Numerical study of a toxin and antibody interaction inside the cell

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Abstract. A nonlinear mathematical model describing toxin penetration into cell from the extracellular domain and its interaction with an antibody placed inside the cell is studied. The protective properties of the antibody against the toxin are investigated numerically choosing the model parameters in case of the well-mixed solution assumption.

Keywords: antibody, toxin, receptor, cell protection.

Introduction

A toxin is a poisonous substance produced by living cells, animals, microorganisms or plants. Toxins can be small molecules, peptides, or proteins that are capable to cause a disease on contact with (or absorption) by body tissues interacting with biological macromolecules such as enzymes or cellular receptors [1, 3]. Many antibodies have been generated to protect from a number of toxins. Recent progress of antibody engineering allows to produce antibodies which are used against one or several of toxins [5]. The design and production of new antibodies usually is an expensive and long process that include experimental investigation of the antibodies properties. A proper mathematical model can be used to predict the antibody properties (parameters) before experimental research [2, 4, 6]. In this paper we use the mathematical model proposed in [4] in case of the well-mixed solution assumption. The antibody is placed inside the cell where it interact with the toxin. The aim of this paper is to study the influence of the Endoplasmic Reticulum (ER) absorption rate, toxin-antibody reaction rate, initial antibody concentration on cell protection parameter and toxin concentration near the ER envelope.

1 The mathematical model

We examine the situation when the spherical cell is encircled by the toxin solution which occupies a domain between a cell membrane and an external surface. Toxin moves toward the cell and interact with receptors on the cell membrane. Some of this toxin penetrates into the cell. The other part of the toxin bounded to membrane receptors leaves them and returns into the extracellular domain. Toxin being inside the cell moves toward endoplasmic reticulum(ER) and interacts with the antibody forming the toxin-antibody complex. Part of the toxin-antibody complex splits up
again in the toxin and antibody. Toxin penetrates across the ER membrane into the
cytosol, reaches ribosomes and damages the protein production mechanism.

The basic notations used in this paper are:

- $\rho$ - distance to the origin,
- $S_m = \{ \rho: \rho = \rho_m \}$ - the surface of the membrane of a spherical cell,
- $S_e = \{ \rho: \rho_m < \rho < \rho_e \}$ - the surface of the external sphere which limits domain $\Omega_e$,
- $\Omega_e = \{ \rho: \rho_m < \rho < \rho_e \}$ - the domain between the cell membrane surface $S_m$ and the
- sphere $S_e$ initially occupied by the toxin,
- $S_n = \{ \rho: \rho_0 = \rho_n \}$, $\rho_n < \rho_m$ - the surface of the spherical domain occupied by the
- endoplasmic reticulum.
- $\Omega_i = \{ \rho: \rho_m < \rho < \rho_e \}$ - the domain between the spheres $S_n$ and $S_m$ where the
- antibody and toxin forward and reverse reactions run,
- $r_0\theta$ - the concentration of receptors confined to the cell membrane,
- $1 - \theta$ - the part of the free receptors,
- $\theta$ - the part of the toxin-bound receptors,
- $u_T e$ - the toxin concentration in $\Omega_e$,
- $u_0 T e$ - the initial toxin concentration in $\Omega_e$,
- $u_T$, $u_A$, $u_C$ - the concentrations of the toxin, antibody and toxin-antibody complex,
- respectively, in the domain $\Omega_i$,
- $u^0_T$ - the initial antibody concentration in $\Omega_i$,
- $k_1$, $k_{-1}$ - the forward and reverse constants of the toxin-antibody reaction rate,
- $k_2$ and $k_{-2}$ - the forward and reverse binding rate constants of the toxin and receptor
- confined on the membrane,
- $k$ - the toxin penetration rate constant from domain $\Omega_e$ across the membrane into the
- cell (into domain $\Omega_i$),
- $\gamma$ - the toxin absorption rate constant describing toxin influx into ER,
- $\delta(t)$ - the antibody protection function.

In [4] the advection-diffusion mathematical model was proposed where the motion
and interaction of species in the extracellular and intracellular domains are described
by a system of nonlinear partial differential equations. In that paper a more simple
model of nonlinear ordinary differential equations was also studied assuming that
the solution of the toxin, antibody and toxin-antibody complex in the intracellular
domain and the toxin solution in the extracellular domain are uniformly mixed and
homogeneously distributed (well-mixed solutions).

In this paper the mathematical model proposed in [4] for case of the well-mixed
solution assumption is used. The change in time of the toxin concentration in the
extracellular domain $\Omega_e$ and the number of toxin-bound receptors is described by the
system of nonlinear ordinary differential equations:

\[
\begin{align*}
\frac{du_T}{dt} &= -k_3 r_0 (k_2 (1 - \theta) u_T - k_{-2} \theta), \\
    u_T |_{t=0} &= u^0_T, \\
\frac{d\theta}{dt} &= k_2 (1 - \theta) u_T - k_{-2} \theta - k \theta, \\
    \theta |_{t=0} &= 0.
\end{align*}
\]
Numerical study of a toxin and antibody interaction inside the cell

Concentrations of the toxin, antibody and toxin-antibody complex in the intracellular domain $\Omega_i$ are calculated from the system of the nonlinear differential equations:

\[
\begin{align*}
\frac{du_T}{dt} &= -k_1 u_T u_A + k_{-1} u_C + k k_4 r_0 \theta - k_5 \gamma u_T, \quad t > 0, \\
\frac{du_A}{dt} &= -k_1 u_T u_A + k_{-1} u_C, \quad t > 0, \\
\frac{du_C}{dt} &= k_1 u_T u_A - k_{-1} u_C, \quad t > 0, \\
u_A|_{t=0} &= u_0^A, \\
u_C|_{t=0} &= 0,
\end{align*}
\]

(2)

where $k_3 = 3\rho_m^2/(\rho_m^3 - \rho_n^3)$, $k_4 = 3\rho_m^2/(\rho_m^3 - \rho_n^3)$, $k_5 = 3\rho_n^2/(\rho_m^3 - \rho_n^3)$. The systems (1) and (2) are coupled by function $\theta$. The term $k k_4 r_0 \theta$ determines an amount of the toxin internalized from domain $\Omega_i$ across the membrane into the cell.

In the case where no antibody is used in domain $\Omega_i$ only the equation describing the toxin concentration is solved instead of system (2):

\[
\begin{align*}
\frac{du_T}{dt} &= k k_4 r_0 \theta - k_5 \gamma u_T, \quad t > 0, \\
\frac{du_T}{dt}|_{t=0} &= 0.
\end{align*}
\]

(3)

The effectiveness of antibody treatment is measured by comparison of the toxin concentrations near ER in the case where antibody is used ($u_A^0 > 0$) and in the case where antibody is absent ($u_A^0 = 0$). The dimensionless parameter (antibody protection factor) is defined by the expression

\[
\delta(t) = \frac{u_T(t, u_A^0)}{u_T(t, 0)}.
\]

(4)

The numerator is determined by solving systems (1) and (2), denominator is solution of problem (1) and (3). By definition, $0 \leq \delta(t) \leq 1$ with the lower values corresponding to the better therapeutic effect of the antibody treatment. Values of $\delta$ close to unity inform that the influence of the antibody treatment is negligible – the difference of concentrations of internalized toxin near ER in cases $u_A^0 > 0$ and $u_A^0 = 0$ is small.

Using characteristic dimensional time, length and concentration units $\tau_\star$, $l_\star$, $u_\star$, respectively, Eqs. (1)–(3) can be rewritten in a dimensionless form. By substituting new variables, $\rho = \hat{l}, \bar{t} = \tau_\star \hat{t}, \bar{r}_0 = l_\star \bar{r}_0, u_{T\star} = u_\star \bar{u}_T, u_T = u_\star \bar{u}_T, u_A = u_\star \bar{u}_A, u_C = u_\star \bar{u}_C, u_{T\star}^0 = u_\star \bar{u}_T^0, u_T^0 = u_\star \bar{u}_T^0, u_A^0 = u_\star \bar{u}_A^0, u_C^0 = u_\star \bar{u}_C^0, k_1 = \tau_\star u_\star k_1, k_2 = \tau_\star u_\star k_2, k = \tau_\star k, k_{-1} = \tau_\star k_{-1}, k_{-2} = \tau_\star k_{-2}, k_3 = l k_3, k_4 = l k_4, k_5 = l k_5$, we deduce the same problem (1)–(3) but in the dimensionless form. In what follows all variables are non-dimensional.

2 Numerical results

To solve the nonlinear problems (1), (2) and (1), (3) we apply the Runge–Kutta fourth order method using a uniform discrete grid. The following non-dimensional
Fig. 1. Dependence of antibody protection factor $\delta(t)$ and toxin concentration $u_T(t)$ on toxin absorption rate constant $\gamma$: 0.01 (1), 0.025 (2), 0.05 (3), 0.1 (4). — $u_A^0 = 1$, - - - $u_A^0 = 0$.

Fig. 2. Dependence of antibody protection factor $\delta(t)$ and toxin concentration $u_T(t)$ on the initial antibody concentration $u_A^0$: 1 (—), 0.5 (- - -) and toxin absorption rate constant $\gamma$: 0.01 (1), 0.025 (2), 0.1 (3). In case $u_A^0 = 0$, absorption rate constant $\gamma$: 0.01 – •, 0.025 – ◦, 0.1 – ▲.

Values were used in calculations $[4, 6]$: $\rho_n = 0.02, \rho_m = 0.1, \rho_c = 0.18, k_1 = 1.3 \times 10^{-2}, k_{-1} = 1.4 \times 10^{-4}, k_2 = 1.25 \times 10^{-2}, k_{-2} = 5.2 \times 10^{-4}, k = 3.3 \times 10^{-5}, r_0 = 2.115 \times 10^{-3}, \rho_e = 10^{-1}, \rho_c = 1, u_A^0 = 1, u_T^0 = 0.5$. If value of parameter differ from indicated then it is specified in the text or in the legends of plots.

Fig. 1 demonstrates the influence of the endoplasmic reticulum (ER) absorption rate $\gamma$ on antibody protection factor $\delta(t)$ and toxin concentration $u_T(t)$. Since all other parameters are fixed, the amount of toxin which penetrates into the cell, $k_d k_r n_0 \theta$, and amount of the toxin neutralized by the antibody is the same for all values of $\gamma$. The toxin concentration near the ER decreases as the absorption rate increases since more toxin crosses the ER envelope (Fig. 1(b)). The toxin concentration is monotonically increasing time function and rapidly reaches a saturation value excluding the case where $\gamma$ is small and the antibody is used (Fig. 1(b), solid curves 1 and 2). The antibody protection factor $\delta(t)$ is monotonically increasing $\gamma$ function, but non-monotonic time function: decreases, reaches the minimum, tends increasing to a steady-state value. There is a time interval for which the application
Numerical study of a toxin and antibody interaction inside the cell

![Graphs showing the effect of toxin-antibody interaction parameters on antibody protection factor δ.](image)

**Fig. 3.** (a) Effect of the toxin-antibody reaction rate constant $k_1$: 0.0026 (1), 0.013 (2), 0.065 (3) and (b) toxin and receptor binding rate constant $k_2$: 0.00125 (1), 0.0125 (2), 0.125 (3) on antibody protection factor δ for γ: 0.01 (—), 0.025 (●), 0.1 (— —).}

...of the antibody is the most effective. The plots in Fig. 1 show that the efficiency of the antibody application is low if absorption rate constant is large – the reduction of the internalized toxin due to application of antibody is negligible (Figs. 1(a) and 1(b), curves 4).

Fig. 2 shows the dependence of antibody protection factor and toxin concentration on the initial antibody concentration $u_0^A$ for different values of the ER absorption rate γ. In Fig. 2(b) we see that the higher initial antibody concentration reduces the toxin concentration at the ER since the amount of the toxin-antibody complex grows. The difference between toxin concentrations grows as γ decreases. For γ = 1 the influence of $u_0^A$ on the toxin concentration is negligible – in cases $u_0^A = 1, 0.5, and 0$ the toxin concentrations are almost the same (Fig. 2(b), curves 3). For small and larger values of γ the cell protection increases as $u_0^A$ grows (Fig. 2a).

The plots in Fig. 3(a) illustrate the effect of the forward toxin-antibody reaction rate constant $k_1$ for two values of the absorption rate γ = 0.01 and 0.1. The cell protection grows as $k_1$ increases. Constant $k_1$ more influence function δ(t) at lower values of the ER absorption rate. Fig. 3(b) demonstrates the behavior of function δ(t) for three values of the toxin and receptors binding rate constant $k_2$ and for three values of γ. δ(t) is non-monotonic function with respect to time and $k_2$. Calculations show that δ(t) weakly depends on $k_2$ for large γ (Fig. 3(b), solid lines).

### 3 Conclusions

The toxin, antibody, and receptors interaction was studied numerically by using a model proposed in [4] in the case of the well-mixed solution assumption. The results of numerical investigation are the following:

1. The efficiency of the antibody placed inside the cell is low when the ER absorption rate is large.
2. The efficiency of the antibody grows as the ER absorption rate decreases, the toxin-antibody reaction rate and initial antibody concentration increase.

References


REZIUMĖ

**Antikūno ir toksino sąveikos ląstelėje skaitinis tyrimas**

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