On Positive Solutions for Some Nonlinear Semipositone Elliptic Boundary Value Problems

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Abstract. This study concerns the existence of positive solutions to classes of boundary value problems of the form

\[-\Delta u = g(x, u), \quad x \in \Omega,\]
\[u(x) = 0, \quad x \in \partial\Omega,\]

where \(\Delta\) denote the Laplacian operator, \(\Omega\) is a smooth bounded domain in \(\mathbb{R}^N\) \((N \geq 2)\) with \(\partial\Omega\) of class \(C^2\), and connected, and \(g(x, 0) < 0\) for some \(x \in \Omega\) (semipositone problems). By using the method of sub-super solutions we prove the existence of positive solution to special types of \(g(x, u)\).

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1 Introduction

In this paper we consider the existence of positive solution to boundary value problems of the form

\[-\Delta u = g(x, u), \quad x \in \Omega,\]
\[u(x) = 0, \quad x \in \partial\Omega,\]

where \(\Delta\) denote the Laplacian operator, \(\Omega\) is a smooth bounded domain in \(\mathbb{R}^N\) \((N \geq 2)\) with \(\partial\Omega\) of class \(C^2\), and connected, and \(g(x, 0) < 0\) for some \(x \in \Omega\) (semipositone problems). In particular, we first study the case when \(g(x, u) = 323\)
\(a(x) \ u - b(x) \ u^2 - ch(x)\), where \(a(x), b(x)\) are \(C^1(\Omega)\) functions that \(a(x)\) is allowed to be negative near the boundary of \(\Omega\), and \(b(x) > b_0 > 0\) for \(x \in \Omega\). Here \(h : \overline{\Omega} \rightarrow \mathbb{R}\) is a \(C^1(\overline{\Omega})\) function satisfying \(h(x) \geq 0\) for \(x \in \Omega\), \(h(x) \not\equiv 0\), and \(\max_{x \in \overline{\Omega}} h(x) = 1\). We prove that there exists a \(c_0 = c_0(\Omega, a, b) > 0\) such that for \(0 < c < c_0\) there exists a positive solution.

The above equation arises in the studies of population biology of one species with \(u\) representing the concentration of the species or the population density, and \(ch(x)\) representing the rate of harvesting (see [1]). The case when \(a(x), b(x)\) are positive constants throughout \(\overline{\Omega}\), has been studied in [1]. In [2] the authors studied the case when \(c = 0\) (non-harvesting case), \(b(x) \equiv 1\) for \(\overline{\Omega}\) and \(a(x)\) is a positive function throughout \(\overline{\Omega}\). However the \(c > 0\) case is a semipositone problem \((g(x, 0) < 0)\) and studying positive solutions in this case is significantly harder. Here we consider the challenging semipositone case \(c > 0\). Semipositone problems have been of great interest during the past two decades, and continue to pose mathematically difficult problems in the study of positive solutions (see [3–6]).

We next study the case when \(g(x, u) = \lambda m(x)f(u)\), where the weight \(m\) satisfying \(m \in C(\Omega)\) and \(m(x) \geq m_0 > 0\) for \(x \in \Omega\), \(f \in C^1[0, \rho)\) is a nondecreasing function for some \(\rho > 0\) such that \(f(0) < 0\) and there exist \(\alpha \in (0, \rho)\) such that \(f(t)(t - \alpha) \geq 0\) for \(t \in [0, \rho]\).

See [7] where positive solution is obtained for large \(\lambda\) when \(m(x) \equiv 1\) for \(x \in \overline{\Omega}\) and \(f\) is sublinear at infinity. We are interested in the existence of a positive solution in a range of \(\lambda\) without assuming any condition on \(f\) at infinity. Our approach is based on the method of sub-super solutions, see [2, 8].

2 Existence results

We first give the definition of sub-super solution of (1). A super solution to (1) is defined as a function \(z \in C^2(\Omega)\) such that

\[-\Delta z \geq \lambda g(x, z), \quad x \in \Omega,\]
\[z \geq 0, \quad x \in \partial \Omega.\]

Sub solutions are defined similarly with the inequalities reversed and it is well known that if there exists a sub solution \(\psi\) and a super solution \(z\) to (1) such that
ψ(x) ≤ z(x) for x ∈ Ω, then (1) has a solution u such that \( ψ(x) ≤ u(x) ≤ z(x) \) for \( x ∈ \bar{Ω} \). Further note that if \( ψ(x) ≥ 0 \) for \( x ∈ Ω \) then \( u ≥ 0 \) for \( x ∈ Ω \).

To precisely state our existence result we consider the eigenvalue problem

\[
-\Delta \phi = \lambda \phi, \quad x ∈ Ω,
\]
\[
\phi = 0, \quad x ∈ ∂Ω. \tag{2}
\]

Let \( φ_1 ∈ C^1(Ω) \) be the eigenfunction corresponding to the first eigenvalue \( λ_1 \) of (3) such that \( φ_1(x) > 0 \) in \( Ω \), and \( \| φ_1 \|_∞ = 1 \). It can be shown that \( \frac{∂φ_1}{∂n} < 0 \) on \( ∂Ω \). This result is well known (see, e.g., [9]), and hence, depending on \( Ω \), there exist positive constants \( k, η, µ \) such that

\[
λ_1 φ_1^2 - |∇φ_1|^2 ≤ -k, \quad x ∈ Ω_η, \tag{3}
\]
\[
φ_1 ≥ µ, \quad x ∈ Ω_0 = Ω \setminus Ω_η, \tag{4}
\]

with \( Ω_η = \{ x ∈ Ω \mid d(x, ∂Ω) ≤ η \} \). Further assume that there exists a constants \( a_0, a_1 > 0 \) such that \( a(x) ≥ -a_0 \) in \( Ω_η \) and \( a(x) ≥ a_1 \) in \( Ω_0 = Ω \setminus Ω_η \).

We will also consider the unique solution, \( ζ ∈ C^1(Ω) \), of the boundary value problem

\[
-\Delta ζ = 1, \quad x ∈ Ω,
\]
\[
ζ = 0, \quad x ∈ ∂Ω,
\]

to discuss our existence result. It is known that \( ζ > 0 \) in \( Ω \) and \( \frac{∂ζ}{∂n} < 0 \) on \( ∂Ω \).

First we obtain the existence of positive solution of (1) in the case when \( g(x, u) = a(x)u - b(x)u^2 - ch(x) \).

**Theorem 1.** Suppose that \( a_0 < 2k \) and \( 2λ_1 < a_1 µ^2 \). Then there exists \( c_0 = c_0(Ω, a_0, a_1, b) > 0 \) such that if \( 0 < c < c_0 \) then the problem (1) has a positive solution \( u \).

**Proof.** To obtain the existence of positive solution to problem (1) we constructing a positive subsolution \( ψ \) and supersolution \( z \). We shall verify that \( ψ = δφ_1^2 \) is a subsolution of (1), where \( δ > 0 \) is small and specified later (note that \( \| ψ \|_∞ ≤ δ \)).

Since \( \nabla ψ = 2δφ_1\nabla φ_1 \), a calculation shows that

\[
-\Delta ψ = -δ\Delta φ_1^2 = -2δ(|∇φ_1|^2 + φ_1 Δφ_1) = 2δ(λ_1 φ_1^2 - |∇φ_1|^2).
\]

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Then $\psi$ is a subsolution if
\[ 2\delta (\lambda_1 \phi_1^2 - |\nabla \phi_1|^2) \leq a(x)\psi - b(x)\psi^2 - ch(x), \]

Now $\lambda_1 \phi_1^2 - |\nabla \phi_1|^2 \leq -k$ in $\bar{\Omega}_\eta$, and therefore
\[ 2\delta (\lambda_1 \phi_1^2 - |\nabla \phi_1|^2) \leq -2k\delta \leq -a_0\delta - \|b\|_\infty \delta^2 - c, \]
if
\[ \delta < \theta_1 = \frac{2k - a_0}{\|b\|_\infty}, \]
\[ c \leq \tilde{c}(\delta) = \delta(2k - a_0 - \|b\|_\infty \delta). \]

Clearly $\tilde{c}(\delta) > 0$.

Furthermore, we note that $\phi_1 \geq \mu > 0$ in $\Omega_0 = \Omega \setminus \bar{\Omega}_\eta$, also in $\Omega_0$ we have
\[ 2\delta (\lambda_1 \phi_1^2 - |\nabla \phi_1|^2) \leq 2\lambda_1 \delta \leq a_1\delta \phi_1^2 - \|b\|_\infty \delta^2 - c, \]
if
\[ \delta < \theta_2 = \frac{a_1\mu^2 - 2\lambda_1}{\|b\|_\infty}, \]
\[ c \leq \bar{c}(\delta) = \delta(a_1\mu^2 - 2\lambda_1 - \|b\|_\infty \delta). \]

Clearly $\bar{c}(\delta) > 0$. Choose $\theta = \min\{\theta_1, \theta_2\}$ and $\delta = \theta/2$. Then simplifying, both $\tilde{c}$ and $\bar{c}$ are greater than $(\frac{\theta}{2})^2\|b\|_\infty$. Hence if $c \leq (\frac{\theta}{2})^2\|b\|_\infty = c_0(\Omega, a_0, a_1, b)$ then $\psi$ is a subsolution.

Next, we construct a supersolution $z$ of (1). We denote $z = N\zeta(x)$, where the constant $N > 0$ is large and to be chosen later. We shall verify that $z$ is a supersolution of (1). A calculation shows that
\[ -\Delta z = N(-\Delta \zeta) = N. \]

Thus $z$ is a supersolution if
\[ N \geq a(x)z - b(x)z^2 - ch(x), \]
and therefore if $N \geq N_0$ where $N_0 = \sup_{[0,\|a\|_\infty /b_0]}(||a||_\infty v - b_0v^2)$, we have
\[ -\Delta z \geq a(x)z - b(x)z^2 - ch(x), \]
and hence \( z \) is supersolution of (1). Since \( \zeta > 0 \) and \( \partial \zeta / \partial n < 0 \) on \( \partial \Omega \), we can choose \( N \) large enough so that \( \psi \leq z \) is also satisfied. Hence Theorem 1 is proven.

Now, we obtain the existence of positive solution of (1) in the case when \( g(x, u) = \lambda m(x) f(u) \). Assume that there exist positive constants \( r_1, r_2 \in (\alpha, \rho] \) satisfying:

(H.1) \( \frac{r_2}{r_1} \geq \max \left\{ \frac{2 \lambda_1 \| \zeta \|_\infty}{\mu^2}, \frac{2 \lambda_1 \| \zeta \|_\infty \| m \|_\infty f(r_2)}{m_0 \mu^2 f(r_1)} \right\} \),

(H.2) \( kf(r_1) > \lambda_1 |f(0)| \).

**Theorem 2.** Let (H.1), (H.2) hold. Then there exist \( \lambda_* < \bar{\lambda} \) such that (1) has a positive solution for \( \lambda \in [\lambda_*, \bar{\lambda}] \).

**Proof.** Let \( \lambda_1, \phi_1, k, \mu \) and \( \zeta(x) \) are the same as in the proof of Theorem 1. We now construct our positive subsolution. Let \( \psi = r_1 (\phi_1 / \mu)^2 \). Using a calculation similar to the one in the proof of Theorem 1, we have

\[
-\Delta \psi = \frac{2r_1}{\mu^2} \left( \lambda_1 \phi_1^2 - |\nabla \phi_1|^2 \right).
\]

Thus \( \psi \) is a subsolution if

\[
\frac{2r_1}{\mu^2} \left( \lambda_1 \phi_1^2 - |\nabla \phi_1|^2 \right) \leq \lambda m(x) f(\psi),
\]

Now \( \lambda_1 \phi_1^2 - |\nabla \phi_1|^2 \leq -k \) in \( \Omega_\eta \), and therefore

\[
\frac{2r_1}{\mu^2} \left( \lambda_1 \phi_1^2 - |\nabla \phi_1|^2 \right) \leq -\frac{2kr_1}{\mu^2} \leq \lambda m(x) f(\psi),
\]

if

\[
\lambda \leq \bar{\lambda} = \frac{2kr_1}{\mu^2 m_0 |f(0)|}.
\]

Furthermore, we note that \( \phi_1 \geq \mu > 0 \) in \( \Omega_0 = \Omega \setminus \Omega_\eta \), and therefore

\[
\psi = r_1 (\phi_1 / \mu)^2 \geq r_1 (\mu / \mu)^2 = r_1,
\]

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thus \( f(\psi) \geq f(r_1) \). Hence if
\[
\lambda \geq \lambda_* = \frac{2\lambda_1 r_1}{\mu^2 m_0 f(r_1)},
\]
we have
\[
\frac{2r_1}{\mu^2} \left( \lambda_1 \phi^2_1 - |\nabla \phi_1|^2 \right) \leq \frac{2\lambda_1 r_1}{\mu^2} \leq \lambda m_0 f(r_1) \leq \lambda m(x) f(\psi).
\]

We get \( \lambda_* < \hat{\lambda} \) by using (H.2). Therefore if \( \lambda_* \leq \lambda \leq \hat{\lambda} \), then \( \psi \) is subsolution.

Next, we construct a supersolution \( z \) of (1) such that \( z \geq \psi \). We denote \( z = \frac{r_2}{\|\xi\|_{\infty}} \xi(x) \). We shall verify that \( z \) is a super solution of (1). We have
\[
-\Delta z = \frac{r_2}{\|\xi\|_{\infty}}.
\]
(6)

Thus \( z \) is a super solution if
\[
\frac{r_2}{\|\xi\|_{\infty}} \geq \lambda m(x) f(z).
\]

But \( f(z) \leq f(r_2) \) and hence \( z \) is a super solution if
\[
\lambda \leq \bar{\lambda} = \frac{r_2}{\|\xi\|_{\infty} \|m\|_{\infty} f(r_2)}.
\]

We easily see that \( \lambda_* < \bar{\lambda} \), by using (H.1). Finally, using (5), (6) and the comparison principle, we see that \( \psi \leq z \) in \( \Omega \) when (H.1) is satisfied. Therefore (1) has a positive solution for \( \lambda \in [\lambda_*, \bar{\lambda}] \), where \( \bar{\lambda} = \min\{\lambda, \bar{\lambda}\} \). This completes the proof of Theorem 2.

**Remark 1.** Theorem 2 holds no matter what the growth condition of \( f \) is, for large \( u \). Namely, \( f \) could satisfy superlinear, sublinear or linear growth condition at infinity.

**References**


