Estimation of Convergence Rate in the Transfer Theorem for Maxima

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Abstract. Let $Z_N$ be a maximum of independent identically distributed random variables. In this paper, a nonuniform estimate of convergence rate in the transfer theorem max-scheme is obtained. Presented results make the estimates, given in [1] and [2], more precise.

Keywords: max-scheme, transfer theorem, limit theorem, rate of convergence.

1 Introduction

Let $X_1, X_2, \ldots, X_n, \ldots$ be a sequence of independent and identically distributed random variables (r.v.’s) with the distribution function $P(X_j \leq x) = F(x)$, $j \geq 1$; let $N_1, N_2, \ldots, N_n, \ldots$ be a sequence of positive integer-valued r.v.’s independent of $X_j$, $j \geq 1$ and $P(N_n \leq x) = A_n(x)$.

Let us define two random variables

$$Z_n = \max(X_1, \ldots, X_n), \quad Z_{N_n} = \max(X_1, \ldots, X_{N_n}),$$

and consider linear normalized maxima

$$\mathcal{Z}_n = b_n^{-1}(Z_n - a_n), \quad \mathcal{Z}_{N_n} = b_n^{-1}(Z_{N_n} - a_n),$$

$-\infty < a_n < +\infty$, $b_n > 0$.

Now, let us denote:

$$u_n(x) = n \left(1 - F(xb_n + a_n)\right).$$

The necessary and sufficient conditions for weak convergence $P(\mathcal{Z}_n \leq x) \Rightarrow H(x)$ to nonsingular distribution function $H$, $n \to \infty$ are presented in the fundamental work of
Gnedenko [3]. In terms of the function $u_n(x)$ the necessary and sufficient condition is defined to be

$$
\lim_{n \to \infty} u_n(x) = u(x).
$$

Then $H(x) = e^{-u(x)}$ is a limit distribution function.

Nonuniform estimate of convergence rate

$$
|P(Z_n \leq x) - H(x)| \leq \Delta_n(x)
$$

is presented in the monograph of Galambos [4]. More general estimates of convergence rate are given in the paper of Aksomaitis [1].

We are interested in nonuniform estimates of convergence rate in the transfer theorem of Gnedenko [5], i.e. in estimates of type

$$
|P(Z_{N_n} \leq x) - \Psi(x)| \leq \Delta_n(x),
$$

where

$$
\Psi(x) = \int_0^\infty H^z(x) dA(z)
$$

and

$$
A(z) = \lim_{n \to \infty} P\left(\frac{N_n}{n} \leq z\right) = \lim_{n \to \infty} A_n(nz).
$$

Let $\rho_n(x) = u_n(x) - u(x)$. When supposing $\rho_n(x) \geq 0$ an estimate of the rate $\Delta_n(x)$ has already been proposed by the author [2, 6] and is given by Aksomaitis.

## 2 Main results

A nonuniform estimate of the convergence rate in transfer theorem is given by the following theorem.

**Theorem.** Let $H$ be the limit distribution of the r.v. $Z_n$ and $\lim_{n \to \infty} P\left(\frac{N_n}{n} \leq x\right) = A(x)$, $A(+0) = 0$. Then, for any $x$ satisfying condition $\frac{u_n(x)}{n} \leq \frac{1}{2}$, the following estimate holds true:

$$
\Delta_n(x) \leq \left(\frac{u_n^2(x)}{n} + |\rho_n(x)|\right) \int_0^\infty z\delta^z_n(x) dA_n(nz) + u(x) \int_0^\infty |A_n(nz) - A(z)| H^z(x) dz,
$$

where $\delta_n(x) = \max(F^n(xb_n + a_n), H(x))$, $\rho_n(x) = u_n(x) - u(x)$.

This theorem simplifies and makes more precise results obtained in the author’s previous publication [2].
Proposition 1. \( \delta_n(x) \leq H(x) \max(e^{-\rho_n(x)}, 1) \).

Proposition 2. If \( EN_n < \infty \), then
\[
\int_0^\infty z \delta_n(x) dA_n(nz) \leq \frac{EN_n}{n}.
\]

Proposition 3. If \( P(N_n = n) = 1 \), then
\[
\int_0^\infty z \delta_n(x) dA_n(nz) = \delta_n(x).
\]

Proposition 4. If \( \rho_n(x) \geq 0 \), then
\[
\int_0^\infty z \delta_n(x) dA_n(nz) \leq \int_0^\infty z H^z(x) dA_n(nz).
\]

3 Proof of the theorem

We have
\[
|P(Z_{N_n} \leq x) - \Psi(x)| \leq \left| P(Z_{N_n} \leq x) - EH \frac{\delta_n}{n}(x) \right| + \left| EH \frac{\delta_n}{n}(x) - \Psi(x) \right| \leq I^{(1)}_n(x) + I^{(2)}_n(x).
\] (1)

The total probability formula implies:
\[
I^{(1)}_n(x) = \left| \sum_{j \geq 1} F^j(xb_n + a_n) P(N_n = j) - \sum_{j \geq 1} H^j(x) P(N_n = j) \right|
\]
\[
= \left| \int_0^\infty F^{nz}(xb_n + a_n) dA_n(nz) - \int_0^\infty H^z(x) dA_n(nz) \right| \leq \int_0^\infty |F^{nz}(xb_n + a_n) - H^z(x)| dA_n(nz).
\] (2)

Since
\[
|F^{nz}(xb_n + a_n) - H^z(x)| = z \left( \int_{H(x)}^{F^n(xb_n + a_n)} t^{z-1} dt \right)
\]
\[
\leq z \left( \max \left( F^n(xb_n + a_n), H(x) \right) \right)^z \left| \ln F^n(xb_n + a_n) - \ln H(x) \right|
\]
\[
\leq z \delta_n(x) \left( n \ln \left( 1 - \frac{u_n(x)}{n} \right) + u_n(x) \right) + \left| u_n(x) - u(x) \right|
\]
then, using the inequality $|\ln(1 - t) + t| \leq t^2$, $0 \leq t \leq \frac{1}{2}$, we obtain

$$|F_n(x_n + a_n) - H^z(x)| \leq z\delta^z(x)\left(\frac{u_n^2(x)}{n} + |\rho_n(x)|\right),$$

(3)

provided $\frac{a_n(x)}{n} \leq \frac{1}{2}$.

From (2) and (3) it follows that

$$I_n^{(1)}(x) \leq \left(\frac{u_n^2(x)}{n} + |\rho_n(x)|\right) \int_0^\infty z\delta^z_n(x) dA_n(nz).$$

Also,

$$I_n^{(2)}(x) = \left|\int_0^\infty H^z(x) dA_n(nz) - \int_0^\infty H^z(x) dA(z)\right| = \left|\int_0^\infty H^z(x) d(A_n(nz) - A(z))\right|.$$

Integrating by parts, we get

$$I_n^{(2)}(x) = \left|\int_0^\infty (A_n(nz) - A(z)) dH^z(x)\right| \leq u(x) \int_0^\infty |A_n(nz) - A(z)| H^z(x) dz.$$

The proof of the theorem follows from (1), (2) and (3).

4 Example

Let $P(N_n = k) = \frac{1}{n}$, $1 \leq k \leq n$. We get:

$$\lim_{n \to \infty} A_n(nx) = A(x) = x, \quad 0 \leq x \leq 1;$$

$$|A_n(nx) - A(x)| \leq \frac{1}{n}, \quad \frac{E N_n}{n} = \frac{n}{2} + \frac{1}{2};$$

$$I_n^{(2)}(x) \leq \frac{u(x)}{n} \int_0^1 H^z(x) dz = \frac{1 - H(x)}{n} \leq \frac{u(x)}{n};$$

$$\Delta_n(x) \leq \left(\frac{u_n^2(x)}{n} + |\rho_n(x)|\right)\left(\frac{1}{2} + \frac{1}{2n}\right) + \frac{u(x)}{n}.$$

If, for instance,

$$F(x) = 1 - \frac{c}{x^\alpha}, \quad x \geq c \frac{1}{\alpha}, \quad \alpha > 0,$$

then

$$u_n(x) = n\left(1 - F(xcn)^\frac{1}{\alpha}\right) = x^{-\alpha}$$
and

\[ \Delta_n(x) \leq \frac{x^{-\alpha}}{n} \left( 1 + \frac{x^{-\alpha}}{2} \left( 1 + \frac{1}{n} \right) \right), \]

provided \( nx^\alpha \geq 2 \).

References


