An $\omega$-Decidable Deductive Procedure for a Restricted First-order Linear Temporal Logic

Regimantas Pliuškevičius
Institute of Mathematics and Informatics,
Akademijos 4, 2600 Vilnius, Lithuania
E-mail: regis@ktl.mii.lt

Abstract
A new type deduction-based procedure $Sat_\omega$ is presented for a restricted first-order linear temporal logic with temporal operators “next” and “always”. The main part of the proposed deductive procedure is an automatic generation of an inductive hypothesis, i.e., an assertion needed to prove a given sequent by the infinitary omega-type rule for the operator “always”. The proposed deductive procedure $Sat_\omega$ consists of four separate decidable deductive procedures replacing the infinitary omega-type rule for the operator “always”. These four decidable parts of $Sat_\omega$ cannot be joined. Therefore $Sat_\omega$ (by analogy with $\omega$-completeness) is only $\omega$-decidable.

Keywords: first order temporal logic, inductive hypothesis, sequent calculi, deduction-based decision procedures, temporal and inductive reasoning.

1. Introduction
A temporal logic is exploited in a variety of application areas of computer science and artificial intelligence, particularly in system verification, knowledge representation, knowledge-based systems, multi-agent systems, program synthesis and execution (see, e.g., [13, 16, 17, 26, 39]). All these applications impose a requirement to have techniques for reasoning on temporal logic formulas. Model-checking methods are effective and automatic for temporal formulas that are propositional. For more complex systems, however, it is necessary or convenient (see, e.g., [8, 40]) to employ a first-order temporal logic ($FTL$, in short). In [18] it was indicated that the model-theoretic approach was not a panacea and that, ultimately, both model-checking and deductive methods would be needed. $FTL$ is a very expressive language (see, e.g., [1]). Unfortunately, $FTL$ is incomplete, in general (see, e.g., [1, 2, 27, 44]). But it becomes complete (see, e.g., [22, 45]) after adding an infinitary $\omega$-type rule (which we present in the sequent version):

$$
\frac{\Gamma \rightarrow \Delta, A; \ldots; \Gamma \rightarrow \Delta, \circ^k A; \ldots}{\Gamma \rightarrow \Delta, \Box A} (\rightarrow \Box_\omega),
$$

where $\circ^k A$ means ”$k$-time next $A$”. So, $FTL$ is $\omega$-complete, in general. In some particular cases, the $FTL$ (and, of course, in the propositional case) is finitary complete and/or decidable (see, e.g., [6, 27, 29–36, 41, 48, 49]). In this case, instead of the rule $(\rightarrow \Box_\omega)$ one can use the rule of the form:

$$
\frac{\Gamma \rightarrow \Delta, R; R \rightarrow \circ R; R \rightarrow A}{\Gamma \rightarrow \Delta, \Box A} (\rightarrow \Box).
$$
induction hypothesis. Instead of these sequent \( S \) hypothesis for the essential infinite called \( \text{FTL} \) to verify that and \( \rightarrow \) rather constructive proof of completeness of the given infinitary calculus; 4) the complexity of reflects the semantics of the temporal operators under consideration; 3) it allows an natural and allows one to establish closer links between model theory and proof theory; 2) it explicitly temporal operators such as “until” and “fixpoints” are widely used (see e.g., [5, 46, 47]).

The infinitary axiomatization of temporal logic has some useful properties, namely: 1) it \( \text{proves(by induction)} \) that \( \text{a conclusion.} \)

\( \text{A} \) \( \rightarrow \) \( \square \text{g} \)

However, the sequents \( S \) is considered. The aim of this report is to construct (for so-called \( \text{D-sequents} \)) a deductive procedure \( S_{\omega} \), consisting of several separate parts, which are carried out automatically. The main part of \( S_{\omega} \) is an automatic generation of an inductive hypothesis.

One of the difficulties in the construction of an effective deduction-based procedure (based on some calculus, e.g., sequent, tableaux, or resolution) is the indeterminate character of the rules of inference. The rules of traditional calculi are nondeterministic – they say what we may do, not what we must do. Another barrier to obtain effective deductive procedures is connected with the necessity of duplication of the main formulas of some rules in sequent calculi (in the tableaux or resolution calculi the situation is connected with a necessity of “looping”). The examples of such rules are: 1) rule \( (\supset \rightarrow) \) (in the left premise) see [12] in the propositional intuitionistic sequent calculus (without the “contraction” rule); 2) rule \( (\square \rightarrow) \) (in which the duplication of the main formula is with some “temporal” shift, see below, section 2) in \( \text{FTL} \); 3) rules \( (\forall \rightarrow), (\rightarrow \exists) \) in the classical first-order sequent calculus. Any looping rule (i) has the following undesirable property: a premise (or premises) of (i) and has a duplication of the
main formula in a premise of the rule (i) is more complex than the conclusion of (i). In [12] the “contraction-free” (i.e., “looping-free”) sequent calculus for a propositional intuitionistic logic was constructed.

The deductive procedure \(Sat_\omega\), proposed here, is based on a revised version of saturation-type [29–36] calculi (devoted to consider some complete classes of FTL) which are “procedural” ones. They reflect a dominating trend in the area of deductive methods: “from calculus to proof procedures”. The proposed deductive procedure \(Sat_\omega\) consists of decidable deductive procedures which say “what we must do”.

The object of consideration of the proposed saturation deductive procedure \(Sat_\omega\) is the so-called \(D\)-sequents, i.e., the sequents of the form \(\Sigma, \square \Omega \rightarrow \square^n \bigvee A_i\), where \(\Sigma\) consists of atomic formulas of the shape \(\circ^k E (k \geq 0)\), \(E\) is an elementary formula, \(A_i\) is an atomic formula, \(\square \Omega\) consists of formulas of the shape \(\forall x (\circ^k E(x) \supset \circ^l P(f(x)))\), called the kernel formulas (where \(k < l, k \geq 0\); \(E(x)\) is an elementary formula without functional symbols, \(P(f(x))\) is an elementary formula with functional symbols). If \(k = 0\), then such a \(D\)-sequent will be called an elementary one. \(D\)-sequents are a certain skolemized version of M. Fisher’s normal form [14]. The general form of \(D\)-sequents was proposed by V. Orevkov in a private conversation with the author. The shape of \(D\)-sequents allows us to construct decidable “looping-free” calculi for “induction-free” \(D\)-sequents, i.e., without the positive occurrence of \(\square\) and for \(D\)-sequents “with induction”, i.e., containing the positive occurrence of \(\square\). Namely, we can replace the rule \((\rightarrow \exists)\) by the corresponding axiom \((\exists)\) and incorporate applications of the rules \((\forall \rightarrow), (\square \rightarrow)\) into the application of an integrated separation rule and a generalized integrated separation rule (see Sections 2, 3).

The proposed deductive procedure \(Sat_\omega\) for \(D\)-sequents consists of four separate decidable parts. The goal of the first part of \(Sat_\omega\) is to obtain (from a given \(D\)-sequent \(S\)) the elementary \(D\)-sequent \(S^*\). The problem of generating the elementary \(D\)-sequent is decidable, i.e., after a finite number of steps we get either \(Sat_\omega \vdash S^*\) (if the given \(D\)-sequent \(S\) is valid) or \(Sat_\omega \vdash S^*\) (if \(S\) is invalid). The goal of the second part of \(Sat_\omega\) is to obtain (from the generated elementary sequent \(S^* = \Sigma, \square \Omega \rightarrow \square A\)) a so-called “saturated” \(D\)-sequent \(S^{**} = \Sigma^*, \square \Omega \rightarrow \square A\), where \(\Sigma^*\) consists of some subformulas of \(\Omega\). The problem of the generating the saturated sequent \(S^{**}\) is decidable, i.e., after a finite number of steps we get either \(Sat_\omega \vdash S^{**}\) (if given \(D\)-sequent \(S\) is valid) or \(Sat_\omega \vdash S^{**}\) (if \(S\) is invalid). The goal of the third part of \(Sat_\omega\) is to construct a so-called similarity substitution \(\sigma\) (each component of which is periodic, for example, \(x^* \leftarrow f_1(\ldots(f_n(x^*))\), it means that the obtained value \(x^*\) is a fixpoint of the functions \(f_1, \ldots, f_n\)) and the sequent \(S^{**}\), which differs from \(S^{**}\) only by the values of variables. The problem of generating the similarity substitution \(\sigma\) and the sequent \(S^{**}\) is decidable, i.e., after a finite number of steps we get either \(Sat_\omega \vdash S^{**}\) (if given \(D\)-sequent \(S\) is valid) or \(Sat_\omega \not\vdash S^{**}\) (if \(S\) is invalid). With the aid of the obtained similarity substitution \(\sigma\), an inductive hypothesis of the shape \(S^{**}\sigma^n (n \in \omega)\) is constructed directly. Then the generated inductive hypothesis \(S^{**}\sigma^n\) is automatically verified, i.e., it is checked that \(Sat_\omega \vdash S^{**}\sigma^n (n \in \omega)\). This verification is carried out by induction on \(n\). The basis case, i.e., that \(Sat_\omega \vdash S^{**}\sigma^n\), is the same as the third stage of \(Sat_\omega\). The step of induction, i.e., that \(Sat_\omega \vdash S^{**}\sigma^n \Rightarrow Sat_\omega \vdash S^{**}\sigma^{n+1}\), is realized in the fourth stage of \(Sat_\omega\). If all the four parts are successful, then \(Sat_\omega \vdash S\), i.e., the given \(D\)-sequent \(S\) is valid. The third and fourth parts of \(Sat_\omega\) cannot be connected: we can not deduce the sequent \(S^{**}\sigma^n\). We can only test that from the assumption \(S^{**}\sigma^n\) it is possible (or not) to generate the sequent \(S^{**}\sigma^{n+1}\). The third and fourth parts make up the main stage of the proposed deductive procedure \(Sat_\omega\). Jointly they replace the \(\omega\)-rule \((\rightarrow \square_\omega)\) for saturated
\(D\)-sequeents. Since all the parts of \(Sat_\omega\) are decidable (but not joimable!), by analogy with the \(\omega\)-completeness we can say that the deductive procedure \(Sat_\omega\) is \(\omega\)-decidable.

In general, we call a deductive procedure for an \(\omega\)-complete logic \(\omega\)-decidable, if it consists of \(n > 1\) separate, not joimable, decidable deductive procedures. Therefore, the \(\omega\)-decidability is a natural extension of the traditional decidability which is applied to a complete logic or a subset of the complete logic.

2. Infinitary calculi \(G_{L\omega}, \ G^*_L\omega\)

The proposed deductive procedure \(Sat_\omega\) is founded using the infinitary calculus \(G_{L\omega}\) containing the \(\omega\)-type rule \((\rightarrow \square_\omega)\) and the traditional “looping” rule \((\square \rightarrow)\). Apart from the traditional infinitary calculus in this section, we introduce the looping-free infinitary calculus \(G^*_L\omega\) containing a non-traditional, non-local looping-free rule instead the traditional looping rule \((\square \rightarrow)\).

Since the objects of consideration of the calculus \(G^*_L\omega\) are \(D\)-sequeents (see below), the calculus \(G^*_L\omega\) does not contain any logical rules.

**Definition 1 (language, term, elementary formula, formula)**

The language is a countable collection of predicate symbols \(P, Q, R, P_1, Q_1, R_1, \ldots\), functional symbols \(f, g, h, f_1, g_1, h_1, \ldots\), constants \(a, b, c, a_1, b_1, c_1, \ldots\), variables \(x, y, z, x_1, y_1, z_1, \ldots\), logical symbols \(\lor, \land, \lor\), quantifiers \(\forall, \exists\), and temporal operators \(\circ\) (“next”), \(\square\) (“always”). The language does not contain the equality symbol. For simplicity we consider only one-place predicate and functional symbols, as well. We assume that all the predicate symbols are flexible (i.e., change their value in time), and the function symbols are rigid (i.e., with time-independent meanings). A term and formula are defined in the usual way. An elementary formula is an expression of the form \(P(t)\), where \(P\) is a predicate symbol, \(t\) is a term. We use only skolemized formulas, i.e., without positive (negative) occurrences of \(\forall\) (\(\exists\), respectively). Throughout the paper \(\omega := \{0, 1, \ldots, n, \ldots\}\).

In the first-order linear temporal logic we have (cf. [26]) that \(\circ(A \circ B) \equiv A \circ \circ B(\circ \in \{\lor, \land, \lor\})\) and \(\circ\sigma A \equiv \sigma \circ A(\sigma \in \{\lor, \square, \forall x, \exists x\})\). Relying on these equivalences, we can consider occurrences of the “next” operator \(\circ\) only entering the formula \(\circ^k E\) \((k\text{-time “next” elementary formula} E)\). For the sake of simplicity, we “eliminate” the “next” operator and the formula \(\circ^k E\) is abbreviated as \(E^k\) (i.e., as an elementary formula with the index \(k\)). We also use the notation \(A^k\) for an arbitrary formula \(A\) in the following meaning.

**Definition 2 (index, atomic, formula)**

1) If \(E\) is an elementary formula, \(i, k \in \omega\), \(k \not= 0\), then \((E^i)^k := E^{i+k}\ (E^0 := E); E^l(l > 0)\) is called an atomic formula, and \(E^l\) becomes elementary if \(l = 0\); 2) \((A \circ B)^k := A^k \circ B^k; if \circ \in \{\lor, \land, \lor\}; (\sigma A)^k := \sigma A^k, if \sigma \in \{\square, \forall x, \lor\}\).

For example, the expression \(\forall x(P^1(x) \supset Q^5(f(x))))^1\) means the formula \(\forall x(P^2(x) \supset Q^4(f(x)))\).

**Definition 3 (sequent)**

A sequent is an expression of the form \(\Gamma \rightarrow \Delta\), where we assume that \(\Gamma, \Delta\) are arbitrary finite multisets (i.e., not sequences or sets) of formulas.

**Definition 4 (normal form, \(F\)-sequeents, parametrical parts, \(\square\)-parts and \(Q\)-parts of \(F\)-sequeents)**

A sequent \(S\) is in the normal form (in brief: \(F\)-sequent), if (see [14]) \(S = \Sigma_1, \forall \Delta_1, \forall \Omega_1 \rightarrow \Sigma_2, \exists \Delta_2, \exists \Omega_2, \) where \(\Sigma_i = \emptyset (i \in \{1,2\})\) or consist of atomic formulas,
\(\Sigma_1, \Sigma_2\) are parametrical parts of \(F\)-sequents; \(\forall \Delta_1 = \emptyset\) or consists of formulas of the form \(\forall x_1, \ldots, x_n A\), where \(A\) does not contain \(\Box, \forall, \exists\); \(\exists \Delta_2 = \emptyset\) or consists of formulas of the form \(\exists x_1, \ldots, x_n A\), where \(A\) does not contain \(\Box, \forall, \exists\); \(\forall \Delta_1, \exists \Delta_2\) are \(Q\)-parts of \(F\)-sequents; \(\forall \Omega_1 (\exists \Omega_2) = \emptyset\) or consists of formulas of the form \(\forall x_1, \ldots, x_n A\) (\(\exists x_1, \ldots, x_n A\), respectively), where \(A\) does not contain \(\forall, \exists\); \(\Box \forall \Omega_1, \Box \exists \Omega_2\) are \(\Box\)-parts of \(F\)-sequents.

From [14] we get the following

**Theorem 1** Let \(S\) be a sequent, then there exists \(S^+\) such that \(S^+\) is an \(F\)-sequent and \(S\) is satisfiable iff \(S^+\) is satisfiable.

**Remark 1** A. Degtyarev in a private communication drew my attention that for getting skolemized normal form [14] it is essential that function symbols be flexible (i.e., change their values in time). So Theorem 1 is valid if \(S^+\) contains corresponding flexible function symbols. However, for simplicity of the considerations, we assume that all constants, function symbols are one-place and rigid (i.e., with time-independent meanings). The consideration of sequents with flexible constants and function symbols require additional deductive tools (see, e.g. [33]) and will not be considered here.

**Definition 5 (calculi \(G_{L\omega}, G\))** A calculus \(G_{L\omega}\) is defined by the following postulates.

Axiom: \(\Gamma, A \rightarrow \Delta, A\).

Rules:

1) temporal rules

\[\frac{A, \Box A^1, \Gamma \rightarrow \Delta}{\Box A, \Gamma \rightarrow \Delta} (\Box \rightarrow)\]

\[\frac{\{\Gamma \rightarrow \Delta, A^k\}_{k \in \omega}}{\Gamma \rightarrow \Delta, \Box A} (\rightarrow \Box)\]

2) logical rules: consist of traditional (see e.g. [20]) invertible rules (i.e., from the derivability of the conclusion of the corresponding rule (i) it follows the derivability of each premise of the rule (i)) for \(\Box, \land, \lor, \forall, \exists\) (since we consider only skolemized \(F\)-sequents, the calculus \(G_{L\omega}\) does not contain the rules \((\exists \rightarrow), (\rightarrow \forall)\).

A calculus \(G\) is obtained from \(G_{L\omega}\) by dropping the rule \((\rightarrow \Box)\).

Analogously as in [22] we get the following

**Theorem 2 (soundness and completeness of \(G_{L\omega}\))** Let \(S\) be an \(F\)-sequent, then \(\forall M \vdash S\) iff \(G_{L\omega} \vdash S\).

Now we shall specify the parametrical and the \(\Box\)-parts of \(F\)-sequents and drop the \(Q\)-parts of \(F\)-sequents obtaining the so-called \(D\)-sequents.

**Definition 6 (\(D\)-sequents, induction-free \(D\)-sequents, kernel of \(D\)-sequents)** An \(F\)-sequent is a \(D\)-sequent if \(S = \Sigma, \Pi^1, \Box \Omega \rightarrow \Box^0 A\), where \(\Box^0 \in \{\emptyset, \Box\}; \Sigma = \emptyset\) or consists of elementary formulas, \(\Pi^1 = \emptyset\) or consists of atomic formulas of the form \(E^l (l > 0)\) (\(\Sigma, \Pi^1\) is a parametrical part of a \(D\)-sequent); \(\Box \Omega = \emptyset\) or consists of formulas of the form \(\forall x (E^k(x) \supset Q^l(f(x)))\) (called kernel formulas), where \(E^k(x)\) is an atomic formula (called a premise of the kernel formula) without functional symbols; if \(k = 0\), the kernel is called an elementary one; \(Q^l(f(\bar{x}))\) is an atomic formula (called a conclusion of the kernel formula), where \(\bar{f}(x) = f_1(f_2 \ldots (f_n(x)) \ldots), f_i (1 \leq i \leq n)\) is a one-place functional symbol;
A = \exists y_1, \ldots, y_n \wedge_{i=1}^m E_i(y_i) \ (m \leq n), \text{ where } E_i(y_i) \text{ is an atomic formula. If } \Box^0 = \emptyset, \text{ i.e., if } S = \Sigma, \Pi^1, \Box \Omega \rightarrow A, \text{ then } S \text{ is called an induction-free } D\text{-sequent. We assume that all kernel premises (conclusions) consist of atomic formulas with different predicate symbols. A } D\text{-sequent satisfies the following conditions:}

1. If \( E^l(t) \) is a parametrical formula and \( E^k(\bar{f}(x)) \) is the conclusion of a kernel formula, then \( k < l \); if \( E^l(p) \) and \( E^k(t) \) are parametrical formulas, then \( l \neq k \) (parametrical index condition);

2. for any kernel formula \( \Box \forall x(E^k(x) \supset Q^l(\bar{f}(x))) \) it must be \( k < l \) (kernel index condition);

3. all kernel premises (conclusions) consist of atomic formulas having different predicate symbols; for each kernel premise there exists a unique kernel conclusion with the same predicate symbol and vice versa (bounded connectivity condition).

**Remark 2** The parametrical index condition and bounded connectivity conditions are non-essential for the construction of the proposed deductive procedure \( \text{Sat}_\omega \). The restrictions that one-place predicate and function symbols are considered only and that the premise of a kernel formula does not contain function symbols are also non-essential. All these restrictions allow us only to simplify the components of \( \text{Sat}_\omega \). But the kernel index condition is essential for the correctness of the separation rules (ISIF) and (GIS) (see below), which are the main deductive tools of \( \text{Sat}_\omega \). The shape of kernel formulas of \( D\text{-sequents} \) can be generalized. For example, it is no problem to consider kernel formulas of the shape \( \Box \forall x(\bigvee_{j=1}^m E^{k_j}_i(x) \supset \bigwedge_{j=1}^m Q^{l_j}(\bar{f}_j(x))) \) for the kernel formulas when \( n = m = 1 \). The presentation will be in principle the same but more tedious.

From the bounded connectivity condition and the notion of sequent we get the following

**Lemma 1** Let \( S = \Sigma, \Pi^1, \Box \Omega \rightarrow \Box A \) be a \( D\text{-sequent} \), then the kernel \( \Box \Omega \) can be “ordered” in the following way \( \Box \Omega = \Box \forall x_1(E^{k_1}_1(x_1) \supset E^{l_1}_1(\bar{f}_1(x_1))), \Box \forall x_2(E^{k_2}_2(x_2) \supset E^{l_2}_2(\bar{f}_2(x_2))), \ldots, \Box \forall x_n(E^{k_n}_{n-1}(x_n) \supset E^{l_n}_{n-1}(\bar{f}_n(x_n))) \) \( (k_i < l_i, 1 \leq i \leq n) \) and \( E = E_n \).

The shape of kernel formulas of \( D\text{-sequents} \) allows us to incorporate an application of the looping-free rule (\( \forall^* \rightarrow \) i.e., without duplication of the main formula) into the applications of the separation rule (SSIF) (see below). The shape of the succedent formula \( A \) makes it possible to replace the rule (\( \rightarrow \exists \) by the corresponding axiom (\( \exists \)) (see below).

Derivations in the calculus \( G_{L\omega}^T \) are constructed in the bottom-up manner in the form of an infinite tree. The values of variables in the separation rule (ISIF) will be indicated alongside with the premise of the rule in the form of substitutions \( x^* \leftarrow t \), where \( x^* \) is a new variable, \( t \) is a corresponding term. According to that, the axiom (\( \exists \)) of the calculus will be enriched by the corresponding substitution.

The shape of \( D\text{-sequents} \) allows us to specify the axiomatic substitution using the matching methodology (see, e.g., \([3, 28]\)) which is more efficient than the universal unification methodology.

To specify the axiomatic substitution let us introduce the following definitions.

**Definition 7 (solution of the substitution)** Let \( \sigma := \{ x_n \leftarrow \bar{f}_n(x_{n-1}); x_{n-1} \leftarrow \bar{f}_{n-1}(x_{n-2}); \ldots; x_1 \leftarrow \bar{f}_1(x_0) \} \). Then the substitution \( \sigma^* := \{ x_n \leftarrow \bar{f}_n(\bar{f}_{n-1}(\ldots(\bar{f}_1(x_0))\ldots)) \} \) is called the solution of the substitution \( \sigma \).
Definition 8 (superterm of a term) Let \( p = f_1(\ldots(f_i(\ldots f_n(x)) \ldots)) \) (where \( x \) is a constant or a variable) and \( q = f_1(\ldots(f_i(y)) \ldots) \) (\( y \) is a variable) \( (1 \leq i \leq n) \) (in a separate case \( n = 0 \)). Then the term \( p \) is called a superterm of the term \( q \) (in symbols \( p \succ q \)).

Definition 9 (matching terms) Let \( p, q \) be terms, \( \sigma \) be a substitution. We say that the term \( p \) matches the term \( q \) if \( p\sigma \succ q \).

Definition 10 (calculi \( G^*_\omega \), \( G^* \)) The calculus \( G^*_\omega \) is defined by the following postulates.

The axiom \((\exists):\) \( \Gamma, E_i(f(x)) \rightarrow \exists y_1, \ldots, y_m \bigvee_{j=1}^n E_j(f_j(y_j)) \) \((m \leq n, \ m \geq 0, \ 1 \leq i \leq n)\), where \( f(x)^\sigma \succ f_j(y_j) \); \( \sigma^* \) is the solution of \( \sigma_1 \), where \( \sigma_1 \) starts from the substitution \( x^* \leftarrow t \ \sigma_1 \subseteq \sigma, \sigma \) is the list of the substitution obtained during the generation of the axiom (\( \exists \)). The traditional axiom \( \Gamma, E \rightarrow E \) is a special case of the axiom (\( \exists \)) when \( m = 0, n = 1 \), \( E_i(f(x)) = E_j(f_j(y_j)) \).

The rules consist of the \( \omega \)-type rule \((\rightarrow \square_\omega)\) and the following looping-free rule:

\[
\frac{\Sigma, \Pi^1, \Box \Omega, \Box \Omega_1 \rightarrow B_{k-1}}{\Sigma, \Pi, \Box \Omega, \Box \Omega_1 \rightarrow B_k} \quad (ISIF), \quad k > 0,
\]

where \( \Sigma = \emptyset \) or consists of elementary formulas; \( \Pi^1 = \emptyset \) or consists of atomic formulas of the shape \( E^l \) \((l > 0)\); \( \Box \Omega = \emptyset \) or consists of elementary kernel formulas; \( \Box \Omega_1 = \emptyset \) or consists of non-elementary kernel formulas. \( B = \exists y_1, \ldots, y_m \bigvee_{i=1}^n E_i(y_i) \) \((m \leq n)\), \( E_i(y_i) \) is an atomic formula. Let us define the operation \((+)\), applied to any elementary formula \( E(t) \) from \( \Sigma : (E(t))^+ := P^{k-1}(f(x)) \) \( \text{(where} \ x^* \leftarrow t, \ x^* \text{is a new variable)} \), if \( \Box \forall x(E(x) \supset P^k(f(x))) \in \Box \Omega \), otherwise \( (E(t))^+ := \emptyset \). Let \( \Sigma = E_1, \ldots, E_n \), then \((\Sigma)^+ = (E_1)^+, \ldots, (E_n)^+\).

The calculus \( G^* \) is obtained from \( G^*_\omega \) by dropping the \( \omega \)-type rule \((\rightarrow \square_\omega)\).

Lemma 2 The rule \((ISIF)\) is admissible and invertible in the calculus \( G \).

Proof: using [37, 38] and the properties of \( D \)-sequents.

Lemma 3 The calculus \( G^* \) is decidable.

Proof Since the rule \((ISIF)\) is invertible, we can bottom-up apply \((ISIF)\) until we get an induction-free \( D \)-sequent \( S^* = \Gamma \rightarrow B_k \) \((k = 0)\). Since the axiom \((\exists)\) is decidable, we can automatically verify whether the sequent \( S^* \) is the axiom \((\exists)\) or not.

Using Lemma 2 we get the following

Lemma 4 Let \( S \) be a \( D \)-sequent, then \( G \vdash S \iff G^* \vdash S \).

Remark 3 The kernel index condition is essential for the correctness of Lemma 4. Indeed, let \( S = P(c), \Box \forall x(P(x) \supset Q^1(f(x))) \rightarrow \exists y Q^1(f(g(y))) \). For the sequent \( S \) the kernel index condition is destroyed. It is easy to verify that \( G \vdash S \) (using the looping rules \((\Box \rightarrow), (\forall \rightarrow)) \) but \( G^* \not\vdash S \).
3. Description of the deductive procedure \( Sat_\omega \)

The presented deductive procedure \( Sat_\omega \) consists of four separate decidable deductive procedures with some constraints (i.e., with some stopping instruments).

Let us define the generalized integrated separation rule (GIS) which is the main tool of the proposed deductive procedure \( Sat_\omega \) and which is applied to any non-induction-free \( D \)-sequent.

**Definition 11 (generalized integrated separation rule: (GIS), successful application of (GIS))** Let \( S = \Sigma, \Pi^1, \Box \Omega, \Box \Omega_1 \rightarrow \Box A \) be a \( D \)-sequent. Let \((\Sigma)^+\) mean the same as in the definition of (ISIF) (see Definition 10), then the generalized integrated separation rule (GIS) is as follows:

\[
\frac{\Sigma, \Pi^1, \Box \Omega, \Box \Omega_1 \rightarrow B; (\Sigma)^+, \Pi, \Box \Omega, \Box \Omega_1 \rightarrow \Box B}{\Sigma, \Pi^1, \Box \Omega, \Box \Omega_1 \rightarrow \Box B} \quad (GIS).
\]

If the left premise of (GIS), i.e., the sequent \( S \) is such that \( G^* \vdash S_1 \), we say that bottom-up application of (GIS) is successful.

**Remark 4** The rule (GIS) incorporates the rule (ISIF) and the following rules

\[
\frac{\Gamma \rightarrow A; \Gamma \rightarrow \Box A^1}{\Gamma \rightarrow \Box A} \quad (\rightarrow \Box^1) \quad \frac{\Pi \rightarrow \Box A}{\Sigma, \Pi^1 \rightarrow \Box A^1} \quad (+1),
\]

which are admissible in \( G^{*}_{L_\omega} \).

Using the rules \((\rightarrow \Box^1), (+1)\), we get the following

**Lemma 5** The rule (GIS) is admissible and invertible in \( G^{*}_{L_\omega} \).

To define the first deductive procedure of \( Sat_\omega \), let us define the kernel premise complexity of \( D \)-sequent \( S : \pi(S) \), which serves as a halting test for the first deductive procedure of \( Sat_\omega \).

**Definition 12 (kernel premise complexity of \( D \)-sequent \( S : \pi(S) \), elementary \( D \)-sequent)** Let \( S = \Sigma, \Pi^1, \Box \Omega, \Box \Omega_1 \rightarrow \Box B \) be a \( D \)-sequent, where \( \Box \Omega \) (\( \Box \Omega_1 \)) consists of elementary (non-elementary, respectively) kernel formulas. Let \( P_1^{k_1}, \ldots, P_n^{k_n} \) be the list of all the kernel premises. Then the kernel premise complexity of \( D \)-sequent \( S \) (in notation: \( \pi(S) \)) is defined as \( \max(k_1, \ldots, k_n) \). For example, let \( S = P^1(c), \forall x(P^1(x) \supset Q^2(f(x))), \forall y(Q^2(y) \supset P^4(g(y))) \rightarrow \Box \exists z R(z), \) then \( \pi(S) = 2 \). If \( \pi(S) = 0 \), then \( S \) is an elementary \( D \)-sequent.

Let us define now the first deductive procedure of \( Sat_\omega \), named the first preliminary \( k \)-th resolvent (denoted by \( P_1 Re^k(S) \)). The aim of \( P_1 Re^k(S) \) is to generate (from a given \( D \)-sequent \( S \)) the elementary \( D \)-sequent \( S^* \).

**Definition 13 (first preliminary \( k \)-th resolvent: \( P_1 Re^k(S) \))** Let \( S \) be a \( D \)-sequent, then the first preliminary \( k \)-th resolvent of the \( D \)-sequent \( S \) (in symbols: \( P_1 Re^k(S) \)) is defined in the following way: \( Re^0(S) = S \). Let \( P_1 Re^k(S) = S_k = \Sigma, \Pi^1, \Box \Omega, \Box \Omega_1 \rightarrow \Box B, \) then \( Re^{k+1}(S) \) is defined in the following way.

1. Let us bottom-up apply the rule (GIS) to \( S_k \) and \( S_{k1}, S_{k2} \) be the left and right premises of the application of (GIS).
2. If \( G^* \not\vdash S_{k1} \), then \( P_1 Re^{k+1}(S) = \bot \) (false) and the calculation of \( P_1 Re^{k+1}(S) \) is stopped.
3. Let \( G^* \vdash S_{k1} \) (i.e., the bottom-up application of (GIS) is successful), then \( P_1 Re^{k+1}(S) = S_{k2} \).
4. If \( P_1 Re^{k+1}(S) = S_{k2} \) and \( k + 1 = \pi(S) \), then the calculation of \( P_1 Re^{k+1}(S) \) is finished.
Lemma 6 For a given $D$-sequent $S$ the problem of generating the elementary sequent $S^*$ is decidable.

Example 1 Let $S = E_2^2(c_0), E_1^1(c_1), Q_1^1(c_2), R_1^1(c_3), □ \Omega \rightarrow □ A$, where $□ \Omega = □ \forall x_1(E_1^1(x_1) \supset Q_3^3(f(x_1)), □ \forall x_2(Q_1^1(x_2) \supset R_1^1(g(x_2))), □ \forall x_3(R_1^1(x_3) \supset E_3^3(h(x_3))); A = \exists y(E(y) \lor Q(y) \lor Q_1^1(y) \lor R(y) \lor R_1^1(y))$. Let us construct an elementary $D$-sequent $S^*$ using the procedure $P_1Re^k(S)$. Since $\pi(S) = 1$, $k = 1$. By definition $P_0^0Re^0(S) = S$. Let us construct $P_1Re^1(S)$. Since the sequent $S_{11} = E_2^2(c_0), E_1^1(c_1), Q_1^1(c_2), R_1^1(c_3), □ \Omega \rightarrow A$ is the axiom $(\exists), G^* \vdash S_{11}$. Therefore, by definition, $P_1Re^1(S) = S^* = E_1^1(c_0), E_1^1(c_1), Q_1^1(c_2), R_1^1(c_3), □ \Omega \rightarrow □ A$, which is the elementary $D$-sequent.

To define the second preliminary procedure of $Sat_\omega$ (denoted by $P_2Re^k(S)$), let us define the following notions.

Definition 14 (rank of elementary $D$-sequent: $r(S)$, non-saturated parametrical formula) Let $S = \Sigma, \Pi^1, □ \Omega \rightarrow □ B$ be an elementary $D$-sequent and let $Q_1^1(f(x))$ be any atomic formula from $\Sigma, \Pi^1$. Let us define the rank of $Q_1^1(f(x))$ in symbols: $r(Q_1^1(f(x))) = 0$, if $□ \forall x(P(x) \supset Q_1^1(f(x))) \in □ \Omega (0 \leq l < k)$, otherwise $r(Q_1^1(f(x))) = l$. Let $E \in \Sigma, \Pi^1$ and $r(E) > 0$, then $E$ is a non-saturated parametrical formula. Let $S = E_1, \ldots, E_n, □ \Omega \rightarrow □ B$, then $r(S) = \sum_{i=1}^n r(E_i)$.

Definition 15 (saturated $D$-sequent) Let $S$ be an elementary $D$-sequent, then $S$ is a saturated $D$-sequent, if $r(S) = 0$.

The aim of the second preliminary $k$-th resolvent $P_2Re^k(S^*)$ is to generate (from the constructed by $P_1Re^k(S)$ elementary $D$-sequent $S^*$) a saturated $D$-sequent $S^{**}$. Let us define the halting test for $P_2Re^k(S)$.

Definition 16 (saturation index: $l(S)$) Let $S = \Sigma, \Pi^1, □ \Omega \rightarrow □ B$ be an elementary $D$-sequent, $n$ be the maximal index of a non-saturated parametrical formula from $\Sigma, \Pi^1$. Then $l(S) = n + 1$ is the saturated index of $S$. For example, let $S^*$ be the elementary sequent obtained in Example 1, then $l(S^*) = 2$.

Definition 17 (second preliminary $k$-th resolvent: $P_2Re^k(S^*)$) Let $S^*$ be an elementary $D$-sequent. Then the definition of $P_2Re^k(S^*)$ is obtained from the definition of $P_1Re^k(S)$ (see Definition 13) replacing $P_1Re^k(S)$ by $P_2Re^k(S^*)$ and replacing the point (4) by a new point (4): If $P_2Re^{k+1}(S^*) = S_{k2}$ and $k + 1 = l(S^*)$, then the calculation of $P_2Re^{k+1}(S^*)$ is completed.

Due to the decidability of the calculus $G^*$ we get the following

Lemma 7 For an elementary $D$-sequent $S^*$ the problem of generating a saturated sequent $S^{**}$ is decidable.
Example 2 Let $S^*$ be the elementary D-sequent obtained in Example 1. Let us construct a saturated D-sequent $S^{**}$. Since $l(S^*) = 1+1 = 2$, the calculation of $P_2R_e^k(S^*)$ is stopped when $k = 2$. By definition $P_2R_e^0(S^*) = S^*$. Since the sequent $S_{11} = E_1(c_0), E(c_1), Q(c_2), R(c_3)$, $\Box \Omega \rightarrow A$ is the axiom $(\exists)$, $P_2R_e^1(S^*) = E_1(c_0), Q^1(f(x_1^1)), R^2(g(x_2^2)), E^3(h(x_3^3)), \Box \Omega \rightarrow \Box A$, where $x_1^1 \leftarrow c_1, x_2^2 \leftarrow c_2, x_3^3 \leftarrow c_3$. Since the sequent $E_1(c_0), Q^1(f(x_1^1)), R^2(g(x_2^2)), E^3(h(x_3^3)), \Box \Omega \rightarrow A$ is the axiom $(\exists)$, $P_2R_e^2(S^*) = S^{**} = Q^1(f(x_1^1)), R^1(g(x_2^2)), E^2(h(x_3^3)), \Box \Omega \rightarrow \Box A$, which is the saturated D-sequent.

Now we are going to define the basic part of Sat,ω – the saturated $k$-th resolvent (in symbols: $SRe^k(S^{**})$). The aim of $SRe^k(S^{**})$ is to generate (from the obtained by means of $P_2R_e^k(S^*)$, saturated D-sequent $S^{**}$) a similarity substitution $\sigma$ and a saturated D-sequent $S_\rho$ such that $S_\rho = S^{**}\sigma$, i.e., $S_\rho$ differs from $S^{**}$ only by the values of the variables which are determined by the substitution $\sigma$. To define $SRe^k(S)$, let us define the halting test for $SRe^k(S^{**})$, namely, the similarity index.

Definition 18 (similarity index) Let $S = \Sigma, \Pi^1, \Box \Omega \rightarrow \Box B$ be a saturated D-sequent and $p_1, \ldots, p_n$ indices from $\Omega$, then $p(S) = \sum_{i=1}^{n} p_i$ is the similarity index of $S$. For example, let $S^{**}$ be the saturated D-sequent obtained in Example 2, then $p(S^{**}) = 2 + 3 + 4 = 9$.

Definition 19 (saturated $k$-th resolvent: $SRe^k(S^{**})$) Let $S^{**}$ be a saturated D-sequent. Then the definition of $SRe^k(S^{**})$ is obtained from the definition of $P_2R_e^k(S^*)$ (see Definition 17) replacing $P_2R_e^k(S^*)$ by $SRe^k(S^{**})$ and point (4) by new point (4): If $SRe^{k+1}(S^{**}) = S_{k2}$ and $k + 1 = p(S^{**})$, then the calculation of $SRe^{k+1}(S^{**})$ is finished.

The notation $SRe^k(S) \neq \perp \ (k \in \omega)$ means that all the possible bottom-up applications of (GIS) in constructing $SRe^k(S)$ are successful.

Lemma 8 (composition of $SRe^k(S)$) Let $SRe^n(S) = S_n, SRe^m(S_n) = S^*$ and $SRe^n(S) \neq \perp, SRe^m(S_n) \neq \perp \ (n, m \in \omega)$, then $SRe^l(S) = S^*$, where $l = n + m$.

Proof by induction on $l$.

Lemma 9 (decomposition of $SRe^k(S)$) Let $SRe^{n+m}(S) = S^*$ and $SRe^{n+m}(S) \neq \perp$, then for each $n$ and $m$ there exists a sequent $S_n$ such that $SRe^n(S) = S_n$, and $SRe^m(S_n) = S^*$.

Proof by induction on $n + m$.

Lemma 10 (“length-preserving” of $SRe^k(S)$) Let $S$ be a saturated D-sequent, i.e., $S = \Sigma, \Pi^1, \Box \Omega \rightarrow \Box A$ and $SRe^k(S) = S^*$, then $S^* = \Sigma_i, \Pi_1^i, \Box \Omega \rightarrow \Box A$, where $|\Sigma, \Pi^1| = |\Sigma_i, \Pi_1^i|$, i.e., the lengths of parametrical parts of $S$ and $S^*$ are the same.

Proof by induction on $k$. If $k = 0$, then $S = S^*$. Let $k \geq 1$ and let us consider the construction of $SRe^1(S) = S_1$. Let us consider an atomic formula $E^l$ from the parametrical part of $S$. Assume $E^l \in \Pi^l$ (where $l > 0$), then, by the definition of (GIS), we have that the descendent of the atomic formula $E^l$ in $S_1$ is of the shape $E^{l-1}$. Let $E^l = E_i(\bar{f}_i(x_1^i)) \in \Sigma$. Since $S$ is the saturated D-sequent, we have $\Box \forall (P(x) \supset E^m(\bar{f}_i(x))) \in \Box \Omega$. By the bounded connectivity condition of D-sequents $\Box \forall y (E_i(y) \supset Q^p(g(y))) \in \Omega$. Therefore, by definition of the operation $(\perp)$ (see Definition 10), the descendent of the elementary formula $E_i(\bar{f}_i(x_1^i))$ in $S_1$ is of the
shape $Q^{−1}(g(y^{∗}))$ (where $y^{∗} ← f_{i}(x_{i}^{∗}))$. Hence $SRe^{1}(S) = S_{1} = \Sigma_{11}, \Pi_{11}^{1}, □Ω → □A$ and
$|Σ_{11}, \Pi_{11}^{1}| = |Σ, Π^{1}| (∗).$ If $k = 1$, then $S^{∗} = S_{1}$. Let $k > 1$, then by Lemma 9 we have
that $SRe^{k−1}(S_{1}) = S^{∗}$. Using induction hypothesis, we have that $S^{∗} = \Sigma_{11}, \Pi_{11}, □Ω → □A$ and
$|Σ_{11}, \Pi_{11}^{1}| = |Σ, Π^{1}| (∗).$ Using Lemma 8 and (∗), (**), we get $SRe^{k}(S) = \Sigma_{11}, \Pi_{11}, □Ω → □A$
and $|Σ_{11}, \Pi_{11}| = |Σ, Π^{1}| (∗).$

Lemma 11 (accessibility of a kernel premise) Let $S = △, □Ω → □A$ be a saturated D-sequent and $Δ$ be the parametrical part of $S$, □Ω = □∀x_{1}(E(x_{1}) ⊃ E_{1}^{k}(f_{1}(x_{1}))), . . . , □∀x_{i+1}(E_{i}(x_{i+1}) ⊃ E_{i+1}^{k_{i}}(f_{i+1}(x_{i+1}))), □∀x_{i+2}(E_{i+1}(x_{i+2}) ⊃ E_{i+2}^{k_{i+2}}(f_{i+2}(x_{i+2}))), . . . , □∀x_{n−1}(E_{n−1}(x_{n−1}) ⊃ E_{n−1}^{k_{n−1}}(f_{n−1}(x_{n−1}))), (where $E_{i} = E; \text{let } E_{1}^{k}(f_{1}(x_{1}^{∗})) ∈ Δ \text{ (i.e., } Δ = E_{1}^{k}(f_{1}(x_{1}^{∗})), Δ_{1} \text{ and } SRe^{k}(S) ≠ \perp (k ∈ Ω).$ Then $SRe^{k}(S) = E_{i+m−1}^{k+i}(f_{i+m}(x_{i+m})), Δ_{i}, □Ω → □A$, where $q = l + 1, x_{i+m} ← f_{i}(x_{i}^{∗}), \text{ where } \overline{g_{i}}(x_{i}^{∗}) = g_{i}((g_{i}(x_{i}^{∗}) . . . ))$, $g_{ir} = f_{jr} (1 ≤ r ≤ n)$ and $f_{jr} \in f_{j} (1 ≤ j ≤ n)$.

Proof: by induction on $m$. The case where $m = 0$ is trivial. Let $m = 1$, then $q = l + 1$. Since
$SRe^{k}(S) ≠ \perp (k ∈ Ω)$, applying (G1S) $q = l + 1$ (l-time for “dropping” the index $l$ from $E_{1}^{k}(x_{1}^{∗})$, and once more for getting $E_{i+1}^{k_{i+1}}(f_{i+1}(x_{i+1})))$, we obtain $SRe^{k}(S) = S_{1} = E_{i+1}^{k_{i+1}}(f_{i+1}(x_{i+1})), Δ_{1}, □Ω → □A$ (where $q = l + 1, x_{i+1} ← f_{i}(x_{i}^{∗})$ (∗). Let $m = 1$. Since $SRe^{k}(S) = \perp (k ∈ Ω), SRe^{k}(S) ≠ \perp (k ∈ Ω)$, too. Applying induction hypothesis to the sequent $S_{1}$, we get $SRe^{k}(S_{1}) = E_{i+1}^{k_{i}}(f_{i+1}(x_{i+1})), Δ_{i}, □Ω → □A$, where $q = k_{i+1} + 1 . . . + k_{i+m} + 1 (x_{i+m} ← h_{i}(x_{i+1})), h_{i}(x_{i+1}) = h_{i}((h_{i}(x_{i+1}) . . . ))$, $h_{ir} = f_{jr}, f_{jr} \in f_{j} (1 ≤ j ≤ m)$ (∗∗). Applying Lemma 8 to (∗), (∗∗), we get $SRe^{k}(S) = E_{i+m−1}^{k+i}(f_{i+m}(x_{i+m})), Δ_{i}, □Ω → □A$, where $q = l + 1 + q = l + k_{i+1} . . . + k_{i+m} + 1, x_{i+m} ← h_{i}(f_{i}(x_{i}^{∗}))$ and $\overline{g_{i}}(x_{i}^{∗}) = g_{i}((g_{i}(x_{i}^{∗}) . . . ))$, $g_{ir} = f_{jr} (1 ≤ r ≤ n)$ and $f_{jr} \in f_{j} (1 ≤ j ≤ n)$.

Lemma 12 (generating a σ-similar sequent) Let $S^{∗∗}$ be a saturated D-sequent, $SRe^{k}(S^{∗∗}) ≠ \perp (k ∈ Ω)$, $p = p(S^{∗∗})$ be the similarity index of $S^{∗∗}$, then $SRe^{p}(S^{∗}) = S_{p}$ such that $S^{∗}σ = S_{p}$, where $σ$ is called a similarity substitution and the sequent $S_{p}$ is called σ-similar to $S^{∗∗}$.

Proof Let $S^{∗∗} = △, □Ω → □A$ and $E_{i}(x_{i}^{∗})$ be any member of $Δ$. Without loss of generality we can assume that □Ω = □∀x_{1}(E(x_{1}) ⊃ E_{1}^{k}(f_{1}(x_{1}))), . . . , □∀x_{n}(E_{n−1}(x_{n}) ⊃ E_{n}^{k}(f_{n}(x_{n}))), where $E_{n} = E$. Applying the Lemma 11 to the sequent $S^{∗∗}$ (setting $E_{i} = E$ and $i = m$), we get
$SRe^{p}(S^{∗∗}) = S_{q} = E_{n}^{k}(f_{n}(x_{n})), Δ_{1}, □Ω → □A$ (where $E_{n} = E, q = l + k_{1} . . . + k_{n−1} + 1$ and $x_{n} ← h_{1}(x_{1}), h_{1}(x_{1}) = h_{1}((h_{1}(x_{1}) . . . ))$, $h_{ir} = f_{jr} (1 ≤ r ≤ n)$ (∗). Now let us reduce $E_{n}^{k}(f_{n}(x_{n}))$ up to $E^{l}$, i.e., let us apply (G1S) $k_{n−1}−l$-time (and having in mind that $E_{n} = E$) we get $SRe^{k}(Q_{q}) = E^{l}(f_{n}(x_{n})), Δ_{n}, □Ω → □A$, where $q = k_{n−1} − 1$ (∗∗). Applying Lemma 8 to (∗), (∗∗), we get $SRe^{p}(S^{∗∗}) = S_{p} = E^{l}(f_{n}(x_{n})), Δ_{n}, □Ω → □A$, where $p = q + p = k_{i+1} . . . + k_{i+m} + 1, x_{i+m} ← h_{i}(f_{i}(x_{i}^{∗}))$ and $\overline{g_{i}}(x_{i}^{∗}) = g_{i}((g_{i}(x_{i}^{∗}) . . . ))$, $g_{ir} = f_{jr} (1 ≤ r ≤ n)$ and $f_{jr} \in f_{j} (1 ≤ j ≤ n)$ (∗∗∗). Adding the equalities $x_{i+m} ← h_{i}(f_{i}(x_{i}^{∗}))$ (to (∗∗∗)), we get a component of the similarity substitution: $x_{i} ← \overline{g_{i}}(x_{i}), \overline{g_{i}}(x_{i}) = h_{i}(f_{i}(x_{i}^{∗}))$. Since $E^{l}(x_{i}^{∗})$ is an arbitrary member of $Δ$, the sequents $S^{∗∗}$ and $S_{p}$ differ only in the substitution $σ$: $x_{i} ← \overline{g_{i}}(x_{i}), . . . , x_{n} ← \overline{g_{n}}(x_{n}).$ Hence, after $p = p(S^{∗∗})$ steps we get a sequent $S_{p}$ such that $S^{∗∗}σ = S_{p}$. The lemma is proved.

The proof of Lemma 12 presents an implicit way for constructing the similarity substitution $σ$. Now we present the explicit way for constructing the similarity substitution $σ$.

Algorithm (SS) (algorithm for constructing the similarity substitution).
Let $S^{**} = E_1(x_1^*), \ldots, E_r(x_r^*), \Box \Omega \rightarrow \Box A$ be a saturated $D$-sequent and let $p(S^{**}) = k$, and $SRe^k(S^{**}) = S_n = E_1(x_{n,1}), \ldots, E_n(x_{n,r}), \Box \Omega \rightarrow \Box A$. Let $\sigma_1$ be a sequence of substitutions obtained during the construction of $SRe^k(S^{**}) = S_n$, i.e., $\sigma_1 = x_{11} \leftarrow t_{11}(x_1^*), \ldots, x_{r1} \leftarrow t_{r1}(x_r^*); \ldots; x_{1r-1} \leftarrow t_{1r-1}(x_{1,r-2}); \ldots; x_{r,r-1} \leftarrow t_{r,r-1}(x_{r,n-2}); x_{1r} \leftarrow t_{1r}(x_{1,r-1}); \ldots; x_{rr} \leftarrow t_{rr}(x_{r,r-1})$. Let us, subsequently, eliminate from $\sigma_1$ the intermediate variables $x_{i1}, \ldots, x_{ir-1}(1 \leq i \leq r)$, i.e., replace these variables by the corresponding values. Continue these transformations until a sequence $\sigma_n$ containing $r$ substitutions of the shape $x_{ni} \leftarrow t_{ni}(\ldots(t_{i1}(x_1^*))\ldots)(1 \leq i \leq r)$ is obtained. Add the equalities of the shape $x_{ni} = x_i^*$ $(1 \leq i \leq r)$ to $\sigma_n$. Then, the desired similarity substitution $\sigma$ has the shape $\sigma := \{x_1^* \leftarrow t_{11}(\ldots(t_{i1}(x_1^*))\ldots); \ldots; x_r^* \leftarrow t_{rr}(\ldots(t_{1r}(x_r^*))\ldots)\}$.

Example 3 Let $S^{**}$ be the same saturated $D$-sequent as in Example 2, i.e., $S^{**} = Q(f(x_1^*)), R^1(g(x_2^*)), E^2(h(x_3^*))$, $\Box \Omega \rightarrow \Box A$, where $\Box \Omega = \Box \forall x_1(E(x_1) \supset Q^2(f(x_1))), \Box \forall x_2(Q(x_2) \supset R^3(g(x_2))), \Box \forall x_3(R(x_3) \supset E^4(h(x_3))); A = \exists y(E(y) \lor Q^2(y) \lor Q^3(y) \lor R^4(y)).$ Therefore, the similarity index $p(S^{**}) = 2 + 3 + 4 = 9$, and the construction of $SRe^k(S^{**})$ stops when $k = 9$. The calculation of $SRe^9(S^{**})$ yields a sequent $S_9 = Q(f(x_{13})), R^1(g(x_{23})), E^2(h(x_{33})), S_0$ (where $S_0 = \Box \Omega \rightarrow \Box A$), such that $S^{**} \sigma = S_9$. The construction of $SRe^9(S^{**})$ is as follows (it is easy to verify that all the applications of the rule (GIS) are successful, therefore we indicate only a temporal premise of the rule (GIS) and substitutions generated by means of the operation (+) (see Definition 10):

$SRe^9(S) = Q(f(x_{13})), R^1(g(x_{23})), E^2(h(x_{33})), S_0$,

$x_{13} \leftarrow h(x_{12})$

$x_{23} \leftarrow f(x_{22})$

$SRe^8(S) = Q^1(f(x_{13})), R^2(g(x_{23})), E^3(h(x_{33})), S_0; x_{33} \leftarrow g(x_{32})$

$SRe^7(S) = E(h(x_{12})), Q(f(x_{22})), R(g(x_{32})), S_0$

$SRe^6(S) = E^1(h(x_{12})), Q^1(f(x_{22})), R^1(g(x_{32})), S_0; x_{22} \leftarrow h(x_{21})$

$SRe^5(S) = E^2(h(x_{12})), E(h(x_{21})), R^2(g(x_{32})), S_0; x_{32} \leftarrow f(x_{31})$

$SRe^4(S) = E^3(h(x_{12})), E^1(h(x_{21})), Q(f(x_{31})), S_0; x_{12} \leftarrow g(x_{11})$

$SRe^3(S) = R(g(x_{11})), E^2(h(x_{21})), Q^1(f(x_{31})), S_0; x_{31} \leftarrow h(x_{3})$

$SRe^2(S) = R^1(g(x_{11})), E^3(h(x_{21})), E(h(x_{3})), S_0; x_{21} \leftarrow g(x_{2})$

$SRe^1(S) = R^2(g(x_{11})), R(g(x_{3})), E^1(h(x)), S_0; x_{11} \leftarrow f(x_{1})$

$SRe^0(S) = S^{**} = Q(f(x_{1}^*)), R^1(g(x_{2}^*)), E^2(h(x_{3}^*)), S_0$

Let us now construct a similarity substitution $\sigma$. First, we construct a substitution each component of which is of the shape $x_{i3} \leftarrow \bar{\tau}(x_i^*)$ ($\bar{\tau}$ is a sequence of functional symbols $f, g, h$, $1 \leq i \leq 3$), i.e., expresses the relation between the variables $x_{i3}$ and $x_i^*$. Let $\sigma_1$ be a sequence of the substitutions obtained during the construction of $SRe^9(S^{**}) = S_9$. i.e., $\sigma_1 = x_{11} \leftarrow f(x_1^*); x_{21} \leftarrow g(x_2^*); x_{31} \leftarrow h(x_3^*); x_{12} \leftarrow f(x_{31}); x_{22} \leftarrow h(x_{21}); x_{13} \leftarrow h(x_{12}); x_{23} \leftarrow f(x_{22}); x_{33} \leftarrow g(x_{32})$. Let us eliminate the intermediate variables $x_{i1}$ and
First, let us eliminate the variables \(x_{i_2}(1 \leq i \leq 3)\), i.e., replace the variables \(x_{i_2}\) by the corresponding values of these variables. So, instead of the sequence \(\sigma_1\) we get \(\sigma_2 = x_{i_1} \leftarrow f(x_{i_1}^1); x_{i_1} \leftarrow g(x_{i_1}^2); x_{i_1} \leftarrow h(g(x_{i_1})); x_{i_1} \leftarrow f(h(x_{i_1})); x_{i_3} \leftarrow g(f(x_{i_3})).\) In the same manner, let us eliminate the variables \(x_{i_1}(1 \leq i \leq 3)\). So, instead of the sequence \(\sigma_2\) we get \(\sigma_3 = x_{i_3} \leftarrow h(g(f(x_{i_3}^1))); x_{i_3} \leftarrow f(h(g(x_{i_3}^2))); x_{i_3} \leftarrow g(f(h(x_{i_3}^3))).\) Now, by adding \(x_{i_1}^* = x_{i_3}(1 \leq i \leq 3)\) to \(\sigma_3\) we get the desired similarity substitution \(\sigma := \{x_{i_1}^* \leftarrow h(g(f(x_{i_1}^1))); x_{i_2}^* \leftarrow f(h(g(x_{i_2}^2))); x_{i_3}^* \leftarrow g(f(h(x_{i_3}^3)))\}.\) Thus we get \(S^*\sigma = S_9.\)

**Lemma 13** For a saturated \(D\)-sequent \(S^*\) the problem of generating a \(\sigma\)-similar \(S^*\sigma\) is decidable.

**Proof** follows from Lemma 12 and Algorithm (SS).

Therefore, using the saturated sequent \(S^*\), the similarity substitution \(\sigma\) (which is generated by means of calculation of \(SRe^k(S^*)\) and Algorithm (SS), we can construct a sequent \(S^*\sigma\). Let us construct a substitution \(\sigma^n\) in the following way. Let \(\sigma = \{x_{i_1}^* \leftarrow \bar{f}_1(x_{i_1}^*), \ldots, x_{i_m}^* \leftarrow \bar{f}_m(x_{i_m}^*)\}\), then \(\sigma^n = \{x_{i_1}^* \leftarrow \bar{f}_1(x_{i_1}^*), \ldots, x_{i_m}^* \leftarrow \bar{f}_m(x_{i_m}^*)\}\), where \(\bar{f}_i(x_{i_1}^*) = \emptyset; \bar{f}_i(x_{i_2}^*) = \bar{f}_i(f_i^{n-1}(x_{i_2}^*)\}\), therefore \(\sigma^0 = \emptyset\) and \(\sigma^1 = \sigma\). For example, if \(\sigma = e^* \leftarrow f(g(e^*))\), then \(\sigma^2 = e^* \leftarrow f(g(f(g(e^*)))\).

We want to use the calculation of saturated \(k\)-th resolvants to obtain not only the sequent \(S^*\sigma\) but also sequents \(S^*\sigma^n\) (for each \(n \in \omega\)). The foundation of this possibility is carried out by induction on \(n\). The basis case (i.e., when \(n = 1\)) is performed out by calculating \(SRe^1(S^*)\) and by means of Algorithm (SS). To found the step of induction, let us introduce the notion of a hypothetical \(k\)-th resolvent of the saturated \(D\)-sequent \(S^*\): \(HRe^k(S^*)\). The deductive procedure \(HRe^k(S^*)\) is the fourth part of \(Sat_\omega\). The aim of \(HRe^k(S^*)\) is as follows: assuming that we can generate the saturated \(D\)-sequent \(S^*\sigma^n\) we must verify the possibility of generating the saturated \(D\)-sequent \(S^*\sigma^{n+1}\). The termination test of \(HRe^k(S^*)\) is the same as for \(SRe^k(S^*)\), namely, the similarity index of the saturated \(D\)-sequent \(S^*\).

**Definition 20 (hypothetical \(k\)-th resolvent: \(HRe^k(S)\))** Let \(S\) be a saturated \(D\)-sequent, \(\sigma\) – the similarity substitution, \(m\) be an arbitrary natural number, then \(HRe^0(S) = S\sigma^m\). Let \(HRe^k(S) = S_k\), then \(HRe^{k+1}(S)\) is defined in the following way. 1. Let us bottom-up apply \((GIS)\) to \(S_k\) and \(S_{k_1}, S_{k_2}\) be the left and the right premise of the application of \((GIS)\). 2. If \(G* \not\vdash S_{k_1}\), then \(HRe^{k+1}(S) = \perp\) (false) and the calculation of \(HRe^{k+1}(S)\) is stopped. 3. Let \(G* \vdash S_{k_1}\), then \(HRe^{k+1}(S) = S_{k_2}\). 4. If \(HRe^{k+1}(S) = S_{k_2}\) and \(k+1 = p(S)\), then the calculation of \(HRe^{k+1}(S)\) is finished.

In the same way as in Lemma 13 we get the following

**Lemma 14** Let \(HRe^0(S) = S\sigma^m\), then the problem of generation of the \(\sigma\)-similar sequent \(S\sigma^{m+1}\) is decidable.

Now we can define the proposed deductive procedure \(Sat_\omega\).

**Definition 21 (deductive procedure \(Sat_\omega\), \(D\)-sequent derivable by means of \(Sat_\omega\))** The deductive procedure \(Sat_\omega\) consists of four decidable procedures: (1) \(P_1Re^k(S)\) (where \(S\) is a given \(D\)-sequent); (2) \(P_2Re^k(S^*)\) (where \(S^*\) is the elementary \(D\)-sequent obtained by means of \(P_1Re^k(S)\)); (3) \(SRe^k(S^{**})\) (where \(S^{**}\) is the saturated \(D\)-sequent obtained by means of \(P_2Re^k(S^*)\)); (4) \(HRe^k(S^{**})\). \(D\)-sequent \(S\) is derivable by the help of \(Sat_\omega\) (in symbols: \(Sat_\omega \vdash S\)) if the four conditions are satisfied: (1) \(P_1Re^n(S) = S^*\) (where \(n = \pi(S)\), \(\pi(S)\) is the kernel
It is easy to verify that it is possible to reduce the sequent reduced by (GIS) for each $m \in \omega$. Thus $Sat_\omega \vdash S$.

Remark 5 The procedures $SRe^k(S)$ and $HRe^k(S)$ of $Sat_\omega$ cannot be connected: we can not deduce the sequent $S^{**}\sigma^n$. We can only test that from the assumption $S^{**}\sigma^n$ it is possible (or not) to generate the sequent $S^{**}\sigma^{n+1}$.

Having in mind the definition of $\omega$-decidability (see the and of Introduction), using Remark 5 and Lemmas 6, 7, 13, 14 we get the following

Lemma 15 The deductive procedure $Sat_\omega$ is $\omega$-decidable.

Example 4 (a) Let $S$ be the $D$-sequent from Example 1. In Example 1, the sequent $S$ was reduced by $P_1Re(S)$ to an elementary $D$-sequent $S^*$. In Example 2, the elementary $D$-sequent $S^*$ was reduced by $P_2Re^k(S)$ to the saturated $D$-sequent $S^{**}$. In Example 3, the similarity substitution $\sigma$ was constructed and the $\sigma$-similar saturated sequent $S^{**}\sigma$ generated. Now, using $HRe^k(S)$ let us verify the possibility to generate $S^{**}\sigma^{m+1}$ from $S^{**}\sigma^m$. So, assume that $HRe^0(S^{**}) = S^{**}\sigma^m$. Analogically as in Example 3, we get that $HRe^0(S^{**}) = S^{**}\sigma^{m+1}$. Therefore, we can generate the sequents $S^{**}\sigma^m$ for each $m \in \omega$. Thus $Sat_\omega \vdash S$.

(b) Let $S = E(c), \Box \Omega \rightarrow \Box A$, where $\Omega = \forall x(E(x) \supset E^1(f(x)))$; $A = (E(c) \lor E(f(c)))$. It is easy to verify that it is possible to reduce the sequent $S$ to the saturated $D$-sequent $S^{**} = E(f(x))$, $\Box \Omega \rightarrow \Box A (x^* \leftarrow c)$. Also, it is possible to reduce $S^{**}$ to the $\sigma$-similar sequent $SRe^1(S^{**}) = E(f(x_1))$, $\Box \Omega \rightarrow \Box A \text{ (where } x_1 \leftarrow f(x))$ and to construct the similarity substitution $\sigma := x^* \leftarrow f(x^*)$. Let us try to construct $HRe^k(S^{**})$. Let $HRe^k(S^{**}) = S^{**}\sigma^m$. Since $G^* \not\vdash E(f(x^*))$, $\Box \Omega \rightarrow A (x^* \leftarrow f^m(x^*))$, $HRe^1(S^{**}) = \bot$. Therefore $Sat_\omega \not\vdash S$.

Lemma 16 Let $S$ be a $D$-sequent and $Sat_\omega \vdash S = \Sigma, \Pi^1, \Box \Omega \rightarrow \Box A$, then $G^*_{L_\omega} \vdash S$.

Proof From $Sat_\omega \vdash S$ it follows that all the possible bottom-up applications of (GIS) are successful in all four parts of $Sat_\omega$. From this fact, using Lemma 5 and Remark 4, by induction on $n$ we can prove that $G^* \vdash S_n = \Sigma, \Pi^1, \Box \Omega \rightarrow A \ (n \in \omega)$. Applying $(\rightarrow \Box_\omega)$ to $S_n$ we get $G^*_{L_\omega} \vdash S$.

Lemma 17 Let $S$ be a $D$-sequent and $G^*_{L_\omega} \vdash S$, then $Sat_\omega \vdash S$.

Proof From $G^*_{L_\omega} \vdash S$ we can prove that all the possible bottom-up applications of (GIS) are successful in all four parts of $Sat_\omega$. Using this fact we can get $Sat_\omega \vdash S$.

From Lemmas 16, 17 we get the following

Theorem 3 Let $S$ be a $D$-sequent, then $G^*_{L_\omega} \vdash S \iff Sat_\omega \vdash S$. 
4. Conclusions, related works, and future investigations

We have presented a new deduction-based $\omega$-decidable saturation procedure $Sat_\omega$ for a restricted first-order linear temporal logic with temporal operators $\Diamond$ ("next") and $\Box$ ("always"). The main part of the proposed deductive procedure $Sat_\omega$ is an automatic generation of the inductive hypothesis. The objects of consideration of $Sat_\omega$ are $D$-sequents. The shape of $D$-sequents allows us to construct "looping-free" decidable calculi for induction-free $D$-sequents. The calculus $Sat_\omega$ consists of four decidable deductive procedures replacing the infinitary rule ($\to \Box_\omega$).

$D$-sequents are a certain skolemized version of M.Fisher’s normal form [14]. Basing on the seminal paper [14] and on [10, 11, 15, 16], an interesting project “Mechanizing First-Order Temporal Logic” is realized at the Manchester Metropolitan University (Department of Computing and Mathematics). Impressive deductive temporal procedures are developed at the Stanford University (see, e.g., [7]). An interesting proof planning method for inductive proofs is presented in [9]. A powerful method for proving inductive theorems of equational theories is described in [21].

In future investigations we are going to extend the proposed procedure for sequents more general than $D$-sequents, including also other temporal operators and other temporal models, e.g., past and branching time cases. However, the main attitude (for essential $\omega$-complete FTL classes) will remain the same: the deductive procedure must consist of separate decidable parts, i.e., it must be $\omega$-decidable! Another aim is to join resolution techniques described, e.g., in [10, 11, 15] with the proposed saturation deductive procedure.

References


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