On the anticrowding population dynamics taking into account a discrete set of offspring

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Abstract. A model of a population dynamics is solved numerically taking into account a discrete set of offsprings and the nonlinear (directed) diffusion. The model consists of a system of integro-partial differential equations subject to conditions of integral type. A spread of initially localized population is studied. Some numerical results are discussed.

Keywords: population dynamics, anticrowding population, nonlinear diffusion, discrete set of offsprings.

1 Introduction

Mammals and birds feed, warm, defend their youngs from enemies, and learn them to find a food and hide from enemies. If one of those duties is non realized young offsprings die. In recent years (see [3, 4, 2]) three models were proposed for populations taking care of their young offsprings. Two of them describe dynamics of two-sex population forming temporal and permanent pairs. In the third one a one-sex population dynamics model is studied taking into account the random spatial diffusion.

In the present paper we consider a one-sex age-structured population deterministic model taking into account a discrete set of offsprings, their care, and nonlinear spatial diffusion of individuals.

In case of linear diffusion initially localised population instantaneously spreads into all habitat. Nonlinear diffusion conditions spread of population into a bounded domain. Goal of the paper is the numerical study of the population spread onto 1-dimensional habitat. Based on the paper [1] we proposed and approved a numerical algorithm.

We assume that all individuals have prereproductive, reproductive, and post-reproductive age intervals. The individuals of reproductive age are divided into groups of singles and those who take care of their young offsprings. All individuals of prereproductive age are divided into young (under maternal care) and juvenile classes. Juveniles can live without maternal care but cannot produce offsprings. The model consists of integro-partial differential equations. Number of these equations depends on a biologically possible maximal newborns number of the same generation produced by an individual.

The paper is organised as follows. In Section 3 we introduce the model, in Section 4 we discuss numerical results. Concluding remarks are given in Section 5.
2 Notation

\( \mathbb{R}^m \): the Euclidean space of dimension \( m \) with \( x = (x_1, \ldots, x_m) \);
\( \kappa \): the diffusion modulus;
\( (0, T) \) and \( (T_1, T_3) \) \((T < T_1 < T_3)\): the child care and reproductive age intervals, respectively;
u(t, \tau_1, x): the age-space-density of individuals aged \( \tau_1 \) at time \( t \) at the position \( x \) who are of juvenile \((\tau_1 \in (T, T_1))\), single \((\tau_1 \in (T_1, T_3))\), or post-reproductive \((\tau_1 > T_3)\) age;
u_k(t, \tau_1, \tau_2, x): the age-space-density of individuals aged \( \tau_1 \) at time \( t \) at the position \( x \) who take care of their \( k \) offsprings aged \( \tau_2 \) at the same time;
\( \nu(t, \tau_1, x): \) the natural death rate of individuals aged \( \tau_1 \) at time \( t \) at the position \( x \) who are of juvenile or adult age;
\( \nu_k(t, \tau_1, \tau_2, x): \) the natural death rate of individuals aged \( \tau_1 \) at time \( t \) at the position \( x \) who take care of their \( k \) offsprings aged \( \tau_2 \);
\( \nu_{ks}(t, \tau_1, \tau_2, x): \) the natural death rate of \( k-s \) young offsprings aged \( \tau_2 \) at time \( t \) at the position \( x \) whose mother is aged \( \tau_1 \) at the same time;
\( \tilde{\nu}_k = \nu_k + \sum_{s=0}^{k-1} \nu_{ks}; \)
\( \alpha_k(t, \tau_1, x)dt: \) the probability to produce \( k \) offsprings in the time interval \([t, t+dt]\) at the location \( x \) for an individual aged \( \tau_1 \);
\( N(t, x): \) sum of spatial densities of juvenile and adult individuals;
u_0(\tau_1, x), u_{k0}(\tau_1, \tau_2, x): \) the initial age distributions;
\( [u|_{\tau_1=\tau}]: \) the jump discontinuity of \( u \) at the point \( \tau_1 = \tau; \)
\( \alpha = \sum_{k=1}^{\alpha_k} \alpha_k \); \( (T_1, \tau_1) = \max(0, \tau_1 - T_3) \); \( \gamma_2(\tau_1) = \min(\tau_1 - T_1, T); \)
\( T_2 = T_1 + T: \) the minimal age of an individual finishing care of offsprings of the first generation;
\( T_4 = T_3 + T: \) the maximal age of an individual finishing care of offsprings of the last generation;
\( \sigma_1 = (T_1, T_3) \); \( \sigma_2 = (T_1, T_4) \); \( \sigma_3 = (T_2, T_4); \)
\( \sigma_1 = (T, \infty) \); \( \sigma_2 = (T, \infty) \); \( \sigma_3 = (T, \infty) \); \( \sigma_3 = (T, \infty) \); \( \sigma_3 = (T, \infty) \); \( \sigma_3 = (T, \infty) \);
\( Q = \{(\tau_1, \tau_2): \tau_1 \in (T_1 + \tau_2, T_3 + \tau_2), \tau_2 \in (0, T)\}. \)

3 The anticrowding population model

In this section, we give the population dynamics model in \( \Omega \) with both the extremely inhospitable and impermeable boundary \( \partial \Omega \) which can be described by the system.

Here \( \partial_t \) and \( \partial_{r_k} \) denote partial derivatives, \( \partial_{r} \) is the operator of the outward normal derivative on the \( \partial \Omega \), \( \text{div}(u_k \nabla N) = \nabla u \nabla N + u \Delta N, \) while \( n \) is the biologically possible maximal number of newborns of the same generation produced by an individual.
The first term on the right-hand side in Eq. (1) describes the fraction of individuals who produce offsprings, the second and third terms describe the fraction of individuals aged \( \tau \) and the unknown ones \( \tau \), respectively. The transition term in the right-hand side of Eq. (1) describes the fraction of individuals whose number after the death of offsprings becomes equal to 1, \( \nu \), whose all young offsprings die and who finish child care, respectively. The condition \( \left. u \right|_{t=0} = u_0, \left. u \right|_{\tau_1 = \tau} = 0, \tau = T_1, T_2, T_3, T_4, \) \( u|_{\partial \Omega} = 0 \) or \( \partial u|_{\partial \Omega} = 0 \),

\[
\begin{align*}
\partial_t u + \partial_{\tau_1} u &= -\nu u + \kappa \text{div}(u \nabla N), \\
&= \left\{ \begin{array}{ll}
0, & \tau_1 \in \sigma_1^*, \\
\alpha u, & \tau_1 \in \sigma_1 \\
0, & \tau_1 \in \sigma_2^*, \\
\int_{\gamma_2(\tau_1)} \sum_{k=1}^n \nu_{k0} u_k d\tau_2, & \tau_1 \in \sigma_2 \\
0, & \tau_1 \in \sigma_3^*, \\
\int_{\gamma_1(\tau_1)} \sum_{k=1}^n \nu_{k0} u_k d\tau_2, & \tau_1 \in \sigma_3, \\
0, & t > 0, x \in \Omega,
\end{array} \right.
\end{align*}
\] (1)

\[
\begin{align*}
\left. u \right|_{\tau_1 = \tau} &= \int_{\sigma_3} \sum_{k=1}^n k u_k |_{\tau_2 = \tau} d\tau_1, \\
\left. u \right|_{t=0} &= u_0, \left. u \right|_{\tau_1 = \tau} = 0, \tau = T_1, T_2, T_3, T_4, \) \( u|_{\partial \Omega} = 0 \) or \( \partial u|_{\partial \Omega} = 0 \),
\end{align*}
\] (2)

\[
N = \int_T^\infty u d\tau_1 + \int_T^{T_1+\tau_2} d\tau_2 \int_{T_1+\tau_2}^{T_1+\tau_3} \sum_{k=1}^n u_k d\tau_1.
\] (3)

The integral term in the right-hand side of Eq. (1) can be written as follows:

\[
\int_{\gamma_2(\tau_1)} \sum_{k=1}^n \nu_{k0} u_k d\tau_2 = \left\{ \begin{array}{ll}
\int_0^{\tau_2-T_1} \sum_{k=1}^n \nu_{k0} u_k d\tau_2, & \tau_1 \in (T_1, T_2), \\
\int_0^{T} \sum_{k=1}^n \nu_{k0} u_k d\tau_2, & \tau_1 \in (T_2, T_3), \\
\int_{T_1+\tau_2}^{T_1+\tau_3} \sum_{k=1}^n \nu_{k0} u_k d\tau_2, & \tau_1 \in (T_3, T_4).
\end{array} \right.
\] (4)

The constant \( \kappa \) is the diffusion modulus. Given functions \( \nu, \nu_k, \nu_{k0}, \alpha, u_0, \) and \( u_{k0} \) and the unknown ones \( u \) and \( u_k \) are to be positive. The positive constants \( T \) and
In this section using computer simulations, we study model (1)–(4) in \( \mathbb{R}^4 \). The initial conditions \( z|_{\partial M} = 0 \) and \( \partial_\tau z = 0 \) now reduces to \( z|_{x=0;1} = 0 \). The model was written on the characteristics lines and then solved numerically. For units of age and length we use \( T \) and \( L \).

We assume that integer \( n \) is equal to 3. This corresponds to species e.g., *Felis yagourandi* (2–3 children), *Pseudocheirus peregrinus* (1–3 children), *Tremarctos ornatus* (1–3 children), and *Artictis binturong* (1–3 children).

4 Initial and vital functions. Numerical results

In all calculations we use the vital rates

\[
\begin{align*}
\nu(\tau_1) &= \mu_1 \tau_1^q + \mu_2, \\
\nu_k(\tau_1, \tau_2) &= \mu_k \tau_1^{q_k} + \mu_k, \\
\nu_k(\tau_1, \tau_2) &= \mu_k (\tau_2 - \tau_0)^q + \mu_k, \\
\alpha_k(\tau_1) &= \alpha_k \exp \left\{ - \left( \tau_1 - (T_3 + T_1)/2 \right)/\alpha_k \right\},
\end{align*}
\]

and initial functions

\[
\begin{align*}
u_0(\tau_1, x) &= F_0(x) \beta_3(\tau_1 + \beta_1) \exp\{-\beta_1 \tau_1\}, \\
u_k(\tau_1, \tau_2, x) &= \alpha_k(\tau_1 - \tau_2) u_0(\tau_1 - \tau_2, x) \tilde{U}_k(\tau_2)
\end{align*}
\]

which are also used in [2]. Here \( \tilde{U}_k(\tau_2) = 1 + \frac{\tau_2}{\kappa}(\tilde{U}_k(T) - 1) \).

Using compatibility conditions (4)1, we determine function

\[
\beta_2(x) = \tilde{U}_k(T) \int_{T_1}^{T_3} \sum_{k=1}^3 k \alpha_k(\tau_1)(\tau_1 - T) \exp\{-\beta_1 \tau_1\} d\tau_1 \\
\times \left\{ \exp\{-\beta_1 T\} - \tilde{U}_k(T) \int_{T_1}^{T_3} \sum_{k=1}^3 k \alpha_k(\tau_1) \exp\{-\beta_1 \tau_1\} d\tau_1 \right\}^{-1} - T.
\]

The positive constants \( \mu_1, \mu_2, \mu_k1, \mu_k2, \mu_k3, \mu_k4, q > 1, \mu_k > 1, \beta_1, \beta_3, \tilde{U}_k(T), \xi_2 \geq 1, \xi_3 \geq 1, T < T_1 < T_3, \alpha_k1, \alpha_k2, \tau_0 < T, \) and \( \rho_0 \) remain free.
4.2 Numerical results

Results of numerical calculations are displayed in Figs. 1–3 for the following constants and initial functions:

\[
\begin{align*}
\beta_1 &= 0.55, \quad \beta_2 = 5.7, \quad \kappa = 0.01; \quad T = 1, \quad T_1 = 2, \quad T_2 = 3, \quad T_3 = 4, \quad T_4 = 5, \\
\alpha_{11} &= 0.7, \quad \alpha_{21} = 0.75, \quad \alpha_{31} = 0.7, \quad \alpha_{12} = \alpha_{22} = \alpha_{32} = 4, \\
\mu_1 &= \mu_2 = 0.01, \quad \mu_{11} = \mu_{12} = \mu_{21} = \mu_{22} = \mu_{31} = \mu_{32} = 0.001, \\
\mu_{312} &= \mu_{321} = \mu_{212} = \mu_{121} = \mu_{101} = \mu_{102} = 0.0012, \\
\mu_{311} &= \mu_{312} = \mu_{201} = \mu_{202} = 0.001, \quad \mu_{301} = \mu_{302} = 0.0008, \\
\tilde{U}_1 &= 0.7, \quad \tilde{U}_2 = 0.6, \quad \tilde{U}_3 = 0.5, \quad \tau_0 = 0.2, \quad \xi_3 = 1.5, \\
q_0 &= 1.5, \quad q_1 = 2, \quad q_2 = 2, \quad q = 2, \\
F_0 &= F_1 = \begin{cases} 0, & x \in [0, 0.375], \\
(16(x - 0.375)(x - 0.625))^{3/2}, & x \in [0.375, 0.625], \\
0, & x \in [0.625, 1], \end{cases} \\
F_0 &= F_2 = \begin{cases} (16(x(0.5 - x)))^{3/2}, & x \in [0, 0.5], \\
0, & x \in [0.5, 1], \end{cases}
\end{align*}
\]

Function \(N(t, x)\) determined by the solution to the Dirichlet problem is exhibited in Figs. 1 and 2. In Fig. 3 the comparison of \(N\) for Dirichlet and Neumann problems is given. All figures demonstrate a spread of \(N(t, x)\) over all interval \([0, 1]\) of initially localized functions.

Fig. 1 demonstrates the influence of diffusivity \(\kappa\) while Fig. 2 illustrates \(N(t, x)\) determined by the solution of the Dirichlet problem for initial function \(F_0 = F_1\). In Fig. 3 we compare the \(N(t, x)\) determined by the solution to the Dirichlet and Neumann problems for the initial function \(F_1\).
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Fig. 3. Comparison of $N(t,x)$ for the Dirichlet (1) and Neumann (2) problems in the case $F_0 = F_1$.

5 Concluding remarks

A one sex-age-structured population dynamics model is presented for Dirichlet and Neuman problems and solved numerically taking into account a discrete set of offsprings, strong maternal care, and nonlinear spatial diffusion. According to strong maternal care all offspring die if their mother dies. Because of the diffusion fluxes, $u \nabla N$ and $u_k \nabla N$, the model is nonlinear and describes an anticrowding population dynamics. A spread of initially localized population over all interval is studied.

Our numerous simulations demonstrate a fast convergence of numerical algorithm used for calculations but it’s mathematical justification is an open problem.

References


REZIUMĖ

Apie nesusispiečiančios populiacijos dinamiką atsižvelgiant į diskrečią palikuonių aibę

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Raktiniai žodžiai: populiacijos dinamika, netiesinė difuzija, diskrečioji palikuonių aibė, nesusispiečianti populacija.