Net-Constrained Facility Location Problem

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The Problem

We want to place M cluster-centers on a net (represented by little-squares) in a way that the sum of distances between cluster-members (represented by little-circles) and their corresponding centers is minimal (see Fig. 1 for an illustration).

More detailed explanation:
Suppose you have a network of roads $\mathcal{N}$ on the map and a set of customers $\mathcal{C} = \{C_1, C_2, \ldots, C_N\}$ with demands $d_1, d_2, \ldots, d_N$; the customers are located at points $C_1, C_2, \ldots, C_N$ in 2-dimensional plane (little-circles in Fig. 1).

To serve the customers, you are allowed to open M facilities $\mathcal{F} = \{F_1, F_2, \ldots, F_M\}$ (little-squares in Fig. 1), each with available max-supply $s_1, s_2, \ldots, s_M$. The places where you can open the facilities are restrited by the network $\mathcal{N}$: the 2-dimensional point F_i, representing facility i, must satisfy $F_i \in \mathcal{N}$ (see in Fig. 1 that each little-square is on a segment).

We are also given the operating cost $f_i \geq 0$ of each facility: this is the cost of opening facility i (a facility has to be opened if it is assigned at least one customer).

The cost of serving a single-unit of demand of customer j from facility i is proportional to the distance between the correspoinding points:

$$c_{ij} \propto dist\,(F_i, C_j) := \|F_i - C_j\|$$

We assume that the distance can be defined by Manhattan, Euclidean or Infinity-norm.

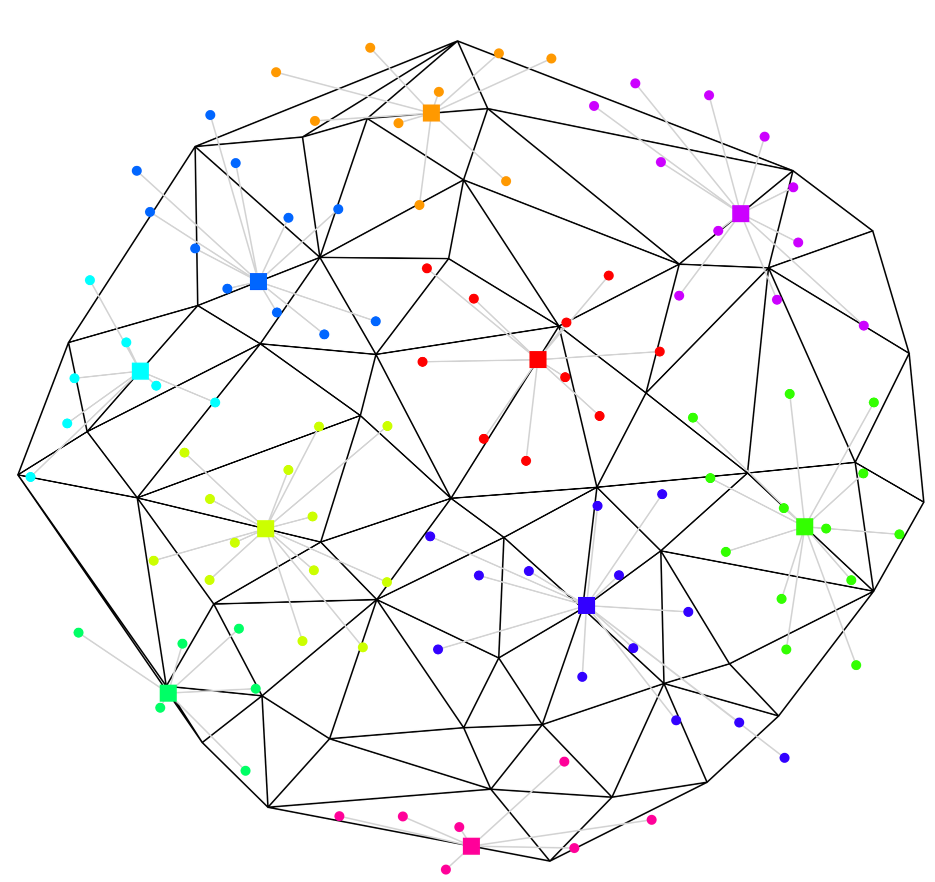

Fig. 1: Problem illustration.

Net-Constraint Formulation

How can we mathematically represent the constraint that a point P belongs to a 2-dimensional net $\mathcal{N}$?

We suppose the net $\mathcal{N}$ consists of K line segments $E_1, E_2, \ldots, E_K$, $\mathcal{N} = \cup_{k=1}^{K} E_k$. Each segment E_k has its starting point P_k and its ending point Q_k and can be represented parametrically as

$$E_k = \{(1-\alpha_k)P_k + \alpha_k Q_k,\ \alpha_k \in [0,1]\}$$

Thus, any point $P \in E_k$ satisfies $P = P_k + \alpha_k(Q_k - P_k)$ for a certain $0 \leq \alpha_k \leq 1$.

Now, lets introduce a binary variable $e_k \in \{0,1\}$ and lets consider the set

$$e_k P_k + \alpha_k(Q_k - P_k),\ 0 \leq \alpha_k \leq e_k$$

When $e_k = 1$, $\{e_k P_k + \alpha_k(Q_k - P_k),\ 0 \leq \alpha_k \leq e_k\} = E_k$.
When $e_k = 0$, $\{e_k P_k + \alpha_k(Q_k - P_k),\ 0 \leq \alpha_k \leq e_k\} = \{0\}$.

Thus, we introduce K binary variables $e_k \in \{0,1\}$, $k = 1, \ldots, K$, which indicate to which segment point P belongs: $e_k = 1$, if the point P belongs to segment E_k and equals 0 otherwise. We assume that a point can only belong to a single segment: this is achieved by adding the constraint $\sum_{k=1}^{K} e_k = 1$. (When P is the intersection point of two or more segments, the segment is chosen arbitrary.)

To conclude, we can see that the net-point-set $\mathcal{N} = \cup_{k=1}^{K} E_k$ is equal to the set

$$S = \left\{ \sum_{k=1}^{K} e_k P_k + \sum_{k=1}^{K} \alpha_k (Q_k - P_k) \right\},$$

here we require that $\forall k, e_k \in \{0,1\}$, $0 \leq \alpha_k \leq e_k$ and $\sum_{k=1}^{K} e_k = 1$.

Other Constraints and Variables

We assume that each customer can be served by only a single facility. Thus, for each pair (i,j), $i = 1, \ldots, M$, $j = 1, \ldots, N$, we introduce a binary variable $z_{ij} \in \{0,1\}$ which equals 1 when facility i supplies customer j and 0 otherwise. The constraint that customer j is served by single facility is then simple to formulate: we require that

$$\sum_{i=1}^{M} z_{ij} = 1,\ j = 1, 2, \ldots, N$$

We also define a binary variable $o_i \in \{0,1\}$, $i = 1, \ldots, M$. $o_i = 1$ if facility i is opened and $o_i = 0$ otherwise.

Mixed-Integer-Program (MIP)

With previous defintions and notations, we see that the net-constrained facility location problem can be formulated as follows:

find facility positions $F_1, F_2, \ldots, F_M$ **which**

minimize $$\left[\sum_{i=1}^{M} f_i o_i\right] + \left[\sum_{j=1}^{N} d_j \left(\sum_{i=1}^{M} z_{ij} \|F_i - C_j\|\right)\right]$$

subject to

net-constraints $$F_1, F_2, \ldots, F_M \in \mathcal{N}$$

capacity-constraints $$\sum_{j=1}^{N} d_j z_{ij} \leq s_i o_i,\ i = 1, \ldots, M$$

single-facility-constraints $$\sum_{i=1}^{M} z_{ij} = 1,\ j = 1, \ldots, N$$

where o_i, z_{ij} are *binary-variables*:

$$o_i, z_{ij} \in \{0,1\}, i = 1, \ldots, M,\ j = 1, \ldots, N$$

It turns out that the above problem can be formulated as

- mixed-integer-programming problem (MIP) in case the distance is Manhattan or Infinity-norm;
- mixed-integer-quadratically-constrained-problem (MIQCP), in case the distance is Euclidean.

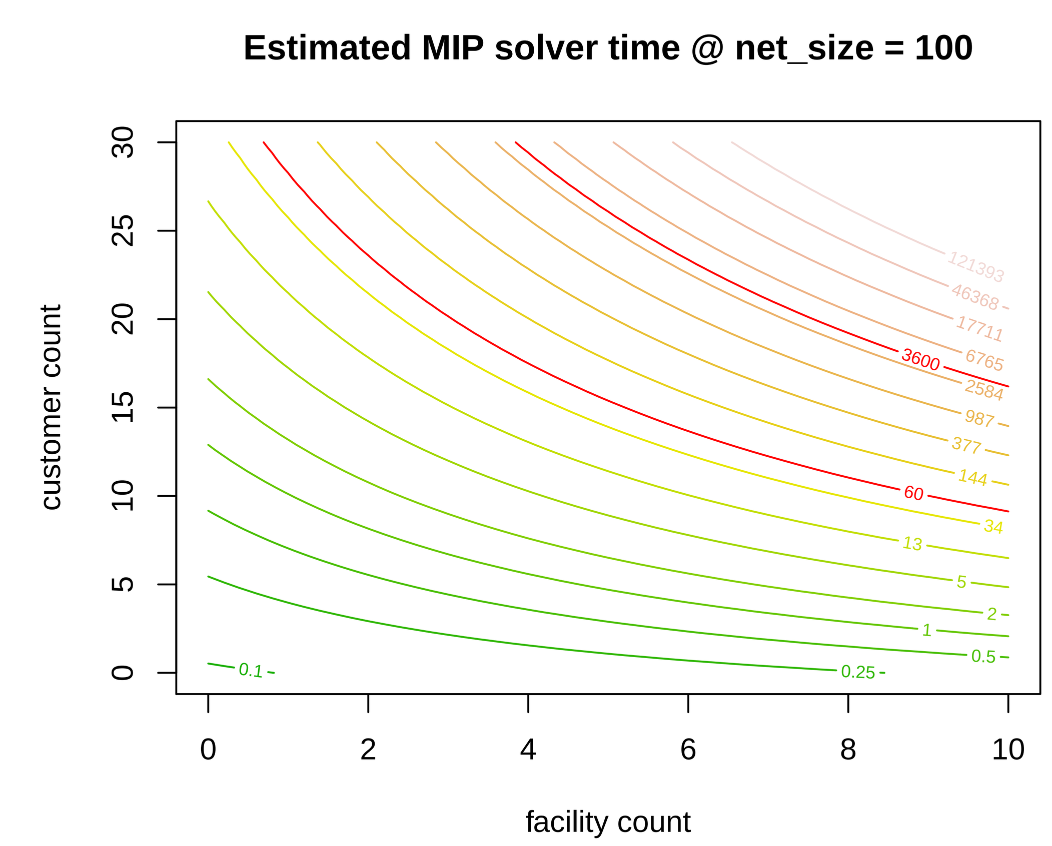

Fig. 2: Time contour-map (time in seconds).

$$T(M,N,K) \approx \exp^{\{-3.911+0.028M+0.196N+0.015K+0.037(M\cdot N)+(0.0012)(M\cdot K)\}}$$
$$\approx 0.020 \cdot (1.029)^M \cdot (1.216)^N \cdot (1.015)^K \cdot (1.037)^{M\cdot N} \cdot (1.0012)^{M\cdot K}$$

R^2-statistic for the model $\ln(T) = \varphi(M,N,K)$: 0.693

MIP Problem Dimensions

Lets analyze the complexity of our MIP problem (in what follows, M is the number of facilities, N is the number of customers and K is the number of segments in network $\mathcal{N}$).

- To define each constraint $F_i \in \mathcal{N}$, $i = 1, \ldots, M$, we need to define *net-variables*:
  - K binary variables for $\{e_k, k = 1, \ldots, K\}$,
  - K continuous variables for $\{\alpha_k, k = 1, \ldots, K\}$,
  - 2 variables for x, y-coordinates of facility F_i.

  net-constraints:
  - K inequality-constraints $\alpha_k \leq e_k$,
  - 2 equality-constraints for x, y-coords of facility F_i,
  - 1 equality-constraint $\sum_{k=1}^{K} e_k = 1$.

  In total:
  Variables: $M \cdot K$ binary, $(M+2) \cdot K$ continuous;
  Constraints: $M \cdot K$ inequality, $M \cdot 3$ equality.
- For each customer C_j, $j = 1, \ldots, N$, we need M binary variables z_{ij}, $i = 1, \ldots, M$. Each customer has a single-facility constraint $\sum_{i=1}^{M} z_{ij} = 1$.
  In total:
  Variables: $N \cdot M$ binary;
  Constraints: N equality.
- For each facility i, customer j pair (i,j) we define a variable $r_{ij} := \|F_i - C_j\|$. For infinity-norm distance (and similarly for Manhattan), we need 4 equations:

  $$F_i.x - r_{ij} \leq C_j.x, \qquad F_i.x + r_{ij} \geq C_j.x \qquad (2a)$$
  $$F_i.y - r_{ij} \leq C_j.y, \qquad F_i.y + r_{ij} \geq C_j.y \qquad (2b)$$

  In total:
  Variables: $N \cdot M$ continuous;
  Constraints: $4 \cdot N \cdot M$ inequality.

MIP Problem Solving Time

MIP problem is NP-hard, thus we assume that time to solve the net-constrained facility location is determined by the law

$$T(M,N,K) = \exp^{\varphi(M,N,K)}$$

Due to the number of variables and constraints discussed in previous block, we take

$$\varphi(M,N,K) = \alpha_0 + \alpha_M M + \alpha_N N + \alpha_K K + \\ + \beta_N^M (M \cdot N) + \beta_K^M (M \cdot K)$$

Coefficients $\alpha_0, \alpha_M, \alpha_N, \alpha_K, \beta_N^M, \beta_K^M$ are estimated from a numerical experiment and results are presented in Fig. 2.