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Regularization algorithms of Conic and Copositive Programming

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ABSTRACT

Conic optimization is a subfield of convex optimization studying the problems of minimizing a convex function over the intersection of an affine subspace and a convex cone. The most important classes of Conic Optimization problems are

- Semidefinite Programming (SDP),
- Second-Order Conic Programming, Copositive Programming (CoP).

Copositive problems consist of optimization over the convex cone of matrices which are positive semi-defined on the non-negative orthant (*copositive matrices*). Copositive models arise in many important applications, including $\mathcal{NP}$-hard problems [1].

In conic optimization, optimality conditions, and duality are formulated under **regularity conditions** (e.g. CQs --- constraint qualifications).

Regularity conditions should guarantee the fulfillment of the Karush-Kuhn-Tucker (KKT) - type optimality conditions and the **strong duality** property consisting in the fact that the primal problem and the corresponding Lagrangian dual one are consistent, the optimal values of these problems are equal and the dual problem reaches its maximum.

Many problems cannot be classified as **regular** (i.e. satisfying some regularity conditions such as, for example, strict feasibility). The loss of a certain CQ is a modeling issue rather than inherent to the problem instance. Thus the **idea of a regularization** appears naturally and is aimed at obtaining an equivalent and more convenient reformulation of the problem with some required regularity properties.

The aims of the study:

- To present a new finite algorithm for regularization of linear CoP problems, based on the *concept of immobile indices;*
- To analogize and compare our approach with others based on the concepts of *minimal face* and *facial reduction*.

Conic and linear copositive programming problems

A general conic problem has the form:

$$\min_x \ c^\top x \quad \text{s.t. } \mathcal{A}(x) \in \mathcal{K},$$

- the constraints matrix function $\mathcal{A}(x)$ is defined as

$$\mathcal{A}(x) := \sum_{i=1}^{n} A_i x_i + A_0,$$

with given $p \times p$ symmetric matrices $A_i,\ i = 0, 1, \ldots, n$;
- $\mathcal{K} \subset \mathbb{R}^p$ is a cone.

If $\mathcal{K} = \mathcal{COP}^p := \{D \in \mathcal{S}(p) : t^\top D t \geq 0 \ \forall t \in \mathbb{R}^p_+\}$,

we get a copositive problem:

$$\min_x c^\top x \quad \text{s.t. } \mathcal{A}(x) \in \mathcal{COP}^p \qquad (CoP)$$

Different approaches to regularization of conic problems:

- Borwein and Wolkowicz [2]: regularization of abstract convex and conic convex problems based on the concept **of the *minimal* cone** of constraints;
- Luo, Sturm, and Zhang [6]: dual regularization or ***conic expansion***;
- Waki and Muramatsu [8]: **the Facial Reduction Algorithm** (FRA) for conic problems in finite dimensional spaces.

Minimal Face Regularization for CoP

(Borwein and Wolkowicz, 1981):

Regularization consists of a unique step:

replace the constraint $\mathcal{A}(x) \in \mathcal{COP}^p$ with an equivalent $\mathcal{A}(x) \in \mathbf{F}_{min}$, where $\mathbf{F}_{min}$ is the smallest face of the cone $\mathcal{COP}^p$ containing the set $\mathcal{D}(x) = \{\mathcal{A}(x) : x \in X\}$ with $X := \{x \in \mathbb{R}^n : \mathcal{A}(x) \in \mathcal{COP}^p$

The regularized problem is:

$$\min_{x \in \mathbb{R}^n} \ c^\top x, \quad \text{s.t. } \mathcal{A}(x) \in \mathbf{F}_{min}.$$

The corresponding *Lagrangian* dual is:

$$\max_{U \in \mathbf{F}^*_{min}} \ -A_0 \bullet U, \quad \text{s.t. } A_j \bullet U = c_j, j = 1, ..., n$$

Here $\mathbf{F}^*_{min}$ is the dual face to $\mathbf{F}_{min}$

Observation:

There is no available information about how one can explicitly construct the cones $\mathbf{F}_{min}$ and $\mathbf{F}^*_{min}$, hence the method is more conceptual than practical.

Face Reduction Algorithm of Waki and Miramatsu for general conic problems [8]

Given the copositive problem (CoP), the algorithm finds repeatedly smaller faces of $\mathcal{COP}^p$, until it stops with the minimal one.

This algorithm can be considered as a *minimal face regularization* applied to a sequence of linear conic problems constructed iteratively:

1: Set $i := 0$ and $\mathcal{F}_0 := \mathcal{COP}^p$.
2: If $\text{Ker}\mathcal{A} \cap \mathcal{F}_i^* \subset \text{span}\{Y_1, ..., Y_i\}$, then STOP: $\mathbf{F}_{min} = \mathcal{F}_i$.
3: Find $Y_{i+1} \in \text{Ker}\mathcal{A} \cap \mathcal{F}_i^* \setminus \text{span}\{Y_1, ..., Y_i\}$.
4: Set $\mathcal{F}_{i+1} := \mathcal{F}_i \cap \{Y_{i+1}\}^\perp$, $i := i + 1$, and go to step **2**.

Note. The algorithm looks very simple, in practice, its implementation presents serious difficulties as there is :

- **no information** concerning how to find matrices Y_i at the steps **2** and **3;**
- **no explicit description** of the set $\mathcal{F}_i^*$.

Regularization of linear copositive problems

In [5], a different algorithm for regularization of linear copositive problems was proposed.

This algorithm**:**

- has a similar structure to the FRA of Waki and Muramatsu, but is more constructive;
- is based on a new approach to semi-infinite problems that exploits the properties of the set of *immobile indices.*

The linear copositive problem (CoP) is equivalent to the semi-infinite problem

$$\min_{x \in \mathbb{R}^n} c^\top x, \text{ s.t. } t^\top \mathcal{A}(x) t \geq 0 \quad \forall t \in T, \ (SIP)$$

with a p-dimensional index set

$$T := \{t \in \mathbb{R}^p_+ : \mathbf{e}^\top t = 1\}, \ \mathbf{e} = (1, 1, ..., 1)^\top \in \mathbb{R}^p.$$

Following the approach to solving SIP problems that was developed in [4] and other papers, we define a normalized set of *immobile indices*

$$T_{im} := \{t \in T : t^\top \mathcal{A}(x) t = 0 \quad \forall x \in X\}.$$

Algorithm REG-LCoP

(Kostyukova, Tchemisova, 2021, [5])

Algorithm **REG-LCoP,** for any copositive problem (CoP) , constructs the equivalent ***regular*** copositive problem satisfying the Slater type regularity condition.

The properties of the algorithm REG-LCoP:

- the algorithm is finite, it runs no more than $\dim(\text{Ker}\mathcal{A})$ iterations;
- on its iterations, the algorithm solves regular semi-infinite problems constructed using the data of the problem;
- the sets constructed on the iterations of the algorithm can be used to construct the minimal face $\mathbf{F}_{min}$ of the cone of constraints of the problem (CoP) ;
- the algorithm is described in all details and justified;
- permits to present explicitly the iterations of the algorithm of Waki and Muramatsu in the case of CoP problems.

Conclusions.

The algorithm **REG-LCoP** is useful for the study of convex copositive problems.
In particular, for the linear copositive problem, it allows to

i) formulate an equivalent (regular) semi-infinite problem which satisfies the Slater type regularity condition and can be solved numerically;

ii) prove new optimality conditions without any additional regularity conditions;

iii) develop strong duality theory based on an explicit representation of the *regularized* feasible cone and the corresponding dual (e.g. the Extended Lagrange Dual Problem suggested for SDP in [7]).

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