Existence results for a class of \((p, q)\) Laplacian systems

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Abstract. We establish the existence of a nontrivial solution for inhomogeneous quasilinear elliptic systems:

\[
\begin{align*}
-\Delta_p u &= \lambda a(x) u |u|^{\gamma-2} + \frac{\alpha}{\alpha+\beta} b(x) u |u|^{\alpha-2} |v|^{\beta} + f \quad \text{in } \Omega, \\
-\Delta_q v &= \mu d(x) v |v|^{\gamma-2} + \frac{\beta}{\alpha+\beta} b(x) |u|^{\alpha} v |v|^{\beta-2} + g \quad \text{in } \Omega, \\
(u, v) &\in W_0^{1,p} (\Omega) \times W_0^{1,q} (\Omega).
\end{align*}
\]

Our result depending on the local minimization.

Keywords: elliptic systems, Nehari manifold, Ekeland variational principle, local minimization.

1 Introduction

In this paper we deal with the nonlinear elliptic system

\[
\begin{align*}
-\Delta_p u &= \lambda a(x) u |u|^{\gamma-2} + \frac{\alpha}{\alpha+\beta} b(x) u |u|^{\alpha-2} |v|^{\beta} + f \quad \text{in } \Omega, \\
-\Delta_q v &= \mu d(x) v |v|^{\gamma-2} + \frac{\beta}{\alpha+\beta} b(x) |u|^{\alpha} v |v|^{\beta-2} + g \quad \text{in } \Omega, \\
(u, v) &\in W_0^{1,p} (\Omega) \times W_0^{1,q} (\Omega),
\end{align*}
\]

where \(1 < p, q < N\) and \(\Omega\) is a regular set of \(\mathbb{R}^N\), \(N \geq 3\), \(\alpha > 0\), \(\beta > 0\), \(\lambda\) and \(\mu\) are positive parameters, functions \(a(x), b(x)\) and \(d(x) \in C(\overline{\Omega})\) are smooth functions with change sign on \(\Omega\), we assume here that \(1 < \gamma < \min(p, q)\), \(\gamma < \alpha + \beta\), \(\alpha + \beta > \max(p, q)\) and \(\alpha/p + \beta/q = 1\). For \(p \geq 1\) \(\Delta_p u\) is the \(p\)-Laplacian defined by \(\Delta_p u = \text{div}(|\nabla u|^{p-2} \nabla u)\) and \(W_0^{1,p} (\Omega)\) is the closer of \(C_0^\infty (\Omega)\) equipped by the norm \(\|u\|_{1,p} := \|\nabla u\|_p\), where \(\|\cdot\|_p\) represent the norm of Lebesgue space \(L^p(\Omega)\). The Lebesgue integral in \(\Omega\) will be denote by the symbol \(\int\) whenever the integration is carried out over all \(\Omega\).

Let \(p'\) be the conjugate to \(p\), \(W_0^{-1,p'} (\Omega)\) is the dual space to \(W_0^{1,p} (\Omega)\) and we denote by \(\|\cdot\|_{-1,p'}\) its norm. We denote by \(\langle \cdot, \cdot \rangle_{X', X}\) the natural duality paring between \(X\) and
Definition 1 (Weak solution). We say that \((u, v) \in X\) is a weak solution of (1) if:

\[
\int \left| \nabla u \right|^{p-2} \nabla u \cdot \nabla w_1 \, dx = \lambda \int a(x) |u|^{\gamma-2} w_1 \, dx + \frac{\alpha}{\alpha + \beta} \int b(x) |u|^{\alpha-2} |v|^{\beta} w_1 \, dx + \int f w_1 \, dx,
\]

\[
\int \left| \nabla v \right|^{q-2} \nabla v \cdot \nabla w_2 \, dx = \mu \int d(x) |v|^{\gamma-2} w_2 \, dx + \frac{\beta}{\alpha + \beta} \int b(x) |u|^\alpha |v|^{\beta-2} w_2 \, dx + \int g w_2 \, dx.
\]

for all \((w_1, w_2) \in X\).

It is clear that problem (1) has a variational structure.

It is well known if the Euler function \(\phi\) is bounded below and \(\phi\) has a minimizer on \(X\), then this minimizer is a critical point of \(\phi\). However, the Euler function \(\phi(u, v)\), associated with the problem (1), is not bounded below on the whole space \(X\), but is bounded on an appropriate subset, and has a minimizer on this set (if it exists) which gives rise to solution to (1). Clearly, the critical points of \(\phi\) are the weak solutions of problem (1).

The associated Euler–Lagrange functional to system (1) \(\phi: X \to R\) is defined by

\[
\phi(u, v) = \frac{1}{p} \|u\|_{1,p}^p + \frac{1}{q} \|v\|_{1,q}^q - \frac{1}{\gamma} \left[ \lambda \int a(x) |u|^\gamma + \mu \int d(x) |v|^\gamma \right] - \frac{\alpha + \beta}{\alpha + \beta} \int b(x) |u|^\alpha |v|^{\beta} - \langle f, u \rangle - \langle g, v \rangle.
\]

Consider the Nehari manifold associated to problem (1) given by

\[\Lambda = \{(u, v) \in X \setminus \{(0, 0)\} : \phi'(u, v)(u, v) = 0\}, \quad m_1 = \inf_{(u, v) \in \Lambda} J(u, v).\]

Consequently, for every \((u, v) \in \Lambda\), (2) becomes

\[
\phi_{|\Lambda}(u, v) = A(p) \|u\|_{1,p}^p + A(q) \|v\|_{1,q}^q - A(\gamma) \left[ \lambda \int a(x) |u|^\gamma + \mu \int d(x) |v|^\gamma \right] - A(1) \langle f, u \rangle - A(1) \langle g, v \rangle,
\]
where for all \( t > 0 \), \( A(t) = 1/t - 1/(\alpha + \beta) \).

We introduce the operators \( J_1, J_2, D_1, D_2, B_1, B_2 : X \rightarrow X^* \) in the following way

\[
\langle J_1(u, v), (w, z) \rangle_X := \int |\nabla u|^{p-2} \nabla u \nabla w,
\]

\[
\langle J_2(u, v), (w, z) \rangle_X := \int |\nabla v|^{q-2} \nabla v \nabla z,
\]

\[
\langle D_1(u, v), (w, z) \rangle_X := \int a(x)|u|^{\alpha-2} uw,
\]

\[
\langle D_2(u, v), (w, z) \rangle_X := \int d(x)|v|^{\beta-2} vz,
\]

\[
\langle B_1(u, v), (w, z) \rangle_X := \int b(x)|u|^{\alpha-2} |v|^\beta uw,
\]

\[
\langle B_2(u, v), (w, z) \rangle_X := \int b(x)|u|^{\alpha} |v|^\beta-2 vz.
\]

2 Main results

Our main result is the following:

**Theorem 1.** Suppose that \((f, g) \in W_{0}^{-1, p'}(\Omega) \times W_{0}^{-1, q'}(\Omega)\), none of the functions \(f\) and \(g\) is identically to zero on \(\Omega\) and:

(a) \(1 < \gamma < \min(p, q)\),
(b) \(\gamma < \alpha + \beta\),
(c) \(\alpha + \beta > \max(p, q)\).

Then, there exists a pair \((u^*, v^*) \in \Lambda\) such that the sequence \((u_n, v_n)\) converges strongly to \((u^*, v^*)\) in \(X\). Moreover, \((u^*, v^*)\) is a solution of system (1) satisfies the property \(\phi(u^*, v^*) < 0\).

**Definition 2.** We say that \(\phi\) satisfies the Palais–Smale condition \((PS)_c\), if every sequence \((u_n, v_n)\) \(\subset X\) such that \(\phi(u_n, v_n)\) is bounded and \(\phi'(u_n, v_n) \rightarrow 0\) as \(m \rightarrow \infty\), is relatively compact in \(X\).

**Lemma 1.** The operators \(J_i, D_i, B_i, i = 1, 2\), are well defined. Also \(J_i, i = 1, 2\), are continuous and the operators \(D_i, B_i, i = 1, 2\), are compact.

**Proof.** This lemma is proved in [3].

**Lemma 2.** Let \((u_n, v_n)\) be a bounded sequence in \(X\) such that \(\phi(u_n, v_n)\) is bounded and \(\phi'(u_n, v_n) \rightarrow 0\) as \(n \rightarrow \infty\). Then \((u_n, v_n)\) has a convergent subsequence.

**Proof.** Since the sequence \((u_n, v_n)\) is bounded in \(X\), we may consider that there is a subsequence (denote again by \((u_n, v_n)\)), which is weakly convergent in \(X\).
Moreover, we have that
\[
\langle \phi'(u_n, v_n) - \phi'(u_m, v_m), (u_n - u_m, v_n - v_m) \rangle \\
= \int \left( |\nabla u_n|^{p-2} \nabla u_n - |\nabla u_m|^{p-2} \nabla u_m \right) (\nabla u_n - \nabla u_m) \\
+ \int \left( |\nabla v_n|^{q-2} \nabla v_n - |\nabla v_m|^{q-2} \nabla v_m \right) (\nabla v_n - \nabla v_m) \\
- \lambda \int a(x) \left( |u_n|^{\gamma-2} u_n - |u_m|^{\gamma-2} u_m \right) (u_n - u_m) \\
- \mu \int d(x) \left( |v_n|^{\gamma-2} v_n - |v_m|^{\gamma-2} v_m \right) (v_n - v_m) \\
- \frac{\alpha}{\alpha + \beta} \int b(x) \left( |u_n|^{\alpha-2} |v_n|^\beta u_n - |u_m|^{\alpha-2} |v_m|^\beta u_m \right) (u_n - u_m) \\
- \frac{\beta}{\alpha + \beta} \int b(x) \left( |u_n|^{\alpha} |v_n|^{\beta-2} v_n - |u_m|^{\alpha} |v_m|^{\beta-2} v_m \right) (v_n - v_m) \\
- \int \left( f(x_n) - f(x_m) \right) (u_n - u_m) - \int \left( g(x_n) - g(x_m) \right) (v_n - v_m). 
\]

Since \((u_n, v_n)\) converges strongly in \(L^p(\Omega) \times L^q(\Omega)\), it is a Cauchy sequence in \(L^p(\Omega) \times L^q(\Omega)\). Using Holder inequality (since \(\alpha/p + \beta/q = 1\) and \((\alpha-1)/\alpha + 1/\alpha = 1\)) we have

\[
\int b(x) |u_n|^{\alpha-2} |v_n|^\beta u_n (u_n - u_m) \\
\leq ||b||_\infty \int |u_n|^{\alpha-1} |v_n|^\beta |u_n - u_m| \\
\leq ||b||_\infty \left[ \int \left( |u_n|^{\alpha-1} |u_n - u_m| \right)^{\frac{\alpha}{\alpha -1}} \right]^{\frac{\alpha -1}{\alpha}} \left[ \int |v_n|^\beta \right]^{\frac{\beta}{\beta}} \\
\leq ||b||_\infty \left[ u_n^{\frac{\alpha-1}{\alpha}} \right]^{\frac{\alpha -1}{\alpha}} \left| u_n - u_m \right| \left\| v_n \right\|_q^{\beta} \\
\leq ||b||_\infty \left[ \int \left( u_n^{\frac{\alpha-1}{\alpha}} \right) \right]^{\frac{\alpha -1}{\alpha}} \left[ \int \left| u_n - u_m \right|^{\frac{\alpha}{\alpha -1}} \right]^{\frac{\alpha -1}{\alpha}} \left\| v_n \right\|_q^{\beta} \\
= ||b||_\infty \left\| u_n \right\|_{\alpha -1}^{\frac{\alpha}{\alpha -1}} \left\| u_n - u_m \right\|_p \left\| v_n \right\|_q^{\beta} \rightarrow 0.
\]

Similarly

\[
\int b(x) \left( |u_n|^{\alpha} |v_n|^{\beta-2} v_n - |u_m|^{\alpha} |v_m|^{\beta-2} v_m \right) (v_n - v_m) \rightarrow 0.
\]

From the compactness of the operators \(B_i, D_i (i = 1, 2), [4]\), continuity of \(f\) and \(g\), we
obtain (passing to a subsequence, if necessary) that

\[
\int \left( |\nabla u_n|^{p-2}\nabla u_n - |\nabla u_m|^{p-2}\nabla u_m \right) (\nabla u_n - \nabla u_m)
+ \int \left( |\nabla v_n|^{q-2}\nabla v_n - |\nabla v_m|^{q-2}\nabla v_m \right) (\nabla v_n - \nabla v_m) \to 0
\]

which implies (see [5]) that \((u_n, v_n)\) converges strongly in \(X\).

**Lemma 3.** Let \(c \in \mathbb{R}\). Then the functional \(\phi(u, v)\) satisfies the \((PS)_c\) condition.

**Proof.** According to Lemma 2, it sufficient to prove that the sequence \((u_n, v_n)\) is bounded in \(X\). Let \((u_n, v_n)\) be such a sequence, that is

\[
\phi(u_n, v_n) = c + o_n(1) \quad \text{and} \quad \phi'(u_n, v_n) = o_n \left(\| (u_n, v_n) \|_X \right),
\]

then

\[
\phi(u_n, v_n) = \frac{1}{\alpha + \beta} \left( \phi'(u_n, v_n), (u_n, v_n) \right)
= A(p) \| u_n \|_{1,p}^p + A(q) \| v_n \|_{1,q}^q - A(\gamma) \left[ \lambda \int a(x)|u_n|^\gamma + \mu \int b(x)|v_n|^\gamma \right]
- A(1)\langle f, u_n \rangle - A(1)\langle g, v_n \rangle
= c + o_n \left(\| (u_n, v_n) \|_X \right) + o_n(1).
\]

Using successively the Holder’s inequality and the Young inequality on the terms \(\langle f, u_n \rangle\) and \(\langle g, v_n \rangle\), we can write

\[
\left[ A(p) \| u_n \|_{1,p}^p - \frac{A(1)}{p} \theta^p \| u_n \|_{1,p}^p - \lambda A(\gamma) \| u_n \|_{1,p}^\gamma \right]
+ \left[ A(q) \| v_n \|_{1,q}^q - \frac{A(1)}{q} \nu^q \| v_n \|_{1,q}^q - \mu A(\gamma) \| v_n \|_{1,q}^\gamma \right]
\leq \frac{A(1)}{p'} \theta^{-p'} \| f \|_{1,p'}^{p'} + \frac{A(1)}{q'} \nu^{-q'} \| g \|_{1,q'}^{q'} + c + o_n \left(\| (u_n, v_n) \|_X \right) + o_n(1).
\]

Since the real numbers \(\theta\) and \(\nu\) being arbitrary, a suitable choose of \(\theta\) and \(\nu\) assure the boundedness of the sequence \((u_n, v_n)\).

**Lemma 4.** The critical value of \(\phi\) on \(\Lambda\), \(m_1 = \inf_{(u,v) \in \Lambda} \phi(u,v)\), has the following property:

\[
m_1 < \min \left[ -\frac{\| f \|_{1,p'}^{p'}}{p'}, -\frac{\| g \|_{1,q'}^{q'}}{q'} \right].
\]
Proof. Let \( u \) be the unique solution of the Dirichlet problem
\[
\begin{align*}
-\Delta_p u &= f & \text{in } \Omega, \\
u &= 0 & \text{on } \partial \Omega,
\end{align*}
\]
and let \( v \) be the unique solution of the problem
\[
\begin{align*}
-\Delta_q v &= g & \text{in } \Omega, \\
v &= 0 & \text{on } \partial \Omega.
\end{align*}
\]
It is clear that \((u,0),(0,v)\) are two elements of \( \Lambda \) and we have
\[
m_1 \leq \phi(u,0) = \left[ \frac{1}{p} \|\nabla u\|^p_p - \langle f, u \rangle \right] = \left( 1 - \frac{1}{p} \right) \|\nabla u\|^p_p = -\frac{1}{p'} \|\nabla u\|^p_p,
\]
\[
m_1 \leq \phi(0,v) = \left[ \frac{1}{q} \|\nabla v\|^q_q - \langle g, v \rangle \right] = \left( 1 - \frac{1}{q} \right) \|\nabla v\|^q_q = -\frac{1}{q'} \|\nabla v\|^q_q.
\]
Similarly to proof of J. Velin [13, 4.2], we can show that
\[
\|f\|^p'_{-1,p'} = \|\nabla u\|^p_p,
\]
\[
\|g\|^q'_{-1,q'} = \|\nabla v\|^q_q.
\]
Then
\[
m_1 \leq \min \left[ \frac{1}{p'} \|f\|^p'_{-1,p'}, \frac{1}{q'} \|g\|^q'_{-1,q'} \right].
\]
Thus, the Lemma is proved.

\( \Box \)

3 Proof of the Theorem 1

We show that \( \phi \) is bounded below on \( \Lambda \). Let \((u, v)\) be an arbitrary element in \( \Lambda \). We have
\[
\phi|_\Lambda (u, v) \geq \left[ A(p) \|u\|^p_p - \frac{A(1)}{p} \|u\|^p_{1,p} \right] + \left[ A(q) \|v\|^q_q - \frac{A(1)}{q} \|v\|^q_{1,q} \right] - \frac{A(1)}{p'} \|f\|^p'_{-1,p'} \cdot \frac{A(1)}{q'} \|g\|^q'_{-1,q'}.
\]
This inequality follows from \( a(x), d(x) \) are sign chaining functions and we can choose \((u, v) \in X\) with these properties that \( \sup u \subseteq \Omega_1 = \{ x \in \Omega; \ a(x) < 0 \} \) and \( \sup v \subseteq \Omega_2 = \{ x \in \Omega; \ d(x) < 0 \} \).

We choose \( \theta = \{ pA(p)/A(1) \}^{1/p} \) and \( \nu = \{ qA(q)/A(1) \}^{1/q} \). Consequently, we have, for every \((u, v) \in \Lambda\)
\[
\phi(u, v) \geq -\frac{A(1)}{p'} \theta^{-p'} \|f\|^p'_{-1,p'} - \frac{A(1)}{q'} \nu^{-q'} \|g\|^q'_{-1,q'}.
\]
Hence, we have shown that \( \phi \) is bounded blow on \( \Lambda \). Then Ekeland variational principle [6] imply the existence of a solution of (1), such that \( \phi(u^*, v^*) < 0 \).
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References


