by

$$D_1 = \frac{1}{n^2}(D_{1,1} + D_{1,2}) \tag{2.5}$$

with

$$D_{1,1} = \int_0^1 \tau^2 (tA)^3 E^{n+1}(\tau t/n) S(t(1-\tau))\, d\tau$$

and

$$D_{1,2} = \int_0^1 \int_0^1 \tau_1^3 \tau_2 (tA)^4 E^{n+1}(\tau_1\tau_2 t/n) S(t(1-\tau_1\tau_2))\, d\tau_1\, d\tau_2.$$

We note that the formal identities (2.4) and (2.8) were also obtained by Bentkus in [5], but we provide them here for the sake of completeness.

To estimate the remainder term D_1 in expansion (2.4), we use conditions (2.6) and (2.7). These conditions are satisfied by bounded holomorphic semigroups by Theorems 2.5.2 and 2.5.3 in [33].

Theorem 2.2. *Assume that there exists a constant K independent of n and t such that*

$$n\|tA(I - tA)^{-n}\| \leq K \tag{2.6}$$

and

$$\|S(t)\| \leq K, \qquad \|tAS(t)\| \leq K \tag{2.7}$$

for all $n = 1, 2, \ldots$ and $t \geq 0$. Then the remainder term D_1 in asymptotic expansion (2.4) *satisfies*

$$\|D_1\| \leq \frac{C_1}{n^2} K^4,$$

where C_1 is an absolute positive constant.

Now we consider the so-called inverse expansions, i.e., expansions where the semigroup $S(t)$ is approximated by $E^n(t/n)$. Again we start with a short expansion with the remainder term $O(n^{-2})$.

Theorem 2.3. *Let $S(t)$ be a differentiable semigroup. Then the following integro-differential identity holds*

$$S(t) = E^n(t/n) + \frac{b_1}{n} + \Delta_1, \tag{2.8}$$

where $b_1 = -\frac{1}{2}(tA)^2 E^n(t/n)$. The remainder term Δ_1 is given by

$$\Delta_1 = -D_1 + \frac{(tA)^2}{2n} D_0,$$

where D_1 is given by (2.5)*, and D_0 is given by* (2.3)*.*

To estimate the remainder term Δ_1 in expansion (2.8), we use the same conditions as in Theorem 2.2.

Theorem 2.4. *Assume that there exists a constant K independent of n and t such that conditions* (2.6) *and* (2.7) *are satisfied for all $n = 1, 2, \ldots$ and $t \geq 0$. Then the remainder term Δ_1 in asymptotic expansion* (2.8) *satisfies*

$$\|\Delta_1\| \leq \frac{C_1}{n^2} K^5,$$

where C_1 is an absolute positive constant.

2.3 The general case

Now we generalize the results of Theorems 2.1–2.4 for asymptotic expansions of any given length k. We first need some additional notation. Henceforth, $\sum_{\alpha}^{*}$ means summation over all integer components $\alpha_1, \ldots, \alpha_k$ of vectors $\alpha = (\alpha_1, \ldots, \alpha_k)$ which satisfy certain conditions. Write

$$c_{m,1} = \frac{1}{m+1} \quad \text{and} \quad c_{m,j} = \frac{1}{m+j} \sum_{i}^{*} \frac{1}{i_2 i_3 \ldots i_j} \quad \text{for} \quad j = 2, \ldots, m, \tag{2.9}$$

where $i = (i_2, i_3, \ldots, i_j)$ satisfy $2 \le i_j \le m - j + 2$ and $i_{n+1} + 2 \le i_n \le m + j - 2(n-1)$ for $n = 2, 3, \ldots, j-1$. Then the coefficients a_m in (2.1) are given by

$$a_m = \sum_{j=1}^{m} c_{m,j} (tA)^{m+j} S(t). \tag{2.10}$$

For example, we have

$$\begin{aligned}
a_1 &= \frac{(tA)^2}{2} S(t), \\
a_2 &= \frac{(tA)^3}{3} S(t) + \frac{(tA)^4}{8} S(t), \\
a_3 &= \frac{(tA)^4}{4} S(t) + \frac{(tA)^5}{6} S(t) + \frac{(tA)^6}{48} S(t).
\end{aligned}$$

We note that an alternative form of the coefficients $c_{m,j}$ is

$$c_{m,j} = \frac{1}{j!} \sum_{i_1 + \cdots + i_j = m+j} \frac{1}{i_1 i_2 \ldots i_j},$$

where $i_1, i_2, \ldots, i_j \ge 2$ and $1 \le j \le m$. Here $\sum_{i_1 + \cdots + i_n = k}$ means summation over all distinct ordered n-tuples of positive integers $i_1, \ldots, i_n$ whose elements sum to k.

We also define the functions

$$\sigma_{k,j} = \sigma_{k,j}(\tau_1, \dots, \tau_j) = \tau_1^{k+j} \sum_i^* \tau_2^{i_2} \dots \tau_j^{i_j} \quad \text{for} \quad j = 2, \dots, k+1, \tag{2.11}$$

where $i = (i_2, i_3, \dots, i_j)$ satisfy $1 \le i_j \le k - j + 2$ and $i_{n+1} + 2 \le i_n \le k + j - 2(n-1)$ for $n = 2, 3, \dots, j-1$. When $j = 1$, $\sigma_{k,1} = \tau_1^{k+1}$.

To shorten the notation for multiple integrals we use a sequence of independent identically distributed random variables $\tau, \tau_1, \tau_2, \dots$ uniformly distributed in the interval $[0,1]$. Then we can write $\int_0^1 f(\tau)\, d\tau = \mathbb{E} f(\tau)$ for any function f. In the case where f is a function of k variables, we write $\mathbb{E} f(\tau_1, \dots, \tau_k)$ instead of a k-tuple integral.

Theorem 2.5. *Let $S(t)$ be a differentiable semigroup. Then we have*

$$E^n(t/n) = S(t) + \frac{a_1}{n} + \dots + \frac{a_k}{n^k} + D_k, \qquad k = 1, 2, \dots, \tag{2.12}$$

where a_m are given by (2.10). *The remainder term is given by*

$$D_k = \frac{1}{n^{k+1}} \sum_{j=1}^{k+1} D_{k,j}, \tag{2.13}$$

where

$$D_{k,j} = \mathbb{E} \sigma_{k,j} (tA)^{k+j+1} E^{n+1}(\tau_1 \dots \tau_j t/n) S(t(1 - \tau_1 \dots \tau_j))$$

with $\sigma_{k,j}$ given by (2.11).

Theorem 2.6. *Assume that there exists a constant K independent of n and t such that conditions* (2.6) *and* (2.7) *are satisfied for all $n = 1, 2, \dots$ and $t \ge 0$. Then the remainder term D_k in asymptotic expansion* (2.12) *satisfies*

$$\|D_k\| \le \frac{C_k}{n^{k+1}} K^{2k+2}, \qquad k = 1, 2, \dots$$

with a positive constant C_k depending only on k.

Now we provide asymptotic expansions of the semigroup $S(t)$ via Euler's approximations $E^n(t/n)$. First, we denote

$$h_0 = 1, \qquad h_m = -\sum_{j=1}^{m} d_j h_{m-j}, \tag{2.14}$$

where

$$d_m = \sum_{i=1}^{m} c_{m,i}(tA)^{m+i}, \tag{2.15}$$

with $c_{m,i}$ are given by (2.9). Then the coefficients b_m in (2.2) are given by

$$b_m = h_m E^n(t/n), \quad m = 1, 2, \ldots. \tag{2.16}$$

We note that h_m can be represented as

$$h_m = (-1)^m \sum_{i=1}^{m} c_{m,i}(-tA)^{m+i}.$$

For example, we have

$$\begin{aligned}
b_1 &= -\frac{(tA)^2}{2} E^n(t/n), \\
b_2 &= -\frac{(tA)^3}{3} E^n(t/n) + \frac{(tA)^4}{8} E^n(t/n), \\
b_3 &= -\frac{(tA)^4}{4} E^n(t/n) + \frac{(tA)^5}{6} E^n(t/n) - \frac{(tA)^6}{48} E^n(t/n).
\end{aligned}$$

Theorem 2.7. *Assume that the semigroup $S(t)$ is differentiable. Then we have*

$$S(t) = E^n(t/n) + \frac{b_1}{n} + \cdots + \frac{b_k}{n^k} + \Delta_k, \qquad k = 1, 2, \ldots, \tag{2.17}$$

where b_m are given by (2.16). The remainder term is given by

$$\Delta_k = -\sum_{j=0}^{k} \frac{h_{k-j}}{n^{k-j}} D_j, \tag{2.18}$$

where h_m are given by (2.14), D_m are given by (2.13) for $m = 1, \ldots, k$ and D_0 is given by (2.3).

Theorem 2.8. *Assume that there exists a constant K independent of n and t such that conditions* (2.6) *and* (2.7) *are satisfied for all $n = 1, 2, \dots$ and $t \geq 0$. Then the remainder term Δ_s in asymptotic expansion* (2.17) *satisfies*

$$\|\Delta_s\| \leq \frac{C_s}{n^{s+1}} K^{2s+3}, \qquad s = 1, 2, \dots$$

with a positive constant C_s depending only on s.

2.4 Proofs

Proof of Theorem 2.1. From (2.3) we have

$$E^n(t/n) = S(t) + D_0, \tag{2.19}$$

where

$$D_0 = \frac{1}{n} \int_0^1 \tau_1 (tA)^2 E^{n+1}(\tau_1 t/n) S(t(1-\tau_1))\, d\tau_1 = \frac{1}{n} D_{0,1}, \tag{2.20}$$

(see also (4.2) in [5]).

It is easy to check the algebraic identity

$$E^{n+1}(\tau_1 t/n) = E^n(\tau_1 t/n) + \frac{1}{n} \tau_1 t A E^{n+1}(\tau_1 t/n). \tag{2.21}$$

Also, from (2.19) and (2.20) we obtain that

$$E^n(\tau_1 t/n) = S(\tau_1 t) + \frac{1}{n} \int_0^1 \tau_1^2 \tau_2 (tA)^2 E^{n+1}(\tau_1 \tau_2 t/n) S(\tau_1 t(1-\tau_2))\, d\tau_2. \tag{2.22}$$

Substituting (2.21) and then (2.22) into expression of $D_{0,1}$ and using the semigroup property, we get

$$D_{0,1} = \frac{1}{n} D_{1,1} + \frac{1}{2}(tA)^2 S(t) + \frac{1}{n} D_{1,2}, \tag{2.23}$$

where

$$D_{1,1} = \int_0^1 \tau_1^2 (tA)^3 E^{n+1}(\tau_1 t/n) S(t(1-\tau_1))\, d\tau_1$$

and

$$D_{1,2} = \int_0^1 \int_0^1 \tau_1^3 \tau_2 (tA)^4 E^{n+1}(\tau_1 \tau_2 t/n) S(t(1-\tau_1\tau_2))\, d\tau_1\, d\tau_2.$$

Substituting expression (2.23) into (2.20), we obtain the asymptotic expansion

$$E^n(t/n) = S(t) + \frac{a_1}{n} + D_1,$$

where

$$a_1 = \frac{1}{2}(tA)^2 S(t)$$

and

$$D_1 = \frac{1}{n^2}(D_{1,1} + D_{1,2}).$$

□

In order to prove Theorem 2.2, we first prove the following:

Lemma 2.1. *Assume that there exists a constant K independent of n and t such that*

$$n\|tA(I - tA)^{-n}\| \leq K$$

and

$$\|tAS(t)\| \leq K$$

for all $n = 1, 2, \ldots$ and $t \geq 0$. Then

$$n^m\|(tA)^m(I - tA)^{-n-1}\| \leq (3m)^m K^m \tag{2.24}$$

and

$$\|(tA)^m S(t)\| \le m^m K^m \tag{2.25}$$

for all $m = 1, 2, \dots$.

Proof. To prove (2.24), write $n = ms + k$ for some $s \in \mathbb{N}$ and $k \in \{1, 2, \dots, m\}$. Regrouping the factors in (2.24) we get

$$\begin{aligned} n^m \|(tA)^m (I - tA)^{-n-1}\| &\le \\ &\le (ms+k)^m \|tA(I-tA)^{-s}\|^{m-k-1} \|tA(I-tA)^{-s-1}\|^{k+1} \\ &\le \frac{(ms+k)^m}{s^{m-k-1}(s+1)^{k+1}} K^m \le (3m)^m K^m. \end{aligned}$$

Now let us prove (2.25). Using the semigroup property $S(t+s) = S(t)S(s)$, we obtain

$$\|(tA)^m S(t)\| \le m^m \|(tA/m) S(t/m)\|^m \le m^m K^m.$$

□

Proof of Theorem 2.2. From (2.5) we have

$$\|D_1\| \le \frac{1}{n^2} (\|D_{1,1}\| + \|D_{1,2}\|).$$

Let us first estimate $\|D_{1,1}\|$. It is clear that $\|D_{1,1}\| \le \theta_{1,1} + \theta_{1,2}$, where

$$\theta_{1,1} = \int_{1/2}^{1} \|\tau^2 (tA)^3 E^{n+1}(\tau t/n) S(t(1-\tau))\| \, d\tau$$

and

$$\theta_{1,2} = \int_{0}^{1/2} \|\tau^2 (tA)^3 E^{n+1}(\tau t/n) S(t(1-\tau))\| \, d\tau.$$

Let $\varrho_{1,1} = \|(\tau t A)^3 E^{n+1}(\tau t/n)\|$. Then

$$\theta_{1,1} = \int\limits_{1/2}^{1} \varrho_{1,1} \frac{\|S(t(1-\tau))\|}{\tau} \, d\tau.$$

By Lemma 2.1 we have $\varrho_{1,1} \leq 9^3 K^3$ and by (2.7) we have $\|S(t(1-\tau))\| \leq K$. Integrating over the interval $[1/2, 1]$, we get

$$\theta_{1,1} \leq 9^3 \ln 2 K^4.$$

Let us estimate $\theta_{1,2}$. Write

$$\varrho_{1,2} = \|(\tau t A)^2 E^{n+1}(\tau t/n)\|, \qquad \varrho_{1,3} = \|(1-\tau) t A S(t(1-\tau))\|.$$

Then

$$\theta_{1,2} = \int\limits_{0}^{1/2} \varrho_{1,2} \varrho_{1,3} \frac{1}{(1-\tau)} \, d\tau.$$

By Lemma 2.1 we have $\varrho_{1,2} \leq 36K^2$ and $\varrho_{1,3} \leq K$. Integrating over the interval $[0, 1/2]$, we get

$$\theta_{1,2} \leq 36 \ln 2 K^3$$

and (note that $K \geq 1$)

$$\|D_{1,1}\| \leq C_{1,1} K^4,$$

where $C_{1,1}$ is an absolute positive constant.

Now we estimate $\|D_{1,2}\|$. It is clear that $\|D_{1,1}\| \leq \theta_{2,1} + \theta_{2,2}$, where

$$\theta_{2,1} = \int\limits_{0}^{1} \int\limits_{1/2}^{1} \|\tau_1^3 \tau_2 (tA)^4 E^{n+1}(\tau_1 \tau_2 t/n) S(t(1-\tau_1\tau_2))\| \, d\tau_1 \, d\tau_2$$

and

$$\theta_{2,2} = \int_0^1 \int_0^{1/2} \|\tau_1^3 \tau_2 (tA)^4 E^{n+1}(\tau_1\tau_2 t/n) S(t(1-\tau_1\tau_2))\| \, d\tau_1 \, d\tau_2.$$

Write

$$\varrho_{2,1} = \|(\tau_1\tau_2 tA)^3 E^{n+1}(\tau_1\tau_2 t/n)\|, \qquad \varrho_{2,2} = \|(1-\tau_1\tau_2) tAS(t(1-\tau_1\tau_2))\|.$$

Then

$$\theta_{2,1} = \int_0^1 \int_{1/2}^1 \varrho_{2,1}\varrho_{2,2} \frac{1}{\tau_2^2(1-\tau_1\tau_2)} \, d\tau_1 \, d\tau_2.$$

By Lemma 2.1 we have $\varrho_{2,1} \le 9^3 K^3$ and $\varrho_{2,2} \le K$. Integrating, we get

$$\theta_{2,1} \le 9^3(1/2 + 2\ln 2) K^4.$$

It remains to estimate $\theta_{2,2}$. Let

$$\varrho_{2,3} = \|\tau_1\tau_2 tAE^{n+1}(\tau_1\tau_2 t/n)\|, \qquad \varrho_{2,4} = \|(1-\tau_1\tau_2)^3 (tA)^3 S(t(1-\tau_1\tau_2))\|.$$

Then

$$\theta_{2,2} = \int_0^1 \int_0^{1/2} \varrho_{2,3}\varrho_{2,4} \frac{\tau_1^2}{(1-\tau_1\tau_2)^3} \, d\tau_1 \, d\tau_2.$$

By Lemma 2.1 we have $\varrho_{2,3} \le K$ and $\varrho_{2,4} \le 27K^3$. Integrating, we get

$$\theta_{2,2} \le 27(7/4 - 2\ln 2) K^4$$

and

$$\|D_{1,2}\| \le C_{1,2} K^4,$$

where $C_{1,2}$ is an absolute positive constant. Then

$$\|D_1\| \le \frac{1}{n^2}(\|D_{1,1}\| + \|D_{1,2}\|) \le \frac{C_1}{n^2} K^4.$$

□

Proof of Theorem 2.3. From Theorem 2.1 we obtain

$$S(t) = E^n(t/n) - \frac{a_1}{n} - D_1,$$

where $a_1 = \frac{1}{2}(tA)^2 S(t)$ and by (2.3) we have

$$S(t) = E^n(t/n) + D_0. \tag{2.26}$$

Substituting (2.26) into expression of a_1, we obtain

$$S(t) = E^n(t/n) - \frac{(tA)^2}{2n}(E^n(t/n) - D_0) - D_1. \tag{2.27}$$

Regrouping the terms in (2.27), we obtain asymptotic expansion (2.8). □

Proof of Theorem 2.4. From Theorem 2.2 we have

$$\|D_1\| \le \frac{c_1}{n^2} K^4,$$

where c_1 is an absolute positive constant.
Let us estimate $\|(tA)^2 D_0\|$. It is clear that $\|(tA)^2 D_0\| \le \frac{1}{n}(\theta_1 + \theta_2)$, where

$$\theta_1 = \int\limits_{1/2}^{1} \|\tau (tA)^4 E^{n+1}(\tau t/n) S(t(1-\tau))\| \, d\tau$$

and

$$\theta_2 = \int\limits_{0}^{1/2} \|\tau (tA)^4 E^{n+1}(\tau t/n) S(t(1-\tau))\| \, d\tau.$$

Write $\varrho_1 = \|(\tau t A)^4 E^{n+1}(\tau t/n)\|$. Then

$$\theta_1 = \int\limits_{1/2}^{1} \varrho_1 \frac{\|S(t(1-\tau))\|}{\tau^3} \, d\tau.$$

By Lemma 2.1 we have $\varrho_1 \le 12^4 K^4$ and $\|S(t(1-\tau))\| \le K$. Integrating over the interval $[1/2, 1]$, we get

$$\theta_1 \le \frac{3}{2} \cdot 12^4 K^5.$$

Now we estimate θ_2. Let

$$\varrho_2 = \|\tau t A E^{n+1}(\tau t/n)\|, \qquad \varrho_3 = \|(1-\tau)^3 (tA)^3 S(t(1-\tau))\|.$$

Then

$$\theta_2 = \int_0^{1/2} \varrho_2 \varrho_3 \frac{1}{(1-\tau)^3}\, d\tau.$$

From (2.6) we have $\varrho_2 \le K$ and by Lemma 2.1 we have $\varrho_3 \le 27K^3$. Integrating over the interval $[0, 1/2]$, we get

$$\theta_2 \le 81K^4/2$$

and (note that $K \ge 1$)

$$\|(tA)^2 D_0\| \le \frac{C_0}{n} K^5,$$

where C_0 is an absolute positive constant. Then

$$\|\Delta_1\| \le \|D_1\| + \frac{\|(tA)^2 D_0\|}{2n} \le \frac{C_1}{n^2} K^5.$$

□

Proof of Theorem 2.5. We prove the theorem using induction with respect to k. In the case $k = 0$, we have

$$E^n(t/n) = S(t) + D_0, \tag{2.28}$$

where

$$D_0 = \frac{1}{n}\int_0^1 \tau_1 (tA)^2 E^{n+1}(\tau_1 t/n) S(t(1-\tau_1))\, d\tau_1 = \frac{1}{n} D_{0,1} \tag{2.29}$$

(see (4.2) in [5]). The case $k = 1$ was proved in Theorem 2.1 (see also [5]).

Assume that (2.12) holds for $0, 1, \dots, k-1$. Let us show that (2.12) holds for k as well. To this end we have to show that

$$D_{k-1} = \frac{a_k}{n^k} + D_k.$$

From (2.13) we have

$$D_{k-1} = \frac{1}{n^k}\sum_{j=1}^{k} D_{k-1,j},$$

where

$$D_{k-1,j} = \mathbb{E}\sigma_{k-1,j}(-tA)^{k+j} E^{n+1}(\tau_1 \dots \tau_j t/n) S(t(1-\tau_1 \dots \tau_j)). \tag{2.30}$$

It is easy to show that the following algebraic identity holds:

$$E^{n+1}(\tau_1 \dots \tau_j t/n) = E^n(\tau_1 \dots \tau_j t/n) + \frac{1}{n}\tau_1 \dots \tau_j tA E^{n+1}(\tau_1 \dots \tau_j t/n). \tag{2.31}$$

Also, from (2.28) and (2.29) we obtain

$$\begin{aligned} &E^n(\tau_1\tau_2 \dots \tau_j t/n) = S(\tau_1\tau_2 \dots \tau_j t) + \\ &\frac{1}{n}\int_0^1 \tau_{j+1}(\tau_1\tau_2 \dots \tau_j tA)^2 E^{n+1}(\tau_1\tau_2 \dots \tau_j\tau_{j+1} t/n) S(\tau_1\tau_2 \dots \tau_j t(1-\tau_{j+1}))\, d\tau_{j+1}. \end{aligned} \tag{2.32}$$

Substituting expressions (2.31) and (2.32) into (2.30), we get

$$D_{k-1,j} = I_{k-1,j,1} + I_{k-1,j,2} + I_{k-1,j,3},$$

where

$$I_{k-1,j,1} = \frac{1}{n}\mathbb{E}\sigma_{k-1,j}\tau_1 \dots \tau_j (tA)^{k+j+1} E^{n+1}(\tau_1 \dots \tau_j t/n) S(t(1-\tau_1 \dots \tau_j)),$$

$$I_{k-1,j,2} = \mathbb{E}\sigma_{k-1,j}(tA)^{k+j} S(t),$$

and

$$I_{k-1,j,3} = \frac{1}{n}\mathbb{E}\sigma_{k-1,j}\tau_{j+1}\tau_1^2 \dots \tau_j^2 (tA)^{k+j+2} E^{n+1}(\tau_1 \dots \tau_{j+1} t/n) S(t(1-\tau_1 \dots \tau_{j+1})).$$

Integrating $I_{k-1,j,2}$ (note that $\mathbb{E}\sigma_{k-1,j} = c_{k,j}$), we get

$$I_{k-1,j,2} = c_{k,j}(tA)^{k+j} S(t),$$

where $c_{k,j}$ is given by (2.9).
Taking the sum, we obtain that $a_k = \sum_{j=1}^{k} I_{k-1,j,2}$.

It is easy to check that

$$\sigma_{k-1,j}\tau_1 \dots \tau_j + \sigma_{k-1,j-1}\tau_1^2 \dots \tau_{j-1}^2 \tau_j = \sigma_{k,j}.$$

From this equality it follows that

$$\begin{aligned} D_{k,j} &= n(I_{k-1,j,1} + I_{k-1,j-1,3}) \qquad \text{for } j = 2, \dots, k, \\ D_{k,1} &= nI_{k-1,1,1}, \quad \text{and} \quad D_{k,k+1} = nI_{k-1,k,3}. \end{aligned}$$

Finally, we get

$$D_{k-1} = \frac{1}{n^k}\sum_{j=1}^{k}(I_{k-1,j,1} + I_{k-1,j,2} + I_{k-1,j,3}) = \frac{a_k}{n^k} + D_k$$

with a_k given by (2.10) and D_k given by (2.13).

□

Proof of Theorem 2.6. From (2.13) we have

$$\|D_k\| \leq \frac{1}{n^{k+1}} \sum_{j=1}^{k+1} \|D_{k,j}\|. \tag{2.33}$$

Each $D_{k,j}$ is the sum of j-tuple integrals of the type

$$J_i = \mathbb{E}\tau_1^{k+j}\tau_2^{i_2}\dots\tau_j^{i_j}(tA)^{k+j+1}E^{n+1}(\tau_1\dots\tau_j t/n)S(t(1-\tau_1\dots\tau_j)),$$

where the sum is taken over all integer components of $i = (i_2, i_3, \dots, i_j)$ such that $1 \leq i_j \leq k-j+2$ and $i_{n+1}+2 \leq i_n \leq k+j-2(n-1)$ for $n = 2, 3, \dots, j-1$.

To simplify the notation we introduce the indicator functions

$$\mathbb{I}_{1,k} = \mathbb{I}\{\tau_2 \geq 1/2, \dots, \tau_k \geq 1/2\}, \quad k \geq 2,$$

$$\mathbb{I}_{m,k} = \mathbb{I}\{\tau_m \leq 1/2, \tau_{m+1} \geq 1/2, \dots, \tau_k \geq 1/2\}, \quad k \geq 3,$$

for $m = 2, \dots, k-1$, and

$$\mathbb{I}_{k,k} = \mathbb{I}\{\tau_k \leq 1/2\}, \quad k \geq 2.$$

Now we rewrite each J_i as the sum of $j \geq 2$ integrals

$$J_i = J_{i,1} + \dots + J_{i,j},$$

where

$$J_{i,m} = \mathbb{E}\mathbb{I}_{m,j}\tau_1^{k+j}\tau_2^{i_2}\dots\tau_j^{i_j}(tA)^{k+j+1}E^{n+1}(\tau_1\dots\tau_j t/n)S(t(1-\tau_1\dots\tau_j))$$

for all $m = 1, \dots, j$.

Then

$$\|J_i\| \leq \|J_{i,1}\| + \dots + \|J_{i,j}\| \leq \theta_{i,1} + \dots + \theta_{i,j},$$

where

$$\theta_{i,m} = \mathbb{E}\mathbb{I}_{m,j} \|\tau_1^{k+j}\tau_2^{i_2}\dots\tau_j^{i_j}(tA)^{k+j+1}E^{n+1}(\tau_1\dots\tau_j t/n)S(t(1-\tau_1\dots\tau_j))\|$$

for $m = 1, ..., j$. Write

$$\varrho_{i,m,1} = \|(\tau_1\dots\tau_j tA)^{i_m}E^{n+1}(\tau_1\dots\tau_j t/n)\|$$

and

$$\varrho_{i,m,2} = \|(t(1-\tau_1\dots\tau_j)A)^{k+j+1-i_m}S(t(1-\tau_1\dots\tau_j))\|.$$

Then we have

$$\theta_{i,m} = \mathbb{E}\mathbb{I}_{m,j}\varrho_{i,m,1}\varrho_{i,m,2}|g_{i,m}(\tau_1,\dots,\tau_j)|,$$

where

$$g_{i,m}(\tau_1,\dots,\tau_j) = \frac{\tau_1^{k+j-i_m}\tau_2^{i_2-i_m}\dots\tau_{m-1}^{i_{m-1}-i_m}}{\tau_{m+1}^{i_m-i_{m+1}}\dots\tau_j^{i_m-i_j}(1-\tau_1\dots\tau_j)^{k+j+1-i_m}}.$$

The function $g_{i,m}(\tau_1,\dots,\tau_j)$, $m = 2,\dots,j$, is bounded for $\tau_1,\dots,\tau_{m-1} \in [0,1]$, $\tau_m \in [0,1/2]$, and $\tau_{m+1},\dots,\tau_j \in [1/2,1]$. By Lemma 2.1 we have

$$\varrho_{i,m,1} \le (3i_m)^{i_m}K^{i_m}, \qquad \varrho_{i,m,2} \le (k+j+1-i_m)^{k+j+1-i_m}K^{k+j+1-i_m},$$

and, integrating, we get

$$\theta_{i,m} \le C_{i,m,k,j}K^{k+j+1},$$

where $C_{i,m,k,j}$ is a positive constant depending only on k, j, m, and $i_1,\dots,i_j$.

In the case $m = 1$, we have

$$\varrho_{i,1,1} = \|(\tau_1\dots\tau_j tA)^{k+j}E^{n+1}(\tau_1\dots\tau_j t/n)\|$$

and

$$\varrho_{i,1,2} = \|(1-\tau_1\dots\tau_j)tAS(t(1-\tau_1\dots\tau_j))\|.$$

Then

$$\theta_{i,1} = \mathbb{E}\mathbb{I}_{1,j}\,\varrho_{i,1,1}\varrho_{i,1,2}|g_{i,1}(\tau_1,\ldots,\tau_j)|,$$

where

$$g_{i,1}(\tau_1,\ldots,\tau_j) = \frac{1}{\tau_2^{k+j-i_2}\ldots\tau_j^{k+j-i_j}(1-\tau_1\ldots\tau_j)}.$$

We note that, for $\tau_2,\ldots,\tau_j \in [1/2,1]$,

$$g_{i,1}(\tau_1,\ldots,\tau_j) \le \frac{2^N}{(1-\tau_1\tau_2)},$$

where $N = i_2 + \cdots + i_j - (k+j)(j-1)$. Integrating over $\tau_3,\ldots,\tau_j \in [1/2,1]$, we have

$$\theta_{i,1} \le 2^{N-j+2}(k+j)^{k+j}K^{k+j+1}\int_0^1\int_{1/2}^1 \frac{1}{(1-\tau_1\tau_2)}\,d\tau_1\,d\tau_2.$$

This integral converges, and from this it follows that

$$\theta_{i,1} \le C_{i,1,k,j}K^{k+j+1},$$

where $C_{i,1,k,j}$ is a positive constant depending only on k, j, and $i_2,\ldots,i_j$. Taking the sums over all m and $i_2,\ldots,i_j$, we obtain

$$\|D_{k,j}\| \le C_{k,j}K^{k+j+1}, \quad j = 2,\ldots,k+1, \tag{2.34}$$

where $C_{k,j}$ is a positive constant depending only on k and j.

It remains to prove the case where $j = 1$. Then

$$D_{k,1} = \mathbb{E}\tau_1^{k+1}(tA)^{k+2}E^{n+1}(\tau_1 t/n)S(t(1-\tau_1)).$$

Write

$$\theta_1 = \mathbb{E}\mathbb{I}\{\tau_1 > 1/2\}\|\tau_1^{k+1}(tA)^{k+2}E^{n+1}(\tau_1 t/n)S(t(1-\tau_1))\|$$

and

$$\theta_2 = \mathbb{E}\mathbb{I}\{\tau_1 \le 1/2\}\|\tau_1^{k+1}(tA)^{k+2}E^{n+1}(\tau_1 t/n)S(t(1-\tau_1))\|,$$

where $\mathbb{I}$ is the indicator function. Then we have $\|D_{k,1}\| \le \theta_1 + \theta_2$.

Let $\varrho_{1,1} = \|(\tau_1 tA)^{k+2}E^{n+1}(\tau_1 t/n)\|$. Then

$$\theta_1 = \mathbb{E}\mathbb{I}\{\tau_1 > 1/2\}\varrho_{1,1}\|S(t(1-\tau_1))\|/\tau_1.$$

By Lemma 2.1 we have $\varrho_{1,1} \le (3(k+2))^{k+2}K^{k+2}$ and $\|S(t(1-\tau_1))\| \le K$. Integrating over the interval $[1/2, 1]$, we get

$$\theta_1 \le C_{1,k,1}K^{k+3}.$$

Now we estimate θ_2. Let $\varrho_{2,1} = \|(\tau_1 tA)^{k+1}E^{n+1}(\tau_1 t/n)\|$ and $\varrho_{2,2} = \|(1-\tau_1)(tA)S(t(1-\tau_1))\|$. Then

$$\theta_2 = \mathbb{E}\mathbb{I}\{\tau_1 \le 1/2\}\varrho_{2,1}\varrho_{2,2}/(1-\tau_1).$$

By Lemma 2.1 we have $\varrho_{2,1} \le (3(k+1))^{k+1}K^{k+1}$ and $\varrho_{2,2} \le K$. Integrating over the interval $[0, 1/2]$, we obtain

$$\theta_2 \le C_{2,k,1}K^{k+2}.$$

Then (note that $K \ge 1$) we have

$$\|D_{k,1}\| \le C_{k,1}K^{k+3}. \tag{2.35}$$

Substituting (2.34) and (2.35) into (2.33), we get

$$\|D_k\| \le \frac{C_k}{n^{k+1}}K^{2k+2}.$$

□

Proof of Theorem 2.7. We prove the theorem using induction with respect to k. For $k = 0$ we have

$$S(t) = E^n(t/n) + \Delta_0, \tag{2.36}$$

where $\Delta_0 = -D_0$. From Theorem 2.5 and (2.15) we have

$$D_m = \frac{a_{m+1}}{n^{m+1}} = \frac{d_{m+1}}{n^{m+1}} S(t) + D_{m+1}, \tag{2.37}$$

for $m = 0, 1, 2, \ldots$. Substituting (2.36) into (2.37) we obtain

$$D_m = \frac{d_{m+1}}{n^{m+1}} E^n(t/n) - \frac{d_{m+1}}{n^{m+1}} D_0 + D_{m+1}. \tag{2.38}$$

For $m = 0$ we have $D_0 = \frac{d_1}{n} E^n(t/n) - \frac{d_1}{n} D_0 + D_1$. Substituting this into (2.36) and denoting $h_1 = -d_1$ we have

$$S(t) = E^n(t/n) + \frac{h_1}{n} E^n(t/n) - \frac{h_1}{n} D_0 - D_1.$$

Denoting $\Delta_1 = -\frac{h_1}{n} D_0 - D_1$ we obtain asymptotic expansion (2.17) for $k = 1$, with coefficient $b_1 = h_1 E^n(t/n)$ given by (2.16) and remainder term given by (2.18).

Now we assume that (2.17) and (2.18) hold for $1, \ldots, k-1$. Let us prove that (2.17) and (2.18) hold for k as well. By our assumption

$$S(t) = \frac{b_1}{n} + \cdots + \frac{b_{k-1}}{n^{k-1}} + \Delta_{k-1},$$

where

$$\Delta_{k-1} = -\sum_{j=0}^{k-1} \frac{h_{k-j-1}}{n^{k-j-1}} D_j. \tag{2.39}$$

Substituting expression (2.38) for D_j into (2.39) we obtain

$$\begin{aligned} \Delta_{k-1} = &- \sum_{j=0}^{k-1} \frac{h_{k-j-1} d_{j+1}}{n^k} E^n(t/n) + \sum_{j=0}^{k-1} \frac{h_{k-j-1} d_{j+1}}{n^k} D_0 - \sum_{j=0}^{k-1} \frac{h_{k-j-1}}{n^{k-j-1}} D_{j+1} = \\ =& \frac{b_k}{n^k} + \Delta_k, \end{aligned}$$

where b_k is given by (2.16) and Δ_k is given by (2.17). From here it follows that our assumption is true for all $k = 1, 2, \ldots$.

□

Proof of Theorem 2.8. From Theorem 2.5 we have

$$\Delta_s = -D_s - \sum_{k=0}^{s-1} h_{s-k} \frac{D_k}{n^{s-k}},$$

where h_m are given by (2.14). From Theorem 2.6 we have

$$\|D_s\| \leq \frac{c_s}{n^{s+1}} K^{2s+2}, \quad s = 1, 2, \dots,$$

where c_s is a positive constant depending only on s. For $s = 0$, we have $\|D_0\| \leq 4K^3/n$ by Theorem 1.3 in [7].

It is not difficult to see that h_{s-k} are linear combinations of $(tA)^{s-k+1}, \dots, (tA)^{2s-2k}$ with some numerical coefficients depending only on k and s. So, in order to prove the theorem we have to show that

$$\|(tA)^p D_k\| \leq \frac{C_{p,k}}{n^{k+1}} K^{2s+3}, \quad k = 0, 1, \dots, s-1,$$

where $p = s-k+1, \dots, 2s-2k$ and $C_{p,k}$ is a positive constant depending only on p and k. The proof is similar to the proofs of Theorems 2.6 and 2.4. In the case $k = 1, \dots, s-1$, we obtain $\|(tA)^p D_k\| \leq \frac{C_{p,k}}{n^{k+1}} K^{2s+2}$ and, in the case $k = 0$, we have $\|(tA)^p D_0\| \leq \frac{C_{p,0}}{n} K^{2s+3}$. We omit the proof here. □

3 Another approach to asymptotic expansions for Euler's approximations of semigroups

In this chapter we describe another interesting approach for obtaining error bounds and asymptotic expansions for Euler's approximations of semigroups. This method was proposed by Bentkus in [6]. We also provide explicit formulas for these asymptotic expansions. The results of this chapter are derived for the semigroups in abstract Banach algebras (they are also valid for bounded holomorphic semigroups of operators).

3.1 Introduction

Let X be a complex Banach algebra with norm $\|\cdot\|$. A family $S(t)$, $t > 0$ of elements of a Banach algebra X is called a semigroup if $S(t+s) = S(t)S(s)$, for all $t, s > 0$ (see [22]). We define the resolvent $R(\lambda)$, $\lambda \in \mathbb{C}$ of the semigroup $S(t)$ as the Laplace transform

$$R(\lambda) = \int_0^\infty \exp\{-\lambda t\} S(t)\, dt.$$

Let

$$f(\lambda) = R(1/\lambda)/\lambda, \qquad E_n(t) = f^n(t/n). \tag{3.1}$$

The functions $t \mapsto E_n(t)$, $n \in \mathbb{N}$ are called the Euler approximations of semigroup $S(t)$. Throughout we assume that the semigroup $S(t)$ is continuous on the open interval $(0, \infty)$. We also define

$$K = \sup_{t>0} \|tS'(t)\|. \tag{3.2}$$

In [6] Bentkus obtained optimal bounds for

$$\|E_n(t) - S(t)\|, \qquad \text{as} \qquad n \to \infty,$$

and provided asymptotic expansions for Euler's approximations of semigroups with explicit and optimal bounds for the remainder terms. Using Laplace transforms Bentkus reduced the problem to the convergence rates and asymptotic expansions in the Law of Large Numbers. Here we demonstrate how his idea was used to prove the following proposition.

Proposition 3.1 (Bentkus 2009). *If a semigroup is bounded then the Euler's approximations $E_n(t)$ converge to the semigroup $S(t)$, that is, $\|E_n(t) - S(t)\| \to 0$, as $n \to \infty$.*

At first we introduce the following notations.

Let $\nu(du) = \mathbb{I}\{u \geq 0\}e^{-u}du$ be a probability measure ($\mathbb{I}(A)$ is the indicator function of the event A). Let ξ be a random variable with the distribution $\mathcal{L}(\xi) = \nu$. Let $\xi_1, \dots, \xi_n$ be independent copies of ξ. Changing the variable, we can rewrite f in (3.1) as follows

$$f(t/n) = \int_0^\infty S(tu/n)\,\nu(du) = \mathbb{E}S(t\xi/n), \tag{3.3}$$

where $\mathbb{E}$ stands for mathematical expectation. Writing

$$Z_n = n^{-1}(\xi_1 + \cdots + \xi_n),$$

and using (3.3) together with semigroup property, we obtain

$$E_n(t) = f^n(t/n) = \mathbb{E}S(t\xi_1/n)\dots\mathbb{E}S(t\xi_n/n) = \mathbb{E}S(tZ_n). \tag{3.4}$$

Note that $\mathbb{E}\xi = 1$. Write

$$\eta_k = \xi_k - 1, \qquad U_n = n^{-1}(\eta_1 + \cdots + \eta_n).$$

Since $Z_n = 1 + U_n$, we can rewrite (3.4) as

$$E_n(t) = \mathbb{E}S(tZ_n) = \mathbb{E}S(t + tU_n). \tag{3.5}$$

Identity (3.5) provides a basic tool for analysis of errors in Euler's approximations of semigroups. Here we demonstrate that the proof of Proposition 3.1 easily follows from the Law of Large numbers.

Proof of Proposition 3.1 (Bentkus 2009). By the Law of Large numbers $U_n \to 0$, as $n \to \infty$ with probability 1 (see, for example, [34]). Since $S(t)$ is bounded and continuous for $t > 0$, this immediately implies that the Euler's approximations $E_n(t)$ converge to the semigroup $S(t)$, that is, that

$$\|E_n(t) - S(t)\| = \|\mathbb{E}S(t + tU_n) - S(t)\| \to 0, \quad \text{as} \quad n \to \infty.$$

□

In a similar manner Bentkus then proves that for differentiable semigroup with $K \leq \infty$

$$\|E_n(t) - S(t)\| \leq \frac{4K^2}{n-1},$$

for $n = 2, 3, \ldots$. This estimate covers and refines all known related results, which are obtained for semigroups of operators in Banach spaces.

3.2 Asymptotic expansions

Using Laplace transforms and results from elementary probability theory Bentkus in [6] then proves the existence of asymptotic expansions and obtains optimal error bounds for the remainder terms. We provide explicit formulas for these expansions.

First we introduce some additional notation. Henceforth $\sum_{i_1+\cdots+i_n=k}$ means summation over all distinct ordered n-tuples of positive integers $i_1, \ldots, i_n$ whose elements sum to k. Write

$$c_{k,j} = \frac{1}{j!} \sum_{i_1+\cdots+i_j=k+j} \frac{1}{i_1 i_2 \ldots i_j}, \tag{3.6}$$

where $i_1, i_2, \ldots, i_j \geq 2$ and $1 \leq j \leq k$.

Theorem 3.1. *(Bentkus 2009) If a semigroup S is differentiable and $K < \infty$, then the Euler approximations $E_n(t)$ allow the asymptotic expansion*

$$E_n(t) = S(t) + \frac{a_1}{n} + \cdots + \frac{a_k}{n^k} + r, \quad for \quad n \geq 2, \tag{3.7}$$

with a remainder term r such that $\|r\| \leq C_k(1 + K^{2k+2})n^{-k-1}$, where C_k is a positive constant depending only on k.

The asymptotic expansion (3.7) and the bounds for the remainder terms r were obtained by Bentkus (Theorem 1.3 in [6]) using the Laplace transforms. In the following Lemma we obtain explicit formulas for the coefficients a_m.

Lemma 3.1. *The coefficients* a_m *in the asymptotic expansion* (3.7) *are*

$$a_m = \sum_{j=m+1}^{2m} c_{m,j-m} S^{(j)}(t) t^j, \tag{3.8}$$

for $m = 1, 2, \ldots,$ *with* $c_{k,j}$ *given by* (3.6).

Also it is easy to obtain the recurrence relations for $c_{k,j}$. From (3.6) we get (this can be easily checked using induction)

$$c_{m,1} = \frac{1}{m+1}, \quad c_{m,j} = \frac{1}{j} \sum_{k=j-1}^{m-1} \frac{c_{k,j-1}}{m-k+1}$$

for $j = 2, \ldots, m$ and $m = 1, 2, \ldots$.

We note that the derivatives $E_n^{(s)}(t)$, $s = 1, 2, \ldots$ allow the asymptotic expansion similar to (3.7). In order to obtain these expansions one can term-wise differentiate (3.7).

Now we provide explicit expressions for asymptotic expansions of the semigroup $S(t)$ in a series of powers of n^{-1} with coefficients b_k depending on derivatives of $E_n(t)$, i.e. asymptotic expansions

$$S(t) = E_n(t) + \frac{b_1}{n} + \cdots + \frac{b_k}{n^k} + r, \quad \text{for} \quad n \geq 2 \tag{3.9}$$

In order to establish these expansions, we have to establish expansions for the derivatives $S^{(m)}(t)$ as well, $m = 1, 2, \ldots$ Then the coefficients in (3.9) are given by

$$b_0 = E_n(t), \qquad b_m = -\sum_{l=1}^{m} \sum_{j=l+1}^{2l} c_{l,j-l} t^j b_{m-l}^{(j)}, \qquad m = 1, 2, \ldots \tag{3.10}$$

where $c_{i,j}$ are given by (3.6). For example, we have

$$b_1 = -\frac{t^2}{2}E_n^{(2)}(t),$$
$$b_2 = \frac{t^2}{2}E_n^{(2)}(t) + \frac{2t^3}{3}E_n^{(3)}(t) + \frac{t^4}{8}E_n^{(4)}(t),$$
$$b_3 = -\frac{t^2}{2}E_n^{(2)}(t) - 2t^3E_n^{(3)}(t) - \frac{3t^4}{2}E_n^{(4)}(t) - \frac{t^5}{3}E_n^{(5)}(t) - \frac{t^6}{48}E_n^{(6)}(t).$$

Theorem 3.2. *(Bentkus 2009) Assume that a semigroup S is differentiable and $K < \infty$. Let $n \geq 2$. Then the derivatives $S^{(s)}(t)$, $s = 0, 1, 2, \ldots$, allow the asymptotic expansion*

$$t^s S^{(s)}(t) = t^s E_n^{(s)}(t) + \frac{t^s b_1^{(s)}}{n} + \cdots + \frac{t^s b_k^{(s)}}{n^k} + r, \tag{3.11}$$

with a remainder term r satisfying

$$\|r\| \leq C_{s,k}(1 + K^{s+sk+2})n^{-k-1},$$

where $C_{s,k}$ is a positive constant depending only on s and k.

We note that when $s = 0$, we have asymptotic expansion (3.9) with $b_m = b_m^{(0)}$. We obtain the expressions for the coefficients of the asymptotic expansion (3.11).

Lemma 3.2. *The coefficients $b_m^{(s)}$ in asymptotic expansion* (3.11) *are given by*

$$b_0^{(s)} = E_n^{(s)}(t), \qquad b_k^{(s)} = -\sum_{l=1}^{k}\sum_{j=l+1}^{2l} c_{l,j-l} \sum_{i=0}^{\min(s,j)} \frac{j!}{(j-i)!} C_s^i t^{j-i} b_{k-l}^{(j+s-i)}, \tag{3.12}$$

for $k = 1, 2, \ldots$ and $s = 0, 1, 2, \ldots$. In case where $s = 0$ we obtain coefficients $b_m = b_m^{(0)}$ given by (3.10)*.*

In case of differentiable strongly continuous semigroups of operators we obtain simpler expressions for coefficients b_m. Recall that if the strongly continuous semigroup is differentiable then its derivate

$$S'(t) = AS(t), \tag{3.13}$$

where A is the generator of the semigroup $S(t)$ (see Section 1.2).

We write

$$h_m = \sum_{i=1}^{m} (-1)^i c_{m,i} (At)^{m+i}, \qquad m = 1, 2, \ldots. \tag{3.14}$$

Lemma 3.3. *If $S(t)$ is strongly continuous semigroup satisfying condition* (3.2) *with $K < \infty$, then it allows the asymptotic expansion* (3.9)*, where the coefficients*

$$b_0 = E_n(t), \qquad b_m = h_m E_n(t), \qquad m = 1, 2, \ldots \tag{3.15}$$

with h_m given by (3.14)*.*

3.3 Proofs

Proof of Lemma 3.1. In the proof of Theorem 1.3 in [6] it was demonstrated that coefficients a_m in (3.7) are linear combinations of $t^s S^{(s)}(t)$ with some numerical coefficients $c_{m,s}$ depending only on m and s. Since coefficients $c_{m,s}$ do not depend on concrete semigroup, we can determine them by taking, for example, semigroup $S(t) = e^{-t}$, $t \geq 0$. We write

$$(1 + t/n)^{-n} - e^{-t} = e^{-t} v_n(t),$$

where $v_n(t) = e^t (1 + t/n)^{-n} - 1$. Using expansions $\exp\{x\} = \sum_{k=0}^{\infty} x^k/k!$ and $\ln(1+x) = \sum_{k=1}^{\infty} (-1)^{k-1} x^k/k$ we get

$$\begin{aligned} v_n(t) &= \exp\left\{ t - n \ln\left(1 + \frac{t}{n}\right)\right\} - 1 \\ &= \exp\left\{ t - n \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \left(\frac{t}{n}\right)^k \right\} - 1 \\ &= \exp\left\{ -t \sum_{k=1}^{\infty} \frac{1}{k+1} \frac{(-t)^k}{n^k} \right\} - 1 = \sum_{j=1}^{\infty} \frac{(-t)^j}{j!} \left(\sum_{k=1}^{\infty} \frac{1}{k+1} \frac{(-t)^k}{n^k} \right)^j. \end{aligned}$$

Raising to the jth power and changing the order of summation we obtain

$$v_n(t) = \sum_{j=1}^{\infty} \frac{(-t)^j}{j!} \sum_{k=j}^{\infty} \frac{(-t)^k}{n^k} \sum_{i_1+\cdots+i_j=k} \frac{1}{(i_1+1)\dots(i_j+1)} = \sum_{k=1}^{\infty} \frac{1}{n^k} \sum_{j=1}^{k} (-t)^{k+j} c_{k,j},$$

where $c_{k,j}$ are given by (3.8). Replacing $(-t)^s e^{-t}$ with $t^s S^{(s)}(t)$ we obtain expression (3.8) for a_m. □

Proof of Lemma 3.2. We prove the theorem using induction with respect to k. In case when $k = 0$ we have $S(t) = b_0 + \Delta_0$, where $b_0 = E_n(t)$, $\Delta_0 = -r_0$, and $t^s S^{(s)}(t) = t^s b_0^{(s)} + \Delta_0^{(s)}$, for all $s = 1, 2, \dots$ When $k = 1$, from (3.7) we have

$$S(t) = E_n(t) - \frac{c_{1,1} t^2 S^{(2)}(t)}{n} - r_1.$$

Substituting $t^2 S^{(2)}(t) = t^2 b_0^{(2)} + \Delta_0^{(2)}$ we obtain

$$S(t) = E_n(t) + \frac{b_1}{n} + \Delta_1,$$

where $b_1 = -c_{1,1} t^2 b_0^{(2)}$ and $\Delta_1 = -r_1 - \frac{c_{1,1}}{n} \Delta_0^{(2)}$. Differentiating we get

$$b_1^{(s)} = -c_{1,1} \sum_{i=0}^{\min(s,2)} \frac{2!}{(2-i)!} C_s^i t^{2-i} b_0^{(2+s-i)},$$

for $s = 1, 2, \dots$

Assume, that (3.9) and (3.11) hold for $0, 1, \dots, k-1$ and $s = 1, 2, \dots$. Let us show that (3.9) and (3.11) hold for k as well. From (3.7) we have

$$S(t) = E_n(t) - \frac{a_1}{n} - \cdots - \frac{a_k}{n^k} - r_k, \tag{3.16}$$

where

$$\frac{a_m}{n^m} = \frac{1}{n^m} \sum_{s=m+1}^{2m} c_{m,s-m} t^s S^{(s)}(t) \tag{3.17}$$

with $c_{m,s}$ given by (3.6). From (3.9) and (3.11) we have

$$S(t) = E_n(t) + \frac{b_1}{n} + \cdots + \frac{b_{k-m}}{n^{k-m}} + \Delta_{k-m}$$

$$t^s S^{(s)}(t) = t^s E_n^{(s)}(t) + \frac{t^s b_1^{(s)}}{n} + \cdots + \frac{t^s b_{k-m}^{(s)}}{n^{k-m}} + \Delta_{k-m}^{(s)}, \quad \text{for} \quad s = m+1, \ldots, 2m \quad (3.18)$$

where $m = 1, 2, \ldots, k-1$. Substituting (3.18) into expression (3.17) we obtain

$$\begin{aligned} \frac{a_m}{n^m} =& \frac{1}{n^m} \sum_{s=m+1}^{2m} c_{m,s-m} t^s \left(E_n^{(s)}(t) + \frac{b_1^{(s)}}{n} + \cdots + \frac{b_{k-m}^{(s)}}{n^{k-m}} \right) + \\ &+ \frac{1}{n^m} \sum_{s=m+1}^{2m} c_{m,s-m} \Delta_{k-m}^{(s)}, \end{aligned} \quad (3.19)$$

for $m = 1, 2, \ldots, k$. Substituting (3.19) into (3.16), then collecting terms with the same powers of n and moving terms containing the remainder terms into total remainder term Δ_k we obtain expression (3.9) with b_k given by (3.10). Differentiating b_k with respect to t we get expression (3.12) and from here we obtain asymptotic expansion (3.11).

□

Proof of Lemma 3.3. We first note that if we take in mind the property (3.13), then coefficients a_m in asymptotic expansion (3.7) take the form $a_m = d_m S(t)$, where $d_m = \sum_{j=1}^{m} c_{m,j} (At)^{m+j}$, for $m = 1, 2, \ldots$. This means that to obtain the inverse expansion (3.9) we do not need to find the asymptotic expansions of the derivatives of $S(t)$ and $E_n(t)$ as in Lemma 3.2. Using induction on k like in the proof of Lemma 3.2 we then obtain the following recurrence expressions for coefficients b_m in (3.9) :

$$b_0 = E_n(t), \qquad b_m = -\sum_{j=1}^{m} d_j b_{m-j}, \quad m = 1, 2, \ldots$$

From here it is easy to obtain another form of these coefficients (this can be checked using induction)

$$b_m = \sum_{r=1}^{m} (-1)^r \sum_{i_1 + \cdots + i_r = m} d_{i_1} \ldots d_{i_r} b_0, \quad m = 1, 2, \ldots.$$

We see that the coefficients b_m have the form $b_m = h_m b_0$, where h_m are the linear combinations of $(At)^{m+1}, (At)^{m+2}, \ldots, (At)^{2m}$ with some numerical coefficients which do not depend on concrete semigroup. Therefore, in order to determine them we can take, as in the proof of Lemma 3.1, the semigroup $S(t) = e^{-t}$, $t \geq 0$. Then we write

$$e^{-t} - (1 + t/n)^{-n} = (1 + t/n)^{-n} u_n(t),$$

where $u_n(t) = e^{-t}(1+t/n)^n - 1$. It's easy to see that $u_n(t) = v_{-n}(-t)$ ($v_n(t)$ was defined in the proof of Lemma 3.1). From here (3.15) follows.

□

4 Optimal error bounds and asymptotic expansions for Yosida approximations of semigroups

4.1 Introduction

Let X be a Banach space and $L(X)$ be the space of bounded linear operators on X. Let A be a generator of a strongly continuous semigroup of contractions $S(t)$. We define the Yosida approximant of A by

$$A_\lambda = \lambda A(\lambda I - A)^{-1},$$

for all $\lambda > 0$.

As we noted in Section 1.3, it can be shown that A_λ is the generator of a uniformly continuous semigroup of contractions $S_\lambda(t)$. Furthermore,

$$S(t)x = \lim_{\lambda\to\infty} S_\lambda(t)x, \quad \text{for } x \in X. \tag{4.1}$$

We call $S_\lambda(t)$, $\lambda > 0$, Yosida approximations of contraction semigroup $S(t)$.

In this chapter we investigate the rate of convergence in (4.1). In particular, for bounded holomorphic semigroups of contractions, we obtain optimal error bounds for

$$\|S_\lambda(t)x - S(t)x\|, \qquad \text{for} \quad x \in D(A), \quad t \geq 0 \quad \text{and} \quad \lambda > 0,$$

and

$$\|S_\lambda(t) - S(t)\|, \qquad \text{for} \quad x \in X, \quad t > 0 \quad \text{and} \quad \lambda > 0.$$

We also provide asymptotic expansions for Yosida approximations $S_\lambda(t)$ of differentiable semigroups $S(t)$, i.e. expansions of type

$$S_\lambda(t) = S(t) + \frac{a_1}{\lambda} + \frac{a_2}{\lambda^2} + \cdots + \frac{a_k}{\lambda^k} + o\left(\frac{1}{\lambda^k}\right), \quad \lambda \to \infty,$$

where coefficients a_m do not depend on λ. We also obtain the inverse expansions

$$S(t) = S_\lambda(t) + \frac{b_1}{\lambda} + \frac{b_2}{\lambda^2} + \cdots + \frac{b_k}{\lambda^k} + o\left(\frac{1}{\lambda^k}\right), \quad \lambda \to \infty,$$

here the coefficients b_k are the linear combinations of the derivatives of $S_\lambda(t)$. We also provide optimal bounds for the remainder terms of these expansions in case of bounded holomorphic semigroups of contractions.

To obtain error bounds and asymptotic expansions we use an approach introduced in Section 1.4. Recall that using this method we express the difference $b-a$ as

$$b - a = \int_0^1 \gamma'(\tau) d\tau, \tag{4.2}$$

where $\gamma(\tau)$ is some smooth curve connecting two close objects $a = \gamma(0)$ and $b = \gamma(1)$. We then estimate the integral in (4.2). Here we investigate how this method works with two different choices of $\gamma(\tau)$: $\gamma_1(\tau)$ and $\gamma_2(\tau)$.

1st case. We choose

$$\gamma_1(\tau) = S_\lambda((1-\tau)t)S(\tau t).$$

Then $b = \gamma_1(0) = S_\lambda(t)$, $a = \gamma_1(1) = S(t)$ and

$$\begin{aligned}\gamma_1'(\tau) =&(S_\lambda((1-\tau)t))'S(\tau t) + S_\lambda((1-\tau)t))(S(\tau t))' = \\ =& - A_\lambda t S_\lambda((1-\tau)t)S(\tau t) + AtS_\lambda((1-\tau)t)S(\tau t) = \\ =& t(A - A_\lambda)\gamma_1(\tau) = -\frac{1}{\lambda} tAA_\lambda \gamma_1(\tau).\end{aligned}$$

So, we have

$$D_0 := S_\lambda(t) - S(t) = -\int_0^1 \gamma_1'(\tau) d\tau = \frac{1}{\lambda}\int_0^1 tAA_\lambda \gamma_1(\tau) d\tau. \tag{4.3}$$

2nd case. Here we choose γ_2 in this manner

$$\gamma_2(\tau) := S_{\lambda/\tau}(t) = \exp\left\{tA\frac{\lambda}{\tau}\left(\frac{\lambda}{\tau}I - A\right)^{-1}\right\} = \exp\left\{tA\lambda(\lambda I - \tau A)^{-1}\right\}. \tag{4.4}$$

Then $\gamma_2(1) = S_\lambda(t)$ and $\gamma_2(0)x = \lim_{\tau \downarrow 0} S_{\lambda/\tau}(t)x = S(t)x$, for all $x \in X$. Differentiating we get

$$\gamma_2'(\tau) = tA^2\lambda(\lambda I - \tau A)^{-2}S_{\lambda/\tau}(t) = \frac{1}{\lambda}tA^2_{\lambda/\tau}S_{\lambda/\tau}(t).$$

So, we obtain another integro-differential identity

$$D_0 = S_\lambda(t) - S(t) = \int_0^1 \gamma_2'(\tau)d\tau = \frac{1}{\lambda t}\int_0^1 (tA_{\lambda/\tau})^2 S_{\lambda/\tau}(t)d\tau. \tag{4.5}$$

We note with the help of γ_2 we obtain better error estimates and shorter proofs than with γ_1.

4.2 Error bounds

Assume there exists a positive constant K independent of n, λ and t such that

$$\|tAS(t)\| \le K, \tag{4.6}$$

and

$$(n+1)\|A\lambda^n(\lambda I - A)^{-n-1}\| \le K, \quad n = 0, 1, 2, \ldots, \tag{4.7}$$

for all $\lambda > 0$, $t \ge 0$.

We note that bounded holomorphic semigroups satisfy the first inequality (4.6) by Theorem 2.5.2 in [33]. For $n = 0$, the second inequality (4.7) follows from $\|\lambda(\lambda - A)^{-1}\| \le 1$ which is true if A is the generator of semigroup of contractions (see Theorem 1.3.1 in [33]). Indeed, then $A(\lambda - A)^{-1} = A(\lambda - A)^{-1} + I - I = \lambda(\lambda - A)^{-1} - I$, so $A(\lambda - A)^{-1}$ is a bounded linear operator. For $n = 1, 2, \ldots,$ the inequality (4.7) holds for bounded holomorphic semigroups by Theorem 2.5.5 in [33]. We note that if the generator A of semigroup $S(t)$ is unbounded operator then $K \ge e^{-1}$ by Theorem 2.5.3 in [33].

We also prove the following lemma:

Lemma 4.1. *Assume that A is a generator of contraction semigroup and there exists a positive constant K independent of n, λ and t such that conditions* (4.6) *and* (4.7) *are satisfied for all $\lambda > 0$, $t \ge 0$ and $n = 0, 1, 2, \ldots$. Then Yosida approximations satisfy*

$$\|tA_\lambda S_\lambda(t)\| \le K, \tag{4.8}$$

for all $\lambda > 0$ *and* $t \geq 0$.

It's easy to show that if (4.6) and (4.8) hold then

$$\|(tA)^m S(t)\| \leq m^m K^m, \quad \text{and} \quad \|(tA_\lambda)^m S_\lambda(t)\| \leq m^m K^m, \tag{4.9}$$

for all $t \geq 0$, $\lambda > 0$ and $m = 1, 2, \ldots$. (See Lemma 2.1 in [38].)

Now we provide the optimal bounds for the convergence rate of $S_\lambda(t)$ to a semigroup $S(t)$ satisfying conditions (4.6) and (4.8).

1st case. For the difference $D_0 x = S_\lambda(t)x - S(t)x$ given by (4.3) we obtain the following.

Theorem 4.1. *Assume that semigroup* $S(t)$ *satisfies conditions* (4.6) *and* (4.8). *Then for all* $\lambda > 0$, *the following inequality holds*

$$\|S_\lambda(t)x - S(t)x\| \leq \frac{\ln(4) K \|Ax\|}{\lambda}, \qquad x \in D(A). \tag{4.10}$$

2nd case. With the help of the identity (4.5) we obtain another type of bound (with optimal constant) for $D_0 = S_\lambda(t) - S(t)$. We also improve the constant in (4.10) to the optimal.

Theorem 4.2. *Assume that contraction semigroup* $S(t)$ *satisfies conditions* (4.6) *and* (4.8). *Then convergence rate in* (4.1) *satisfies:*

$$\|S_\lambda(t) - S(t)\| \leq \frac{4K^2}{\lambda t}, \tag{4.11}$$

for all $t > 0$ *and* $\lambda > 0$. *Besides, for all* $x \in D(A)$ *the following inequality holds:*

$$\|S_\lambda(t)x - S(t)x\| \leq \frac{K\|Ax\|}{\lambda}, \tag{4.12}$$

where $t \geq 0$ *and* $\lambda > 0$.

Remark 4.1. Both bounds (4.11) and (4.12) are optimal. Let Q_K be the set of all contraction semigroups $S(t)$ satisfying conditions (4.6) and (4.8) for a fixed K. To prove that the bound in (4.11) is optimal it suffices to find a semigroup $S(t) \in Q_K$ such that $\lim_{\lambda\to\infty} \lambda \|S_\lambda(t) - S(t)\| = 4K^2/t$ for all $t > 0$. Consider the case $X = \mathbb{C}$ and $S(t) = e^{-zt}$ with $\operatorname{Re} z > 0$. Then $|S(t)| \leqslant 1$, i.e., $S(t)$ is a contraction semigroup. Write $\Delta_\lambda(t, z) = \|S_\lambda(t) - S(t)\| = |e^{-tz\lambda(\lambda+z)^{-1}} - e^{-tz}|$. Take $z_1 = (2 + i2\sqrt{K^2e^2 - 1})/t$, where

$K \geqslant e^{-1}$. It is easy to check that $S(t) = e^{-z_1 t}$ satisfies conditions (4.6) and (4.8) with the same constant K (here the generator $A = -z_1$), i.e., $S(t) \in Q_K$. Then it is easy to see that

$$\lim_{\lambda\to\infty} \lambda\Delta_\lambda(t, z_1) = \frac{4K^2}{t}.$$

Note that the constants in (4.11) are also optimal.
Similarly, to prove that the bound (4.12) is optimal, take $z_2 = x + ix\sqrt{K^2e^2 - 1}$ with $x > 0$ and $t_1 = 1/x$. Then $S(t) = e^{-z_2 t} \in Q_K$ and $\lim_{\lambda\to\infty} \lambda\Delta_\lambda(t_1, z_2) = |z_2|K$. Note that the constants in (4.12) are optimal as well.

Remark 4.2. Taking $z_0 = \lambda + i\lambda\sqrt{K^2e^2 - 1}$ and $t_0 = 1/\lambda$ in the previous example, we can show that $\sup_{z\in Q_K, t>0} \Delta_\lambda(t, z)$ does not converge to 0, i.e. the uniform bound for $\Delta_\lambda(t, z)$ with respect to z and t does not exist. Indeed, we have

$$\sup_{z\in Q_K, t>0} \Delta_\lambda(t, z) \geq \Delta_\lambda(t_0, z_0) = c,$$

where c is a positive constant independent of λ. So,

$$\lim_{\lambda\to\infty} \sup_{z\in Q_K, t>0} \Delta_\lambda(t, z) \neq 0.$$

4.3 Short asymptotic expansions

Using integro-differential identities (4.3) and (4.5) we obtain asymptotic expansions for a semigroup $S(t)$ and its Yosida approximations $S_\lambda(t)$. At first we provide short asymptotic expansions with remainders $O(\lambda^{-2})$.

Theorem 4.3. *Assume that the semigroup $S(t)$ is differentiable. Then the following integro-differential identity holds*

$$S_\lambda(t) = S(t) + \frac{a_1}{\lambda} + D_1, \tag{4.13}$$

where coefficient a_1 is given by

$$a_1 = tA^2S(t), \tag{4.14}$$

and the remainder term D_1 satisfies

$$D_1 = D_{1,1} + D_{1,2}, \tag{4.15}$$

with

$$D_{1,1} = \frac{tA^2A_\lambda}{\lambda^2}S(t),$$

and

$$D_{1,2} = \frac{1}{\lambda^2}\int_0^1 \tau(tAA_\lambda)^2\gamma_1(\tau)d\tau.$$

Theorem 4.4. *Assume that strongly continuous semigroup of contractions $S(t)$ satisfies conditions* (4.6) *and* (4.8)*. Then the remainder term D_1 satisfies*

$$\|D_1x\| \leq \frac{C_1K(1+K)\|A^2x\|}{\lambda^2}, \qquad x \in D(A^2), \tag{4.16}$$

where $\lambda > 0$, $t \geq 0$ and C_1 is an absolute positive constant.

We note that the identity (4.13) and the estimate (4.16) were obtained using (4.3). Next we provide the inverse asymptotic expansion using (4.5), but we note that it is possible to obtain the inverse expansions from (4.13) as well.

Theorem 4.5. *Assume that strongly continuous semigroup $S(t)$ of contractions satisfies conditions* (4.6) *and* (4.8)*. Then*

$$S(t) = S_\lambda(t) + \frac{b_1}{\lambda} + d_1,$$

where coefficient b_1 is given by

$$b_1 = -tA_\lambda^2S_\lambda(t),$$

and the remainder term d_1 satisfies

$$\|d_1\| \leq \frac{c_1K^3(1+K)}{\lambda^2t^2}, \quad \textit{for} \quad t > 0 \quad \textit{and} \quad \lambda > 0,$$

where c_1 is an absolute positive constant.

4.4 The general case

In this section we provide asymptotic expansions of any length k, i.e. the expansions

$$S_\lambda(t) = S(t) + \frac{a_1}{\lambda} + \frac{a_2}{\lambda^2} + \cdots + \frac{a_k}{\lambda^k} + D_k, \tag{4.17}$$

for $k = 1, 2, \ldots$.

First we denote

$$d_{m,1,1} = 1, \quad m = 1, 2, \ldots,$$

$$d_{m,m,j} = \frac{1}{m!}, \quad m = 1, 2, \ldots, \quad j = 1, 2, \ldots, m, \tag{4.18}$$

$$d_{m,k,j} = \sum_{i=1}^{j} d_{m-1,k,i}, \quad m = 2, 3, \ldots, \quad k = 1, 2, \ldots, m-1, \quad j = 1, 2, \ldots, k.$$

Theorem 4.6. *Let $S(t)$ be a differentiable semigroup. Then the coefficients a_m in* (4.17) *are given by*

$$a_m = \sum_{k=1}^{m} d_{m,k,k} t^k A^{m+k} S(t), \tag{4.19}$$

and the remainder terms D_m are

$$D_m = D_{m,1} + D_{m,2} \tag{4.20}$$

where

$$D_{m,1} = \frac{1}{\lambda^{m+1}} \sum_{k=1}^{m} \sum_{j=1}^{k} d_{m,k,j} t^k A^{m+j} A_\lambda^{k+1-j} S(t),$$

and

$$D_{m,2} = \frac{1}{\lambda^{m+1}} \int_0^1 \frac{\tau^m}{m!} (tAA_\lambda)^{m+1} S_\lambda((1-\tau)t) S(\tau t) d\tau,$$

with coefficients $d_{m,k,j}$ given by (4.18).

For example, the first three coefficients of the expansion are

$$
\begin{aligned}
a_1 =& tA^2S(t),\\
a_2 =& tA^3S(t) + \frac{t^2A^4}{2}S(t),\\
a_3 =& tA^4S(t) + t^2A^5S(t) + \frac{t^3A^6}{6}S(t).
\end{aligned}
$$

Theorem 4.7. *Assume that semigroup $S(t)$ satisfies conditions* (4.6) *and* (4.8). *Then the remainder terms D_m in* (4.17) *satisfy*

$$
\|D_m x\| \le \frac{C_m(1+K^{m+1})\|A^{m+1}x\|}{\lambda^{m+1}}, \quad m = 1, 2, \ldots
$$

for $\lambda > 0$, $x \in D(A^{m+1})$ and some positive constant C_m depending only on m.

Next we provide the inverse asymptotic expansions. At first we obtain the integro-differential identity

$$
S(t) = S(t)_\lambda + \frac{b_1}{\lambda} + \frac{b_2}{\lambda^2} + \cdots + \frac{b_k}{\lambda^k} + d_k, \tag{4.21}
$$

and then we provide the optimal bounds for the remainder terms d_k.

Denote

$$
h_{m,1} = m!, \qquad h_{m,m} = 1, \qquad m = 1, 2, \ldots,
$$

and

$$
h_{m,k} = (m+k-1)h_{m-1,k} + h_{m-1,k-1}, \;\; m = 3, 4, \ldots, \;\; k = 2, \ldots, m-1. \tag{4.22}
$$

Theorem 4.8. *Let $S(t)$ be a differentiable semigroup. Then the coefficients b_m in* (4.21) *are given by*

$$
\frac{b_m}{\lambda^m} = \frac{(-1)^m}{m!}\gamma_2^{(m)}(1), \tag{4.23}
$$

and the remainder terms d_m are

$$
d_m = (-1)^{m+1} \int\limits_0^1 \frac{\tau^m}{m!}\gamma_2^{(m+1)}(\tau)d\tau, \tag{4.24}
$$

where $\gamma_2(\tau)$ *is given by* (4.4) *and* $\gamma_2^{(m)}(\tau)$ *is its* m*th derivative, given by*

$$\gamma_2^{(m)}(\tau) = \frac{1}{\lambda^m} \sum_{k=1}^{m} h_{m,k} t^k A_{\lambda/\tau}^{m+k} S_{\lambda/\tau}(t),$$

with coefficients $h_{m,k}$ *given by* (4.22).

For example, the first three coefficients of the expansion are

$$\begin{aligned}
b_1 =& - tA_\lambda^2 S_\lambda(t),\\
b_2 =& tA_\lambda^3 S_\lambda(t) + \frac{t^2 A_\lambda^4}{2} S_\lambda(t),\\
b_3 =& - tA_\lambda^4 S_\lambda(t) - t^2 A_\lambda^5 S_\lambda(t) - \frac{t^3 A_\lambda^6}{6} S_\lambda(t).
\end{aligned}$$

Theorem 4.9. *Assume that semigroup* $S(t)$ *satisfies conditions* (4.6) *and* (4.8). *Then the remainder terms* d_m *in* (4.21) *satisfy*

$$\|d_m\| \le \frac{c_m(1 + K^{2m+2})}{(t\lambda)^{m+1}}, \quad m = 1, 2, \dots$$

for $\lambda > 0$, $t > 0$ *and some positive constant* c_m *depending only on* m.

4.5 Proofs

Proof of Lemma 4.1. The proof is similar to the proof of Lemma 2.1 in [7]. We have $A_\lambda = \lambda A(\lambda I - A)^{-1} = \lambda^2(\lambda I - A)^{-1} - \lambda I$. Expanding $e^{t\lambda^2(\lambda I - A)^{-1}}$ into the Taylor series we get

$$tA_\lambda S_\lambda(t) = tA_\lambda e^{tA_\lambda} = e^{-\lambda t} \sum_{n=0}^{\infty} \frac{(\lambda t)^{n+1}}{n!} A\lambda^n (\lambda I - A)^{-n-1}.$$

From (4.6) we have $(n+1)\|A\lambda^n(\lambda I - A)^{-n-1}\| \le K$, so that

$$\|tA_\lambda S_\lambda(t)\| \le Ke^{-\lambda t} \sum_{n=0}^{\infty} \frac{(\lambda t)^{n+1}}{(n+1)!} = K(1 - e^{-t\lambda}) \le K,$$

for all $\lambda > 0$ and $t \ge 0$.

□

Proof of Theorem 4.1. From (4.3) we have

$$D_0 x = \frac{1}{\lambda} \int_0^1 t A A_\lambda \gamma_1(\tau) x d\tau.$$

We denote

$$J_1 x = \int_0^{1/2} t A A_\lambda \gamma_1(\tau) x d\tau \quad \text{and} \quad J_2 x = \int_{1/2}^1 t A A_\lambda \gamma_1(\tau) x d\tau.$$

Then $\|D_0 x\| \leq \frac{1}{\lambda}(\|J_1 x\| + \|J_2 x\|)$. First we estimate $\|J_1 x\|$. We have

$$\|J_1 x\| \leq \int_0^{1/2} \|t A A_\lambda \gamma_1(\tau) x\| d\tau \leq \int_0^{1/2} \frac{\delta_1 \delta_2}{1-\tau} d\tau,$$

where $\delta_1 = \|AS(\tau t)x\|$ and $\delta_2 = \|(1-\tau)tA_\lambda S_\lambda((1-\tau)t)\|$. Since $S(t)$ is semigroup of contractions, we have $\delta_1 \leq \|Ax\|$ and from (4.8) we also have $\delta_2 \leq K$. We obtain

$$\|J_1 x\| \leq K\|Ax\| \int_0^{1/2} \frac{1}{1-\tau} d\tau = \ln(2) K \|Ax\|, \qquad x \in D(A). \tag{4.25}$$

Next we estimate $\|J_2 x\|$. We have

$$\|J_2 x\| \leq \int_{1/2}^1 \|t A A_\lambda \gamma_1(\tau) x\| d\tau \leq \int_{1/2}^1 \frac{\delta_3 \delta_4}{\tau} d\tau,$$

where $\delta_3 = \|A_\lambda S_\lambda((1-\tau)t)x\|$ and $\delta_4 = \|\tau t A S(\tau t)\|$. By Theorem 1.3.1 in [33] we have that the resolvent of semigroup of contractions satisfies $\|\lambda(\lambda I - A)^{-1}\| \leq 1$ for all $\lambda > 0$. It follows that $\|A_\lambda x\| = \|\lambda A(\lambda I - A)^{-1} x\| = \|\lambda(\lambda I - A)^{-1} A x\| \leq \|Ax\|$, for $x \in D(A)$ and $\delta_3 \leq \|Ax\|$. From condition (4.6) we have $\delta_4 \leq K$. Then

$$\|J_2 x\| \leq K\|Ax\| \int_{1/2}^1 \frac{1}{\tau} d\tau = \ln(2) K \|Ax\|, \qquad x \in D(A), \tag{4.26}$$

and substituting (4.25) and (4.26) into $\|D_0 x\| \leq \frac{1}{\lambda}(\|J_1 x\| + \|J_2 x\|)$ we obtain (4.10).

□

Proof of Theorem 4.2. To obtain the error bounds we use the identity (4.5):

$$D_0 x = \frac{1}{\lambda t} \int_0^1 (tA_{\lambda/\tau})^2 S_{\lambda/\tau}(t) x d\tau.$$

From (4.9) we have $\|(tA_{\lambda/\tau})^2 S_{\lambda/\tau}(t)x\| \le 4K^2\|x\|$ for all $\tau \in (0,1)$. Then

$$\|D_0 x\| \le \frac{4K^2\|x\|}{\lambda t}, \quad \text{for all} \quad x \in X,$$

and thus

$$\|D_0\| \le \frac{4K^2}{\lambda t}.$$

Also, from (4.8) we have $\|tA_{\lambda/\tau}S_{\lambda/\tau}(t)\| \le K$ for all $\tau \in (0,1)$. From definition of A_λ we obtain $\|A_{\lambda/\tau}x\| \le \|\lambda/\tau(\lambda/\tau I - A)^{-1}\| \cdot \|Ax\|$. By Theorem 1.3.1 in [33] we have $\|\lambda(\lambda I - A)^{-1}\| \le 1$ for $\lambda > 0$, so that $\|A_{\lambda/\tau}x\| \le \|Ax\|$ for $\tau \in (0,1)$. Then

$$\|D_0 x\| \le \frac{1}{\lambda} \int_0^1 \|tA_{\lambda/\tau}S_{\lambda/\tau}(t)\| \cdot \|A_{\lambda/\tau}x\| d\tau \le \frac{K\|Ax\|}{\lambda},$$

for all $x \in D(A)$. □

Lemma 4.2. *Let A_λ be a Yosida approximant of A. Then for any $k = 1, 2, \ldots$ the following identity holds:*

$$A_\lambda^k = A^k + \frac{1}{\lambda}\sum_{j=1}^{k} A^j A_\lambda^{k-j+1}. \tag{4.27}$$

Proof. Using the definition of A_λ, it's easy to prove the identity (4.27) for $k = 1$.

$$A_\lambda - A = \lambda A(\lambda I - A)^{-1} - A = \frac{AA_\lambda}{\lambda}. \tag{4.28}$$

Suppose, that the identity (4.27) holds for $k = m$, i.e.

$$A_\lambda^m = A^m + \frac{1}{\lambda}\sum_{j=1}^{m} A^j A_\lambda^{m-j+1}. \tag{4.29}$$

We demonstrate that (4.27) holds for $k = m+1$. We have

$$A_\lambda^{m+1} - A^{m+1} = A_\lambda^{m+1} - A^{m+1} + AA_\lambda^m - AA_\lambda^m = (A_\lambda - A)A_\lambda^m + A(A_\lambda^m - A^m). \quad (4.30)$$

Substituting (4.28) and (4.29) into (4.30) we obtain

$$A_\lambda^{m+1} - A^{m+1} = \frac{1}{\lambda} \sum_{j=1}^{m+1} A^j A_\lambda^{m+1-j+1}.$$

□

Proof of Theorem 4.3. From (4.3) we have $S_\lambda(t) = S(t) + D_0$ where

$$D_0 = \frac{1}{\lambda} \int_0^1 tAA_\lambda \gamma_1(\tau) d\tau.$$

Integrating D_0 by parts we obtain

$$D_0 = \frac{1}{\lambda} tAA_\lambda S(t) + \frac{1}{\lambda^2} \int_0^1 \tau (tAA_\lambda)^2 \gamma_1(\tau) d\tau. \quad (4.31)$$

By Lemma 4.2 we have $A_\lambda = A + \frac{AA_\lambda}{\lambda}$. Substituting this identity into the first term of the sum in (4.31) we have

$$D_0 = \frac{tA^2}{\lambda} S(t) + \frac{tA^2 A_\lambda}{\lambda^2} S(t) + \frac{1}{\lambda^2} \int_0^1 \tau (tAA_\lambda)^2 \gamma_1(\tau) d\tau = \frac{a_1}{\lambda} + D_1,$$

where a_1 is given by (4.14) and D_1 is given by (4.15).

□

Proof of Theorem 4.4. From Theorem 4.3 we have $D_1 = D_{1,1} + D_{1,2}$, where

$$D_{1,1} = \frac{tA^2 A_\lambda}{\lambda^2} S(t),$$

and

$$D_{1,2} = \frac{1}{\lambda^2} \int_0^1 \tau (tAA_\lambda)^2 \gamma_1(\tau) d\tau.$$

At first we estimate $\|D_{1,1}x\|$. From (4.6) we have $\|tAS(t)\| \leq K$. From definition of A_λ we obtain $\|AA_\lambda x\| \leq \|\lambda(\lambda I - A)^{-1}\| \cdot \|A^2x\|$ for $x \in D(A^2)$. By Theorem 1.3.1 in [33] we have $\|\lambda(\lambda I - A)^{-1}\| \leq 1$ for $\lambda > 0$, so that $\|AA_\lambda x\| \leq \|A^2x\|$ for $x \in D(A^2)$. Then

$$\|D_{1,1}x\| = \frac{K\|A^2x\|}{\lambda^2}, \qquad x \in D(A^2).$$

To estimate $\|D_{1,2}x\|$, we write $D_{1,2} = I_1 + I_2$, where

$$I_1 = \int_0^{1/2} \tau(tAA_\lambda)^2\gamma_1(\tau)d\tau, \quad \text{and} \quad I_2 = \int_{1/2}^{1} \tau(tAA_\lambda)^2\gamma_1(\tau)d\tau.$$

Then $\|D_{1,2}x\| \leq \frac{1}{\lambda^2}(\|I_1x\| + \|I_2x\|)$. First we estimate $\|I_1x\|$. We have

$$\|I_1x\| \leq \int_0^{1/2} \frac{\tau}{(1-\tau)^2}\delta_1\delta_2 d\tau,$$

where $\delta_1 = \|A^2S(\tau t)x\|$ and $\delta_2 = \|(1-\tau)^2(tA_\lambda)^2S_\lambda((1-\tau)t)\|$. Since $S(t)$ is a semigroup of contractions, we have $\delta_1 \leq \|A^2x\|$ and from (4.9) we have $\delta_2 \leq 4K^2$. Then

$$\|I_1x\| \leq 4K^2\|A^2x\| \int_0^{1/2} \frac{\tau}{(1-\tau)^2} d\tau \leq 4\ln(e/2)K^2\|A^2x\|, \qquad x \in D(A^2).$$

Similarly, we estimate $\|I_2x\|$. We have

$$\|I_2x\| \leq \int_{1/2}^{1} \frac{\delta_3\delta_4}{\tau} d\tau,$$

where $\delta_3 = \|A_\lambda^2 S_\lambda((1-\tau)t)$ and $\delta_4 = \|(\tau tA)^2S(\tau t)\|$. Since $S(t)$ is a semigroup of contractions and $\|A_\lambda^2 x\| \leq \|A^2x\|$, we have $\delta_3 \leq \|A^2x\|$. From (4.9) we have $\delta_4 \leq 4K^2$. So, we obtain

$$\|I_2x\| \leq 4K^2\|A^2x\| \int_{1/2}^{1} \frac{1}{\tau} d\tau \leq 4\ln(2)K^2\|A^2x\|, \qquad x \in D(A^2).$$

Finally, we obtain

$$\|D_1x\| \leq \|D_{1,1}x\| + \|D_{1,2}x\| \leq \frac{C_1K(1+K)\|A^2x\|}{\lambda^2}, \qquad x \in D(A^2),$$

where C_1 is an absolute positive constant. □

Proof of Theorem 4.5. From (4.5) we have

$$S(t) = S_\lambda(t) + d_0,$$

where $d_0 = -D_0 = -\int\limits_0^1 \gamma_2'(\tau)d\tau$. Integrating by parts we obtain

$$d_0 = \frac{b_1}{\lambda} + d_1,$$

where $b_1 = -\gamma_2'(1) = -tA_\lambda^2 S_\lambda(t)$ and

$$d_1 = \int\limits_0^1 \tau\gamma_2''(\tau)d\tau = \frac{1}{t^2\lambda^2}\int\limits_0^1 \tau S_{\lambda/\tau}(t)\left((tA_{\lambda/\tau})^4 + 2(tA_{\lambda/\tau})^3\right)d\tau.$$

From (4.9) we have $\|(tA_{\lambda/\tau})^4 S_{\lambda/\tau}(t)x\| \leq 4^4K^4\|x\|$ and $\|(tA_{\lambda/\tau})^3 S_{\lambda/\tau}(t)x\| \leq 3^3K^3\|x\|$ for all $\tau \in (0,1)$. Then

$$\|d_1x\| \leq \frac{CK^3(1+K)\|x\|}{\lambda^2t^2}, \quad \text{for all} \quad x \in X,$$

and thus

$$\|d_1\| \leq \frac{CK^3(1+K)}{\lambda^2t^2}.$$

□

Proof of Theorem 4.6. We proved (4.19) and (4.20) for $m = 1$ in Theorem 4.3. Using induction on m we obtain the general result. Suppose, that (4.19) and (4.20) hold for m. Let us prove that (4.19) and (4.20) hold for $m+1$ as well.
We have

$$D_{m,1} = \frac{1}{\lambda^{m+1}}\sum_{k=1}^{m}\sum_{j=1}^{k} d_{m,k,j}t^kA^{m+j}A_\lambda^{k+1-j}S(t),$$

and

$$D_{m,2} = \frac{1}{\lambda^{m+1}} \int_0^1 \frac{\tau^m}{m!} (tAA_\lambda)^{m+1} \gamma_1(\tau) d\tau,$$

By (4.27), we have $A_\lambda^{k+1-j} = A^{k+1-j} + \frac{1}{\lambda} \sum_{i=1}^{k+1-j} A^i A_\lambda^{k+1-j-i+1}$. Substituting this identity into expression of $D_{m,1}$, we obtain

$$D_{m,1} = a_{m+1,1}/\lambda^{m+1} + D_{m+1,1,1},$$

where

$$a_{m+1,1} = \sum_{k=1}^{m} \sum_{j=1}^{k} d_{m,k,j} t^k A^{m+k+1} S(t), \tag{4.32}$$

and

$$D_{m+1,1,1} = \frac{1}{\lambda^{m+2}} \sum_{k=1}^{m} t^k \sum_{j=1}^{k} d_{m,k,j} \sum_{l=1}^{k+1-j} A^{m+j+l} A_\lambda^{k-j-l+2} S(t).$$

Changing the variable of summation from l to $i = l + j - 1$ in the third sum of $D_{m,1,2}$ we obtain

$$D_{m+1,1,1} = \frac{1}{\lambda^{m+2}} \sum_{k=1}^{m} t^k \sum_{j=1}^{k} d_{m,k,j} \sum_{i=j}^{k} A^{m+i+1} A_\lambda^{k-i+1} S(t).$$

Changing the order of summation we get

$$D_{m+1,1,1} = \frac{1}{\lambda^{m+2}} \sum_{k=1}^{m} t^k \sum_{i=1}^{k} \sum_{j=1}^{i} d_{m,k,j} A^{m+i+1} A_\lambda^{k-i+1} S(t). \tag{4.33}$$

By (4.18) we have $d_{m,k,i} = \sum_{j=1}^{i} d_{m-1,k,j}$, for $m = 2, 3, \ldots,$ $k = 1, 2, \ldots, m-1$ and $i = 1, 2, \ldots, k$. Substituting this into (4.33) and (4.32), we get

$$D_{m+1,1,1} = \frac{1}{\lambda^{m+2}} \sum_{k=1}^{m} \sum_{i=1}^{k} d_{m+1,k,i} t^k A^{m+i+1} A_\lambda^{k-i+1} S(t).$$

and

$$a_{m+1,1} = \sum_{k=1}^{m} d_{m+1,k,k} t^k A^{m+k+1} S(t).$$

Integrating $D_{m,2}$ by parts we obtain

$$D_{m,2} = D_{m,2,1} + D_{m+1,2} = \frac{1}{\lambda^{m+1}} \int_0^1 \frac{\tau^m}{m!} (tAA_\lambda)^{m+1} \gamma_1(\tau) d\tau,$$

where

$$D_{m,2,1} = \frac{1}{\lambda^{m+1}} \frac{(tAA_\lambda)^{m+1}}{(m+1)!} S(t),$$

and

$$D_{m+1,2} = \frac{1}{\lambda^{m+2}} \int_0^1 \frac{\tau^{m+1}}{(m+1)!} (tAA_\lambda)^{m+2} \gamma_1(\tau) d\tau. \qquad (4.34)$$

Substituting the expression for A_λ^{m+1} from (4.27) we get $D_{m,2,1} = a_{m+1,2}/\lambda^{m+1} + D_{m+1,1,2}$ where

$$a_{m+1,2} = \frac{t^{m+1} A^{2m+2}}{(m+1)!} S(t), \qquad (4.35)$$

and

$$D_{m+1,1,2} = \frac{1}{\lambda^{m+2}} \sum_{i=1}^{m+1} \frac{t^{m+1} A^{m+i+1} A_\lambda^{m-i+2}}{(m+1)!} S(t). \qquad (4.36)$$

By (4.18) we have $d_{m+1,m+1,i} = \frac{1}{(m+1)!}$, for $m = 1, 2, \dots$ and $i = 1, 2, \dots, m$. Substituting this into (4.36) and (4.35), we obtain

$$D_{m+1,1,2} = \frac{1}{\lambda^{m+2}} \sum_{i=1}^{m+1} d_{m+1,m+1,i} t^{m+1} A^{m+i+1} A_\lambda^{m-i+2} S(t)$$

and

$$a_{m+1,2} = d_{m+1,m+1,m+1} t^{m+1} A^{2m+2} S(t),$$

It's easy to see that

$$D_{m+1,1} = D_{m+1,1,1} + D_{m+1,1,2},$$

which coincides with (4.20) for $k = m + 1$. $D_{m+1,2}$ is given by (4.34) and

$$a_{m+1} = a_{m+1,1} + a_{m+1,2}.$$

□

Proof of Theorem 4.7. From (4.6) and (4.8) it easily follows that

$$\|D_{m,1}x\| \le C_{m,1}(1+K^m)\|A^{m+1}x\|/\lambda^{m+1},$$

where $C_{m,1}$ is some positive constant depending only on m. The bound

$$\|D_{m,2}x\| \le C_{m,2}K^{m+1}\|A^{m+1}x\|/\lambda^{m+1}$$

can be obtained in the similar manner as the bound for $\|D_{1,2}x\|$ in the proof of Theorem 4.4.

□

Proof of Theorem 4.8. We proved (4.23) and (4.24) for $m = 1$ in Theorem 4.5. Using induction on m we obtain the general result. Suppose, that (4.23) and (4.24) hold for m. Let us prove that (4.23) and (4.24) hold for $m + 1$ as well.
We have

$$d_m = (-1)^{m+1} \int_0^1 \frac{\tau^m}{m!} \gamma_2^{(m+1)}(\tau) d\tau,$$

where

$$\gamma_2^{(m)}(\tau) = \frac{1}{\lambda^m} \sum_{k=1}^{m} h_{m,k} t^k A_{\lambda/\tau}^{m+k} \gamma_2(\tau),$$

with coefficients $h_{m,k}$ given by (4.22).
Then it is easy to show that the $(m+1)$th derivative of $\gamma_2(\tau)$ is

$$\gamma_2^{(m+1)}(\tau) = \frac{1}{\lambda^{m+1}} \sum_{k=1}^{m+1} h_{m+1,k} t^k A_{\lambda/\tau}^{m+1+k} \gamma_2(\tau).$$

Integrating d_m by parts we obtain

$$d_m = \frac{(-1)^{m+1}}{(m+1)!} \gamma_2^{(m+1)}(1) + (-1)^{m+2} \int_0^1 \frac{\tau^{m+1}}{(m+1)!} \gamma_2^{(m+2)}(\tau) d\tau,$$

i.e.,

$$d_m = \frac{b_{m+1}}{\lambda^{m+1}} + d_{m+1}.$$

□

Proof of Theorem 4.9. From (4.24) we have

$$d_m = \frac{(-1)^{m+1}}{(t\lambda)^{m+1}} \int_0^1 \frac{\tau^m}{m!} \sum_{k=1}^{m+1} h_{m+1,k} (tA_{\lambda/\tau})^{m+k+1} S_{\lambda/\tau}(t) d\tau.$$

Then

$$\|d_m\| \leq \frac{1}{(t\lambda)^{m+1}} \int_0^1 \frac{\tau^m}{m!} \sum_{k=1}^{m+1} h_{m+1,k} \|(tA_{\lambda/\tau})^{m+k+1} S_{\lambda/\tau}(t)\| d\tau.$$

From (4.9) we have $\|(tA_{\lambda/\tau})^{m+k+1} S_{\lambda/\tau}(t)\| \leq (m+k+1)^{m+k+1} K^{m+k+1}$ and from here

$$\|d_m\| \leq \frac{c_m(1+K^{2m+2})}{(t\lambda)^{m+1}}, \quad m = 1, 2, \ldots,$$

easily follows. □

5 Conclusions

In our work, we investigate the convergence of Euler's and Yosida approximations of semigroups.

1. We provide asymptotic expansions for Euler's approximations of differentiable strongly continuous semigroups. We also provide the inverse asymptotic expansions, i.e. the expansions of the semigroup via it's Euler's approximations.

2. We obtain optimal bounds for the remainder terms of both expansions in case of bounded holomorphic semigroups.

3. Using alternative approach introduced by Bentkus in [6] we obtain explicit formulas for asymptotic expansions for Euler's approximations of semigroups in Banach algebras.

4. We obtain two optimal error bounds (with optimal constants) for Yosida approximations of bounded holomorphic semigroups of contractions.

5. We provide asymptotic expansions for Yosida approximations of differentiable strongly continuous semigroups and provide optimal bounds for the remainder terms of these expansions. We provide the inverse expansions as well.

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