Visualisation of 2D Fractal Structures Associated with the Riemann Zeta Function

Igoris Belovas, Martynas Sabaliauskas, Lukas Kuzma

E-mails: igoris.belovas@mif.vu.lt, martynas.sabaliauskas@mif.vu.lt, lukas.kuzma@mif.vu.lt

Vilnius University, Institute of Data Science and Digital Technologies

Abstract

We present visualizations of Mandelbrot sets associated with the Riemann zeta function. These complex plane fractal structures offer additional information on the properties of the Riemann zeta function, such as the arrangement of the non-trivial zeros.

Introduction

The Riemann Hypothesis is one of the seven Millennium Prize Problems - greatest unsolved problems in mathematics - selected by the Clay Mathematics Institute. Currently, the only problem to have been solved is the Poincaré conjecture mastered by Grigori Perelman in 2010. The Riemann zeta function is extremely important, playing a central role in analytic number theory. The properties of the function are essential in determining the distribution of primes. Prime numbers, in turn, have a broad spectrum of applications, ranging from quantum mechanics to information security. Let $s = \sigma + it$ be a complex variable. The Riemann zeta-function is defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p\in\mathbb{P}} \left(1 - \frac{1}{p^s}\right)^{-1}, \quad \sigma > 1,$$

and by analytic continuation elsewhere except for a simple pole at $s = 1$. The Riemann zeta function satisfies the functional equation

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)\zeta(1-s), \quad s \in \mathbb{C}.$$

The equation implies that $\zeta(s)$ has simple zeros at $s = -2n$, $n \in \mathbb{N}$, known as trivial zeros. It is known that non-trivial zeros belong to the critical strip $0 < \Re s < 1$. The Riemann hypothesis asserts that all non-trivial zeros are on the critical line $\Re s = 1/2$.

Fractal heuristic (FH) approach

This approach is based on applying the Mandelbrot sets associated with the Riemann zeta function for visual investigation of its underlying nature. Suppose we aim to visualize the fractal geography of the Riemann zeta function for $(\sigma, t) \in (\sigma_1, \sigma_2) \times (t_1, t_2)$. First, we introduce the log-transformation for each point (x, y) of the graph,

$$\begin{cases} x = L(\Re(\zeta(\sigma + it))), \\ y = L(\Im(\zeta(\sigma + it))), \end{cases} \tag{1}$$

thus obtaining the set $Q = (x_{min}, y_{min}) \times (x_{max}, y_{max})$. Here

$$L(x) = \begin{cases} \log|x|, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases} \tag{2}$$

Next, we transform $(x, y) \in Q \to (x^*, y^*) \in S$ using special parameters $a, b, c, d \in \mathbb{R}$:

$$x^* = \log(xy) \cdot a - b, \quad y^* = \log(x/y) \cdot c - d.$$

Then we use an algorithm to generate the Mandelbrot set, setting the starting position at $z_0 = 0$ and taking $z^* = (x^*, y^*)$:

$$z_{k+1} \leftarrow z_k^2 + z^*. \tag{3}$$

The final result depends on the number of iterations in (3), since we seek to get the most precise fractal image based on yellow-black-blue color palette.

Conclusion and discussion

Figures 1 and 2 were generated using FH-method and reveal the arrangement of non-trivial zeros. Can you locate them?

Visualizations of Mandelbrot sets

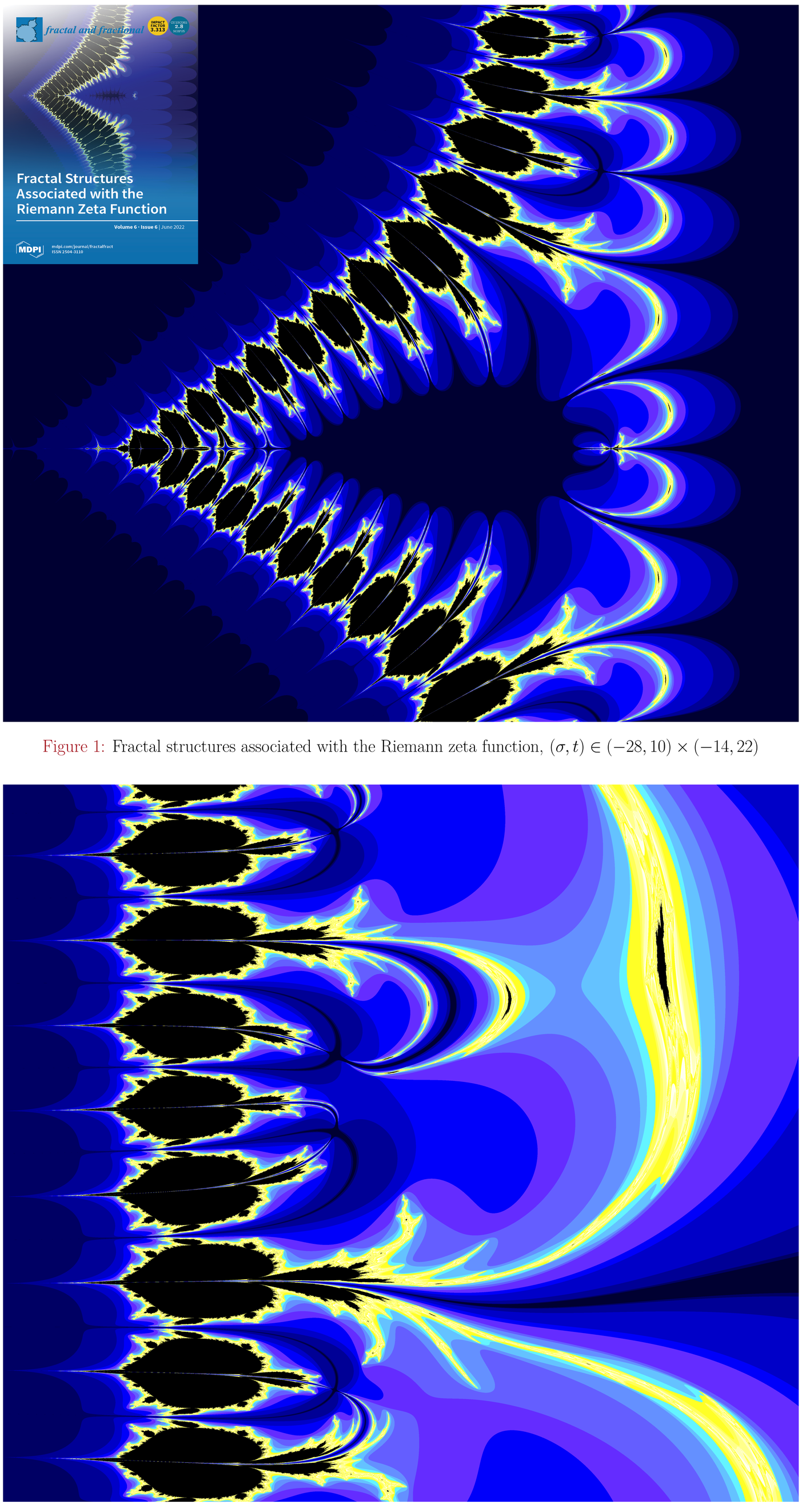

Figure 1: Fractal structures associated with the Riemann zeta function, $(\sigma, t) \in (-28, 10) \times (-14, 22)$

Figure 2: Fractal structures associated with the Riemann zeta function, $(\sigma, t) \in (-2, 4) \times (514.5, 520.5)$