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SOME LINEAR MODELS OF TIME SERIES WITH NONSTATIONARY LONG MEMORY

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KAI KURIE TIESINIAI LAIKO EILUČIŲ MODELIAI SU NESTACIONARIA ILGĄJA ATMINTIMI

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Introduction

1.1 The actuality of the thesis

Long memory, or long-range dependence, is a well-established empirical fact, which appears in various scientific fields (finance, astronomy, chemistry, hydrology, telecommunications, statistical physics etc.); see e.g. monographs Beran [8], Doukhan *et al.* [28], Palma [46] and the numerous references therein. Statistical inference under long memory is more difficult since observations are strongly dependent and their limit laws may be different from the classical i.i.d. r.v. set-up. Most of the studies in the area of long memory focus on the stationary situation. It is clear that in the case of a long (and sometimes very long) sample, the stationarity assumption might be often violated and not realistic. Therefore the study of nonstationary long memory is important to theory and applications. In particularly, parametric and semiparametric models of time series with nonstationary long memory should be developed together with inferential procedures for analyzing such series.

A natural class and most studied class of time series form – linear models. The parametric class FARIMA(p, d, q) is probably the most important class of stationary long memory processes. Therefore nonstationary and time-varying generalizations of this class present considerable interest.

It is well-known that the asymptotic properties of various tests and statistics rely on the limit distribution of partial sums process of observations, through the invariance principle. The study of the limit distribution of partial sums process of linear models with nonstationary long memory is an essential step towards their applications.

1.2 The aims and the problems of the thesis

The main object of the thesis is the study of the limit distribution of partial sums of certain linear time series models with nonstationary long memory and certain statistics which involve partial sums processes. In particularly, we focus on the following problems:

1. The description of the limit distribution of partial sums processes of infinite variance time-varying fractionally integrated (tv-FARIMA) filters. These filters were

introduced in Philippe, Surgailis, Viano [53], [51], who studied this problem under finite variance set-up.

More specifically, we assume that the innovations belong to the domain of attraction of an α-stable law ($1 < \alpha < 2$) and show that the partial sums process of filtered tv-FARIMA series converges to some α-stable self-similar process.

2. The limit of the Increment Ratio (IR) statistic for Gaussian observations superimposed on a slowly varying deterministic trend. The IR statistic was introduced in Surgailis, Teyssière, Vaičiulis [61] and its limit distribution was studied under the assumption of stationarity of observations. The IR statistic can be used for testing nonparametric hypotheses about d-integrated ($-1/2 < d < 3/2$) behavior of the time series which can be confused with deterministic trends and change-points. This statistic is written in terms of partial sums process and its limit is closely related to the limit of partial sums. In particularly, the consistency of the IR statistic uses asymptotic independence of distant partial sums, the fact is established in the thesis for a wide class of linear processes.

1.3 The methods of the thesis

The proofs of the limit behavior of partial sums are based on the so-called "scheme of discrete stochastic integrals" (introduced in [59]), and the properties of the weak convergence of probability measures. The asymptotic behavior of the IR statistic uses the method of Hermite expansions and the so-called Arcones' inequality (see [1]).

1.4 The novelty of the thesis

All results of the thesis are new.

1.5 The history of the problem and the main results

Definition 1. A covariance-stationary time series $(X_t) = (X_t,\ t \in \mathbb{Z})$ is said to be *covariance long memory* (or covariance long–range dependent) if the sum of its covariances absolutely diverges:

$$\sum_{t\in\mathbb{Z}} |\mathrm{cov}(X_0, X_t)| \ = \ \infty; \tag{1}$$

otherwise the process (X_t) is called *covariance short memory.*

A related definition of long memory is given in terms of spectral density $f(\lambda), \lambda \in [-\pi, \pi]$, see e.g. Beran [8], p. 42. While these definitions are simple and intuitive, they are limited to stationary processes with finite second moment. Moreover, condition (1) is not very constructive and further assumptions on the decay of the covariances are necessary to show the limit distribution of simplest nonlinear statistics of observations $(X_t, 1 \le t \le N)$ even if (X_t) is a Gaussian process.

A different notion of long memory (called distributional long memory) is given in Cox [21], Dehling and Philipp [23] and other works.

Definition 2. A strictly stationary time series (X_t) is called *distributional long memory* if its partial sums process, when suitably normalized, weakly converges to some random process with stationary dependent increments. More precisely, this means that there exist some constants $A_N \to \infty$ $(N \to \infty)$ and B_N and a stochastic stationary increment process $(J(\tau), \tau \geq 0) \not\equiv 0$ with dependent increments, such that

$$A_N^{-1} \sum_{t=1}^{[N\tau]} (X_t - B_N) \ \to_{\mathrm{FDD}} \ J(\tau), \tag{2}$$

as $N \to \infty$, where $[a]$ stands for the integer part of a real number a, and $\to_{\mathrm{FDD}}$ denotes the weak convergence of finite dimensional distributions.

Lamperti [40] showed that under mild additional assumptions the normalizing constants in (2) grow as N^H (with some $H > 0$), more precisely,

$$A_N = L(N) N^H \tag{3}$$

where $L(N))$ is a slowly at infinity varying function, and the limit process $(J(\tau), \tau \geq 0)$ is *self-similar with index H*. The last property means that for any $a > 0$, finite dimensional distributions of processes $(J(\tau), \tau \geq 0)$ and $(a^{-H} J(a\tau), \tau \geq 0)$ coincide:

$$(J(\tau), \tau \geq 0) \ =_{\mathrm{FDD}} \ (a^{-H} J(a\tau), \tau \geq 0),$$

where $=_{\mathrm{FDD}}$ denotes equality of finite dimensional distributions. The exponent H in (3) is called the *Hurst index* of time series (X_t). In the finite variance case $EX_t^2 < \infty$, usually $A_N^2 = N^2 \mathrm{var}(\bar{X}) = E\left(\sum_{t=1}^N (X_t - EX_t)\right)^2$ and the variance $\mathrm{var}(\bar{X})$ of the sample mean is called the *Allen variance* (see Heyde and Yang [35]).

Definition 3. (Heyde and Yang [35]) A time series (X_t) with finite variance is called *LRD(AV)* (Long-Range Dependence (Allen Variance)) if

$$\lim_{N\to\infty} N \mathrm{var}(\bar{X}) \ = \ \lim_{N\to\infty} N^{-1} E\left(\sum_{t=1}^{N} (X_t - EX_t)\right)^2 \ = \ \infty, \tag{4}$$

otherwise (X_t) is called *SRD(AV)* (Short-Range Dependence (Allen Variance)).

Heyde and Yang [35] note that the LRD(AV) definition allows for departure from stationarity. For nonstationary processes with finite variance, the Hurst index is defined by

$$H \ = \ \inf\left\{ \limsup_{N\to\infty} N^{-2h} \mathrm{var}\Big(\sum_{t=1}^{N} X_t\Big) = 0 \right\}, \tag{5}$$

see Philippe *et al.* [52]. Heyde and Yang [35] extended the characterization property (4) to infinite variance (nonstationary) processes. Accordingly, (X_t) is called *LRD(SAV)* (Long-Range Dependence (Sample Allen Variance)) if

$$\frac{\left(\sum_{t=1}^{N} X_t\right)^2}{\sum_{t=1}^{N} X_t^2} \to_P \infty. \tag{6}$$

For stationary processes, (6) essentially amounts to the observation that a central limit theorem does not hold for partial sums of (X_t) ([35], p. 883).

The above definitions (4)-(6) present a theoretical interest but they are not very useful for modeling and statistical analysis of time series with nonstationary long memory. In fact, there are few "genuinely nonstationary" times series models with long memory discussed in the literature. Philippe *et al.* [52] discuss several classes of almost periodically correlated processes constructed from the well-known FARIMA (Fractional Autoregressive Moving Average) class by amplitude modulation (AM), phase modulation (PM), memory modulation (MM) and coefficient modulation (CM). The most interesting from these classes is CM, also called *time-varying FARIMA* (tv-FARIMA). This model was introduced in [51] and [53]. It is the main object of the first part of the thesis.

We recall that the most important class of long memory models form fractionally integrated autoregressive processes FARIMA(p, d, q), defined as stationary solutions of the difference equation

$$\varphi(L)(I-L)^d X_t = \vartheta(L)\varepsilon_t, \tag{7}$$

where L is the backward shift operator, $\varphi(L), \vartheta(L)$ are polynomials in L of degree p, q, respectively, and the operator $(I-L)^d$ is defined by the binomial expansion

$$(I-L)^d := \sum_{j=0}^{\infty} \psi_j(d) L^j,$$

where $\psi_0(d) := 1$ and

$$\psi_j(d) := \frac{\Gamma(-d+j)}{j!\Gamma(-d)} \qquad (j \geq 1).$$

For properties of FARIMA(p, d, q) processes we refer to Brockwell and Davis [15]. It is well-known that in the case $0 < d < 1/2$ and under suitable conditions on the polynomial $\varphi(\cdot)$ and the i.i.d. noise $(\varepsilon_t,\ t \in \mathbb{Z})$, the autocovariance function of the FARIMA(p, d, q) process decays as t^{2d-1} and its partial sum process converges in distribution to a fractional Brownian motion (fBm) $W_H(\tau)$ with Hurst parameter $H = d + (1/2)$. The last result is a particular case of a more general result due to Davydov (1970, [22]) for partial sums of general second order linear processes. Several authors (Astrauskas (1983, [3]), Kasahara and Maejima (1986, [37]), Avram and Taqqu (1992, [5]), Vaičiulis (2003, [64])) discussed partial sums limits of linear processes with *infinite variance*, in particular, stationary solutions of FARIMA(p,d,q) in (7) with i.i.d. noise ε_t belonging to the domain of attraction of α-stable law $(1 < \alpha < 2)$. In the latter case, a stationary solution of (7) exists for $0 < d < 1 - 1/\alpha$ and the corresponding limit process of $N^{-d-1/\alpha} \sum_{t=1}^{[N\tau]} X_t$ is a so-called fractional stable motion.

Recently, Philippe et al. [51, 53] (hereafter: PSV) introduced time-varying fractionally differentiating filters

$$A(\mathbf{d})x_t \ := \ \sum_{j=0}^{\infty} a_j(t) x_{t-j}, \qquad B(\mathbf{d})x_t \ := \ \sum_{j=0}^{\infty} b_j(t) x_{t-j}, \tag{8}$$

where $\mathbf{d} = (d_t, t \in \mathbb{Z})$ is a given function of $t \in \mathbb{Z}$,

$$a_j(t) \ := \ \Big(\frac{d_{t-1}}{1}\Big)\Big(\frac{d_{t-2}+1}{2}\Big)\Big(\frac{d_{t-3}+2}{3}\Big)\cdots\Big(\frac{d_{t-j}+j-1}{j}\Big), \tag{9}$$

$$b_j(t) \ := \ \Big(\frac{d_{t-1}}{1}\Big)\Big(\frac{d_{t-j}+1}{2}\Big)\Big(\frac{d_{t-j+1}+2}{3}\Big)\cdots\Big(\frac{d_{t-2}+j-1}{j}\Big), \qquad j \ge 1, \tag{10}$$

$a_0(t) = b_0(t) := 1$. If $d_t = d$ is constant, then

$$a_j(t) \ = \ b_j(t) \ = \ \Big(\frac{d}{1}\Big)\Big(\frac{d+1}{2}\Big)\Big(\frac{d+2}{3}\Big)\cdots\Big(\frac{d+j-1}{j}\Big) \ = \ \psi_j(-d)$$

and (8) coincide with FARIMA filter $(I-L)^{-d}$. The operators $A(\mathbf{d}), B(\mathbf{d})$ are related by $B(-\mathbf{d})A(\mathbf{d}) = A(-\mathbf{d})B(\mathbf{d}) = I$, where $-\mathbf{d} := (-d_t, t \in \mathbb{Z})$.

PSV [51, 53] (see also PSV [52]) studied partial sums limits of time-varying fractionally integrated processes X_t and Y_t defined by

$$X_t \ = \ A(-\mathbf{d})^{-1} G\varepsilon_t \ = \ B(\mathbf{d}) G\varepsilon_t \ = \ \sum_{j=0}^{\infty} (b \star g)_j(t) \varepsilon_{t-j}, \tag{11}$$

$$Y_t \ = \ B(-\mathbf{d})^{-1} G\varepsilon_t \ = \ A(\mathbf{d}) G\varepsilon_t \ = \ \sum_{j=0}^{\infty} (a \star g)_j(t) \varepsilon_{t-j}, \tag{12}$$

where $(\varepsilon_t, t \in \mathbb{Z})$ is an i.i.d. (or martingale difference) sequence, with zero mean and unit variance,

$$(b \star g)_j(t) := \sum_{i=0}^{j} b_i(t) g_{j-i}, \qquad (a \star g)_j(t) := \sum_{i=0}^{j} a_i(t) g_{j-i}$$

are the impulse responses of the product operators $B(\mathbf{d})G$, $A(\mathbf{d})G$, respectively, and G is a short-memory filter with absolutely summable coefficients:

$$Gx_t = \sum_{j=0}^{\infty} g_j x_{t-j}, \qquad \text{with} \quad \sum_{j=0}^{\infty} |g_j| < \infty \text{ and } \sum_{j=0}^{\infty} g_j \ne 0.$$

PSV [51, 53] discussed two classes of sequences $\mathbf{d}$, namely: (I) the class of almost periodic sequences $\mathbf{d}$ having a mean value $\bar{d} \in (0, 1/2)$, and (II) the class of (asymptotic) sequences $\mathbf{d} = (d_t, t \in \mathbb{Z})$ having limits $d_\pm = \lim_{t\to\pm\infty} d_t \in (0, 1/2)$. They showed that the case (I), "averaging of long memory" of nonstationary processes (X_t) and (Y_t) occurs and their partial sums converge to a usual fBm with Hurst parameter $H = \bar{d} + (1/2)$. In the case (II), the partial sums of (X_t) and (Y_t) converge to two different Gaussian self-similar processes depending essentially on the asymptotic parameters $d_\pm$ only and having asymptotically stationary or asymptotically vanishing increments (see Chapter 1, Definition 1.3).

In Chapter 1 we extend the results of PSV [51, 53] in two directions. Firstly, we consider the class of time-varying processes (X_t) and (Y_t) in (11), (12) with infinite variance, by assuming that innovations $(\varepsilon_t, t \in \mathbb{Z})$ are i.i.d. r.v. belonging to the domain of attraction of α-stable law ($1 < \alpha \le 2$). We show that in this case, partial sums of (X_t) and (Y_t) converge to some α-stable self-similar processes which are α-stable counterparts of the Gaussian process introduced in PSV [51]. Secondly, we combine the classes of almost periodic and asymptotic sequences $\mathbf{d} = (d_t, t \in \mathbb{Z})$ (which were discussed separately in PSV [51, 53]) into a more general class of sequences $\mathbf{d} = (d_t, t \in \mathbb{Z})$ admitting possibly different Cesaro limits $\bar{d}_\pm \in (0, 1 - (1/\alpha))$ at $\pm\infty$:

$$\bar{d}_+ := \lim_{n\to\infty} n^{-1} \sum_{i=1}^{n} d_i, \qquad \bar{d}_- := \lim_{n\to\infty} n^{-1} \sum_{i=1}^{n} d_{-i}, \tag{13}$$

and satisfying some additional conditions (see Chapter 1, Definition 1.1 and 1.2 for precise formulation). Clearly, the existence of $d_\pm = \lim_{t\to\pm\infty} d_t$ implies the existence of the limits in (13), with $\bar{d}_\pm = d_\pm$. On the other hand, if $(d_t, t \in \mathbb{Z})$ is almost periodic with mean value $\bar{d}$, then (13) hold with $\bar{d}_+ = \bar{d}_- = \bar{d}$.

The main results (Theorems 1.1 and 1.2) are given in Sec. 1.2, together with main auxiliary Lemmas 1.1 and 1.2. Since the formulations of these theorems are rather involved, here we present a corollary from Theorems 1.1 and 1.2 (Corollary 1 below), which does not require complex notation.

Consider the case of tv-FARIMA filter corresponding to "change-point in memory", i.e.

$$d_t = \begin{cases} d_+, & \text{if } t \ge t_0, \\ d_-, & \text{if } t < t_0, \end{cases} \tag{14}$$

where $t_0 \in \mathbb{Z}$ is a fixed integer, and $d_\pm \in (0, 1/2)$ are some values. Note (14) satisfies (13) with $\bar{d}_\pm = d_\pm$. According to the definitions (9)-(10),

$$a_{t-s}(t) = \begin{cases} \prod_{s\le k<t} \frac{t-k-1+d_-}{t-k}, & \text{if } s < t \le t_0, \\ \prod_{s\le k<t_0} \frac{t-k-1+d_-}{t-k} \prod_{t_0\le k<t} \frac{t-k-1+d_+}{t-k}, & \text{if } s < t_0 < t, \\ \prod_{s\le k<t} \frac{t-k-1+d_+}{t-k}, & \text{if } t_0 \le s < t \end{cases}$$

and

$$b_{t-s}(t) \;=\; \begin{cases} d_- \prod_{s-1<k\le t-2} \frac{k-s+1+d_-}{k-s+2}, & \text{if } s<t\le t_0, \\ d_+ \prod_{s-1<k<t_0} \frac{k-s+1+d_-}{k-s+2} \prod_{t_0\le k\le t-2} \frac{k-s+1+d_+}{k-s+2}, & \text{if } s<t_0<t, \\ d_+ \prod_{s-1<k\le t-2} \frac{k-s+1+d_+}{k-s+2}, & \text{if } t_0\le s<t. \end{cases}$$

From these equations, one easily obtains the asymptotics

$$a_{t-s}(t) \;\sim\; \begin{cases} \psi_{t-s}(-d_+) & \sim\; \frac{1}{\Gamma(d_+)}(t-s)^{d_+-1}, & s\ge 0,\ t\to\infty, \\ \psi_{t-s}(-d_-)\frac{\psi_t(-d_+)}{\psi_t(-d_-)} & \sim\; \frac{1}{\Gamma(d_+)}(t-s)^{d_--1}t^{d_+-d_-}, & s\le 0,\ t\to\infty, \end{cases}$$

and

$$b_{t-s}(t) \;\sim\; \begin{cases} \psi_{t-s}(-d_+) & \sim\; \frac{1}{\Gamma(d_+)}(t-s)^{d_+-1}, & s\ge 0,\ t\to\infty, \\ \psi_{t-s}(-d_+)\frac{-d_+\psi_t(-d_-)}{-d_-\psi_t(-d_+)} & \sim\; \frac{-d_+}{-d_-\Gamma(d_-)}(t-s)^{d_+-1}t^{d_--d_+}, & s\le 0,\ t\to\infty. \end{cases}$$

Corollary 1. *Let*

$$Y_t = A(\mathbf{d})\varepsilon_t \;=\; \sum_{s\le t} a_{t-s}(t)\varepsilon_s$$

and

$$X_t = B(\mathbf{d})\varepsilon_t \;=\; \sum_{s\le t} b_{t-s}(t)\varepsilon_s$$

be time-varying fractionally integrated filters in (11) and (12), respectively, corresponding to $\mathbf{d}$ *as in (14), with symmetric* α*-stable innovations* (ε_t)*. Let*

$$1<\alpha\le 2, \qquad d_\pm \in (0, 1-(1/\alpha)).$$

Then

$$N^{-d_+-(1/\alpha)}\sum_{t=1}^{[N\tau]} Y_t \;\to_{\mathrm{FDD}}\; c_A(J(\tau)+U(\tau)), \tag{15}$$

$$N^{-d_+-(1/\alpha)}\sum_{t=1}^{[N\tau]} X_t \;\to_{\mathrm{FDD}}\; c_B^+ J(\tau), \qquad \textit{if}\quad d_+>d_-, \tag{16}$$

$$N^{-d_--(1/\alpha)}\sum_{t=1}^{[N\tau]} X_t \;\to_{\mathrm{FDD}}\; c_B^- V(\tau), \qquad \textit{if}\quad d_+<d_-, \tag{17}$$

$$N^{-d-(1/\alpha)}\sum_{t=1}^{[N\tau]} X_t \;\to_{\mathrm{FDD}}\; c_B(J(\tau)+V(\tau)), \quad \textit{if}\quad d_+=d_-=d. \tag{18}$$

Here, $c_A, c_B^{\pm}$ are some constants, and the limiting processes J, U, V are defined below, as stochastic integrals with respect to a symmetric α-stable Lévy process Z on the real line:

$$\begin{aligned} J(\tau) &= \int_0^\tau Z(\,dx) \int_x^\tau (y-x)^{d_+-1}\,dy, \\ U(\tau) &= \int_{-\infty}^0 Z(\,dx) \int_0^\tau (y-x)^{d_--1} y^{d_+-d_-}\,dy, \\ V(\tau) &= \int_{-\infty}^0 (-x)^{d_--d_+} Z(\,dx) \int_0^\tau (y-x)^{d_+-1}\,dy. \end{aligned}$$

Some comments about Corollary 1 are in order. In the absence of "jump $d_+ - d_-$ in memory" (i.e., in the case $d_+ = d_- = d$), the processes (X_t) and (Y_t) are the classical FARIMA$(0,d,0)$ and Corollary 1 is well-known; see [3], [5], [37], [64] and other papers which discuss weak convergence of partial sums of infinite variance stationary processes. We note that in this case, the limit process in (15) and (18) is a *fractional stable motion* (see [54]). On the other hand, the corollary exhibits a rather simple parametric class of time series models with *nonstationary* distributional long memory (c.f. Definition 2): not only the processes (X_t) and (Y_t) are nonstationary, but the nonstationarity persists in the distributional limit, since for $d_+ \neq d_-$, all three limit processes in $J+U$, J and V have nonstationary (and dependent) increments. A surprising limit process is V in (26), which is a.s. infinitely differentiable on $(0,\infty)$ and so very unusual from the point of view of limit theorems for partial sums processes. Further properties of these limit processes are listed in Chapter 1 below.

The Increment Ratio (IR) statistic was introduced by Surgailis, Vaičiulis, Teyssière [61]. It is defined for given observations $X_1, \dots, X_N$ as the sum of ratios of partial sums

$$IR := \frac{1}{N-3m} \sum_{k=0}^{N-3m-1} \frac{\left|\sum_{t=k+1}^{k+m}(X_{t+m}-X_t) + \sum_{t=k+m+1}^{k+2m}(X_{t+m}-X_t)\right|}{\left|\sum_{t=k+1}^{k+m}(X_{t+m}-X_t)\right| + \left|\sum_{t=k+m+1}^{k+2m}(X_{t+m}-X_t)\right|} \qquad (19)$$

with the convention $0/0 = 1$; here $m = 1, 2, \dots$ is bandwidth parameter (see [17] for generalization of the IR statistic). The IR statistic can be used for testing nonparametric hypotheses for d-integrated $(-1/2 < d < 5/4)$ behavior of time series $(X_t, 1 \le t \le N)$, including short memory $(d = 0)$, (stationary) long memory $(0 < d < 1/2)$ and unit roots $(d = 1)$. If partial sums process of X_t's asymptotically behaves as an (integrated) fractional Brownian motion with parameter $H = d + 1/2$, the IR statistic converges (as $N, m, N/m \to \infty$) to the expectation

$$\Lambda(d) := \mathrm{E}\left[\frac{|Z_1 + Z_2|}{|Z_1| + |Z_2|}\right], \qquad (20)$$

where (Z_1, Z_2) have a jointly Gaussian distribution with zero mean, unit variances, and the covariance

$$\varrho(d) := \mathrm{cov}(Z_1, Z_2) \;=\; \frac{-9^{d+.5} + 4^{d+1.5} - 7}{2(4 - 4^{d+.5})}.$$

The function $\Lambda(d)$ in (20) is strictly monotone increasing on the interval $(-1/2, 3/2)$ and is explicitly written in [61]. For Gaussian observations $\{X_t\}$, in [61], a rate of decay of the bias $\mathrm{E}IR - \Lambda(d)$ and a central limit theorem (see below) in the region $-1/2 < d < 5/4$ are obtained. The corresponding IR test rejecting the null hypothesis $\mathrm{H}_0 : d = d_0$ in favor of $\mathrm{H}_1 : d \neq d_0$ has the critical region

$$|IR - \Lambda(d_0)| \;>\; z_{\alpha/2}\sigma(d_0)\sqrt{\frac{m}{N-3m}}, \tag{21}$$

where z_α is a standard normal quantile, and the function $\sigma(d)$ is numerically tabulated in Stoncelis and Vaičiulis [57] (see also the graph in [61]). A simulation study in [61] shows that the IR test for short memory ($d = 0$) against stationary long memory alternatives ($0 < d < 1/2$) has good size and power properties and is robust against changes in mean, slowly varying trends, and nonstationarities.

In the thesis (Chapter 2), we assume that the observed sample comes from the model

$$X_t \;=\; g_{N,t} + X_t^0 \qquad (1 \le t \le N), \tag{22}$$

where $g_{N,t}$ is a slowly varying deterministic trend, and $\{X_t^0\}$ is a stationry/stationary increment Gaussian process. We want to study the impact of the trend on the limit distribution of the IR statistic. In particular, we obtain conditions on the trend and stationary component guaranteeing that the limit distribution of the IR statistic under the model (22) follows the same central limit theorem as in the absence of trend.

Let us recall the main result of [61]. For brevity, we formulate it under slightly stronger assumptions than in [61].

Assumption A. $\{X_t^0\}$ is a zero mean stationary Gaussian sequence with spectral density $f(x), x \in [-\pi, \pi]$, of the form

$$f(x) \;=\; |x|^{-2d}\left(c_0 + O(|x|^\beta)\right) \qquad (x \to 0),$$

where $c_0 > 0, 0 < \beta < 2d + 1$, and $d \in (-1/2, 1/2)$ are some constants. Moreover, $f(x)$ is differentiable on $(0, \pi)$ and $|f'(x)| \le C|x|^{-1-2d}$, where $C > 0$ is some positive constant.

Assumption B. The differences $\{X_t^0 - X_{t-1}^0\}$ form a zero-mean stationary Gaussian sequence whose spectral density satisfies

$$f(x) \;=\; |x|^{2-2d}\left(c_0 + O(|x|^\beta)\right) \qquad (x \to 0)$$

for some constants $c_0 > 0, 0 < \beta < 2d-1, 1/2 < d < 5/4$. Moreover, $f(x)$ is differentiable on $(0,\pi)$ and $|f'(x)| \le C|x|^{1-2d}$, where $C > 0$ is some positive constant.

Let IR^0 denote the IR statistic in (19) with $X_t = X_t^0$.

Theorem 1 [see [61]] **.** *Suppose that $\{X_t^0\}$ satisfies Assumption A or Assumption B. Then, as N, m, $N/m \to \infty$,*

$$\mathrm{E}IR^0 - \Lambda(d) = O(m^{-\beta}), \tag{23}$$

$$\mathrm{E}\left(IR^0 - \Lambda(d)\right)^2 = o(1), \tag{24}$$

$$(N/m)^{1/2}(IR^0 - \mathrm{E}IR^0) \Rightarrow \mathcal{N}(0, \sigma^2(d)), \tag{25}$$

where $\sigma^2(d) > 0$ is defined in [61], and $\Rightarrow$ denotes the convergence in distribution.

Introduce the following notation

$$G_m(k) := V_m^{-1}\Big|\sum_{t=k+1}^{k+m}(g_N(t+m) - g_N(t))\Big|,$$

$$\overline{G_m^i} := \frac{1}{N-2m}\sum_{k=0}^{N-2m-1} G_m^i(k) \qquad (i = 1, 2),$$

$$V_m^2 := \mathrm{E}\Big(\sum_{t=1}^{m}(X_{t+m}^0 - X_t^0)\Big)^2.$$

Under Assumptions A or B, for any $d \in (-1/2, 5/4), d \ne 1/2$,

$$V_m^2 \sim c(d)m^{1+2d} \qquad (m \to \infty), \tag{26}$$

where $c(d) > 0$ is a constant which is explicitly written in [61] (Eqs. (2.20) and (2.22)).

The main result of Chapter 2 is Theorem 2, which gives a bound of the bias of the IR statistic and a central limit theorem for the centered IR statistic for trended observations as in (22).

Theorem 2. *Suppose that observations X_t, $t = 1, \dots, N$, follow the model as in (22). Let N and $m = m(N)$ both tend to ∞ so that $m = o(N)$.*

(i) *Let $\{X_t^0\}$ satisfy Assumption A or Assumption B. Then*

$$\mathrm{E}IR - \Lambda(d) = O\left(\max\left(m^{-\beta}, \overline{G_m^2}\right)\right).$$

In addition, if $\overline{G_m^1} \to 0$, then

$$\mathrm{E}\left(IR - \Lambda(d)\right)^2 \to 0.$$

(ii) *Let* $\{X_t^0\}$ *satisfy Assumption A or Assumption B. If*

$$\overline{G_m^i} \;=\; o\left((m/N)^{1/2}\right) \qquad (i=1,2), \tag{27}$$

then

$$(N/m)^{1/2}\,(IR\;-\mathrm{E}IR)\;\Rightarrow\;\mathcal{N}\left(0,\sigma^2(d)\right),$$

where $\sigma^2(d)$ *is the same as in Theorem 1.*

Corollary 2. *Let* $\{X_t^0\}$ *satisfy conditions of Theorem 1,* $m^{-\beta}=o((m/N)^{1/2})$ *and* $g_N(t)$ *satisfy (27). Then the IR test of the hypothesis* $H_0: d=d_0,\;d_0\in(-1/2,5/4),\;d_0\neq 1/2$ *under the model (22) follows the same asymptotic confidence intervals in (21) as in the absence of trend.*

It is well-known that various tests and graphical methods often confuse trends with long memory. This phenomenon is known as "spurious long memory" (see e.g. Lobato and Savin [43]). Two natural questions in this context are (I) "how small a trend must be to be no longer asymptotically detectable (by a given test)?" and (II) "how large a trend must be to be distinguished from stationary observations?". Bhattacharya *et al.* [10] studied these questions for the R/S statistic and weakly dependent observations. Shimotsu [56] provided a test to distinguish between a true and spurious FARIMA(0,d,0) processes. See also Künsch [39], Teverovsky and Taqqu [63], Diebold and Inoue [26], Giraitis *et al.* [30], Leipus and Viano [41], Giraitis *et al.* [32], and the references therein. In particular, Giraitis *et al.* [30] studied questions (I) and (II) for the V/S and related R/S-type statistics and a (general) weakly dependent stationary process $\{X^0\}$ (case $d=0$). They showed "small trends" (corresponding to case (I)) can be roughly characterized by the requirement

$$\|g_N\|_2 := \left(\sum_{t=1}^{N} g_N^2(t)\right)^{1/2} \;=\; O(1). \tag{28}$$

On the other hand, if the trend g_N satisfies $\|g_N\|_2\to\infty$ and some additional conditions, the V/S statistic converges to a different limit than in the absence of trend, and so it is fooled by the trend. For hyperbolic trend

$$g_N(t)=c_1|t+c_2N|^{\gamma}, \tag{29}$$

the two cases (I) and (II) correspond to $\gamma<-1/2$ and $\gamma>-1/2$, respectively (see Giraitis *et al.* [30] for details). Another important example is the case of change point in mean:

$$g_N(t)=\begin{cases}0, & 1\le t\le[\tau N],\\ \mu_N, & [\tau N]<t\le N,\end{cases} \tag{30}$$

for some $0<\tau<1$. Clearly, condition (28) for g_N in (30) is equivalent to $|\mu_N|=O(N^{-1/2})$. On the other hand, if $|\mu_N|N^{1/2}\to\infty$, then Assumption 2.3 of Giraitis *et al.* [30] is satisfied, and so the V/S test for short memory is again fooled by the trend.

Example 1. Consider regression-type trend $g_N(t) = g(t/N)$, where g is a continuously differentiable function in the interval $[0,1]$. Then $\|g_N\|_2 \sim N^{1/2}\int_0^1 g^2(\tau)\,\mathrm{d}\tau \to \infty$ and so condition (28) is violated. On the other hand, $\overline{G^1_m} \le \sup_{\tau\in[0,1]} |g'(\tau)|(m^2/NV_m) = O(m^2/NV_m) = O(m^{3/2}/N)$ for $d = 0$ (see (26)) and so condition (27) for $i = 1$ becomes $m = o(N^{1/2})$. In a similar way, condition (27) for $i = 2$ follows from $m = o(N^{3/5})$. Since $m^{-\beta}(N/m)^{1/2} = o(1)$ is needed for (21) in the absence of trend (see Theorem 1, (23), (25), or Corollary 2), we obtain that the IR test for testing the short memory hypothesis $\mathrm{H}_0 : d = 0$ is not affected by any regression-type trend for bandwidths $m \sim N^{\lambda}$, $1/(1+2\beta) < \lambda < 1/2$.

Example 2. Consider the hyperbolic trend depending on N as in (29). Let $-1/2 < \gamma < 1/2$ and $c_i > 0$ $(i = 1,2)$. Then

$$\begin{aligned} |g_N(t+m) - g_N(t)| &= c_1|t+m+c_2N|^{\gamma}\left|\left|1+\frac{m}{t+c_2N}\right|^{\beta}-1\right| \\ &= O\left(m|t+c_2N|^{\gamma-1}\right) = O(mN^{\gamma-1}), \end{aligned}$$

implying $\overline{G^1_m} = O(m^2N^{\gamma-1}/V_m) = O(m^{3/2}N^{\gamma-1})$ and, similarly, $\overline{G^2_m} = O(m^3N^{2\gamma-2})$ for $V_m \sim \mathrm{const.}m^{1/2}$. Therefore a similar conclusion about this trend being asymptotically ignored by the IR test in (21) (with $d_0 = 0$) as in the previous example applies for bandwidths $m \sim N^{\lambda}$, $1/(1+2\beta) < \lambda < (1/2) - \gamma$.

Example 3. Consider the change-point trend (30) with μ_N *arbitrary*. Since

$$|IR - IR^0| \le 4m/N \tag{31}$$

by the definition (19) of the IR statistic, together with Theorem 1, (23), and (25), this immediately implies that the IR test is asymptotically insensitive to such change points (and also to any finite number of such change points). The last observation can be applied to a couple of important trends. The first of them is so-called local trend defined by

$$g_N(t) = L(t)\begin{cases} 1, & \tau \in [\tau', \tau''], \\ 0, & \tau \notin [\tau', \tau''], \end{cases}$$

where $L(t)$ is a function slowly varying at infinity, and the difference $\tau'' - \tau'$ does not depend on N. Then

$$|IR - IR^0| \le \frac{4m + (\tau'' - \tau')}{N},$$

and thus the local trend is asymptotically ignored by the IR test. The second one is the change-point trend in the scale (volatility) model. One obtains (31) for this model by using the scale invariance of the IR statistic.

We do not study problem (II) "how large a trend must be to be detected by the IR statistic?", since our empirical simulations and the above discussion confirm that the IR statistic is insensitive to trends and hence not good for detecting trends. Certainly, some other statistics (e.g., the V/S statistic) are better fitted for such purpose.

The main result is given in Chapter 2. The proof of Theorem 2 is given in Section 2.3. Monte Carlo simulations are given in Section 2.2.

In Chapter 3 of the thesis we discuss the joint weak convergence (f.d.d. and functional) of the vector-valued process $(U_n^{(1)}(\tau), U_n^{(2)}(\tau))$, $\tau \in [0,1]$, where $U_n^{(1)}(\tau) := A_n^{-1}\sum_{t=1}^{[n\tau]} X_t$, $U_n^{(2)}(\tau) := A_n^{-1}\sum_{t=1}^{[n\tau]} X_{t+m}$ are the normalized partial sums processes separated by a large lag (m, $m/n \to \infty$) and $(X_t,\ t \in \mathbb{Z})$ is stationary moving average process in i.i.d. (or martingale difference) innovations with finite variance. The cases of long memory, short memory and negative memory moving average (X_t) are discussed. We show that in each cases the bivariate partial sums process $(U_n^{(1)}(\tau), U_n^{(2)}(\tau))$ tends to bivariate fractional Brownian motion with mutually independent components. This result is applied to prove consistency of certain increment type statistics in moving averages observations.

1.6 Approbation of the thesis

The result of this thesis were presented at the Conferences of Lithuanian Mathematical Society (2005, 2006, 2007, 2008) and the 9th International Vilnius Conference on Probability Theory and Mathematical Statistics (Vilnius, Lithuania, June 25–30, 2006).

One of the talk is published in:

K. Bružaitė, M.Vaičiulis, Time-varying fractionally integrated processes with infinite variance and nonstationary long memory, 9th International Vilnius Conference on Probability Theory and Mathematical Statistics, Abstracts of Communications, p.114, 2006.

Moreover, the results of the thesis were presented at the seminar on Probability Theory and Mathematical Statistics of Institute of Mathematics and Informatics, at the seminar on Econometrics of the Department of Mathematics and Informatics of Vilnius University and at the seminar of the Department of Mathematics and Informatics of Šiauliai University.

1.7 Principal publications

The main results of the thesis are published in the following papers:

1. K. Bružaitė, M. Vaičiulis, Asymptotic independence of distant partial sums of linear process, *Liet. Matem. Rink.*, **45**(4), 479–500 (2005) (in Russian) == *Lith. Math. J.* **45**(4), 387–404 (2005).

2. K. Bružaitė, D. Surgailis, M. Vaičiulis, Time-varying fractionally integrated processes with finite or infinite variance and nonstationary long memory, *Acta Appl. Math.*, **96**, 99–118, (2007).

3. K. Bružaitė, M. Vaičiulis, The increment ratio statistic under deterministic trends, *Lith. Math. J.*, **48**(3), 1–15 (2008).

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Chapter 1

Time-varying fractionally integrated processes with finite or infinite variance and nonstationary long memory

1.1 Some preliminaries

Fractionally integrated autoregressive processes FARIMA(p, d, q) are defined as stationary solutions of the difference equation

$$\varphi(L)(I-L)^d X_t = \vartheta(L)\varepsilon_t, \tag{1}$$

where L is the backward shift operator, $\varphi(L), \vartheta(L)$ are polynomials in L of degree p, q, respectively, and the operator $(I-L)^d$ is defined by the binomial expansion

$$(I-L)^d \ := \ \sum_{j=0}^{\infty} \psi_j(d) L^j,$$

where $\psi_0(d) := 1$ and

$$\psi_j(d) \ := \ \frac{\Gamma(-d+j)}{j!\Gamma(-d)} \qquad (j \geq 1). \tag{2}$$

For properties of FARIMA(p, d, q) processes we refer to Brockwell and Davis [15]. It is well-known that in the case $0 < d < 1/2$ and under suitable conditions on the polynomial $\varphi(\cdot)$ and the i.i.d. noise $(\varepsilon_t,\ t \in \mathbb{Z})$, the autocovariance function EX_0X_t of the FARIMA(p, d, q) process in (1) decays as t^{2d-1} with $t \to \infty$ and its partial sum process converges in distribution to a fractional Brownian motion (fBm) $W_H(\tau)$ with Hurst parameter $H = d + (1/2)$.

Time-varying fractionally differentiating filters, introduced in Philippe et al. [51, 53] (hereafter PSV), are defined as

$$A(\mathbf{d})x_t \;:=\; \sum_{j=0}^{\infty} a_j(t)x_{t-j}, \qquad B(\mathbf{d})x_t \;:=\; \sum_{j=0}^{\infty} b_j(t)x_{t-j}, \tag{3}$$

where $\mathbf{d} = (d_t, t \in \mathbb{Z})$ is a given function of $t \in \mathbb{Z}$,

$$a_j(t) \;:=\; \Big(\frac{d_{t-1}}{1}\Big)\Big(\frac{d_{t-2}+1}{2}\Big)\Big(\frac{d_{t-3}+2}{3}\Big)\cdots\Big(\frac{d_{t-j}+j-1}{j}\Big), \tag{4}$$

$$b_j(t) \;:=\; \Big(\frac{d_{t-1}}{1}\Big)\Big(\frac{d_{t-j}+1}{2}\Big)\Big(\frac{d_{t-j+1}+2}{3}\Big)\cdots\Big(\frac{d_{t-2}+j-1}{j}\Big), \qquad j \ge 1, \tag{5}$$

$a_0(t) = b_0(t) := 1$. If $d_t = d$ is constant, then

$$a_j(t) \;=\; b_j(t) \;=\; \Big(\frac{d}{1}\Big)\Big(\frac{d+1}{2}\Big)\Big(\frac{d+2}{3}\Big)\cdots\Big(\frac{d+j-1}{j}\Big) \;=\; \psi_j(-d)$$

and (3) coincide with FARIMA filter $(I-L)^{-d}$. The operators $A(\mathbf{d}), B(\mathbf{d})$ are related by $B(-\mathbf{d})A(\mathbf{d}) \;=\; A(-\mathbf{d})B(\mathbf{d}) \;=\; I$, where $-\mathbf{d} := (-d_t, t \in \mathbb{Z})$.

Finally, let us define time-varying fractionally integrated processes X_t and Y_t (see PSV [51, 53], also PSV [52]) by:

$$X_t \;=\; A(-\mathbf{d})^{-1}G\varepsilon_t \;=\; B(\mathbf{d})G\varepsilon_t \;=\; \sum_{j=0}^{\infty}(b\star g)_j(t)\varepsilon_{t-j}, \tag{6}$$

$$Y_t \;=\; B(-\mathbf{d})^{-1}G\varepsilon_t \;=\; A(\mathbf{d})G\varepsilon_t \;=\; \sum_{j=0}^{\infty}(a\star g)_j(t)\varepsilon_{t-j}, \tag{7}$$

where $(\varepsilon_t, t \in \mathbb{Z})$ is an i.i.d. (or martingale difference) sequence, with zero mean and unit variance,

$$(b\star g)_j(t) := \sum_{i=0}^{j} b_i(t)g_{j-i}, \qquad (a\star g)_j(t) := \sum_{i=0}^{j} a_i(t)g_{j-i} \tag{8}$$

are the impulse responses of the product operators $B(\mathbf{d})G$, $A(\mathbf{d})G$, respectively, and G is a short memory filter with absolutely summable coefficients:

$$Gx_t = \sum_{j=0}^{\infty} g_j x_{t-j}, \qquad \text{with} \quad \sum_{j=0}^{\infty}|g_j| < \infty \qquad \text{and} \qquad \sum_{j=0}^{\infty} g_j \ne 0.$$

Definition 1.1 *A bounded sequence* $\mathbf{d} = (d_t, t \in \mathbb{Z})$ *will be called*

(i) Averageable at $+\infty$ *if the following limit exists*

$$\bar{d}_+ = \lim_{n\to\infty} n^{-1} \sum\nolimits_{k=s}^{s+n} d_k \qquad \textit{uniformly in } s \geq 0; \tag{9}$$

(ii) Averageable at $-\infty$ *if the following limit exists*

$$\bar{d}_- = \lim_{n\to\infty} n^{-1} \sum\nolimits_{k=s-n}^{s} d_k \qquad \textit{uniformly in } s \leq 0; \tag{10}$$

(iii) Averageable if the following limit exists

$$\bar{d} = \lim_{n\to\infty} n^{-1} \sum\nolimits_{k=s}^{s+n} d_k \qquad \textit{uniformly in } s \in \mathbb{Z}. \tag{11}$$

We call the limits $\bar{d}_+$ *in (9)-(10) the mean value of* $\mathbf{d}$ *at* $\pm\infty$*, respectively, and* $\bar{d}$ *in (11) the mean value of* $\mathbf{d}$*.*

Definition 1.2 *A bounded sequence* $\mathbf{d} = (d_t, t \in \mathbb{Z})$ *will be called*

(i) Almost periodic at $+\infty$ *if for each* $\varepsilon > 0$ *there exist* $k_\varepsilon > 0$ *and a periodic sequence* $\mathbf{d}^\varepsilon = (d_t^\varepsilon, t \in \mathbb{Z})$ *such that* $\sup_{t>k_\varepsilon} |d_t - d_t^\varepsilon| < \varepsilon$*,*

(ii) Almost periodic at $-\infty$ *if the sequence* $\mathbf{d} = (-\mathbf{d_t}, \mathbf{t} \in \mathbb{Z})$ *is almost periodical at* $+\infty$*,*

(iii) Almost periodic if for each $\varepsilon > 0$ *there exists a periodic sequence* $\mathbf{d}^\varepsilon = (d_t^\varepsilon, t \in \mathbb{Z})$ *such that* $\sup_{t\in\mathbb{Z}} |d_t - d_t^\varepsilon| < \varepsilon$*.*

Denote $\mathcal{AP}$ (respectively, $\mathcal{AP}(+\infty)$ and $\mathcal{AP}(-\infty)$) the class of all almost periodic sequences which are almost periodic (respectively, almost periodic at $+\infty$ and almost periodic at $-\infty$). Denote $\mathcal{A}$ (respectively, $\mathcal{A}(+\infty)$ and $\mathcal{A}(-\infty)$) the class of all sequences which are averageable (respectively, averageable at $+\infty$ and averageable at $-\infty$).

Proposition 1.1 [see [16]] $\mathcal{AP} \subset \mathcal{A}$, $\mathcal{AP}(+\infty) \subset \mathcal{A}(+\infty)$, $\mathcal{AP}(-\infty) \subset \mathcal{A}(-\infty)$.

Proof. The first inclusion well known. Let us prove part (i). Let $\mathbf{d}$ be almost periodic at $+\infty$. For any $p \ge 1$, there exist $k_p \ge 1$ and a periodic sequence $\mathbf{d}^{(p)} = (d_t^{(p)}, t \in \mathbb{Z})$ such that $\sup_{t>k_p} |d_t - d_t^{(p)}| < 2^{-p}$. This implies $\sup_{t>k_p \vee k_q} |d_t^{(p)} - d_t^{(q)}| < 2^{-p} + 2^{-q}$ and therefore $\sup_{t\in\mathbb{Z}} |d_t^{(p)} - d_t^{(q)}| < 2^{-p} + 2^{-q}$ by periodicity. The same inequality holds for the mean values of the periodic functions:

$$|\overline{d^{(p)}} - \overline{d^{(p)}}| < 2^{-p} + 2^{-q} \qquad (p, q \ge 1)$$

and therefore

$$\lim_{p\to\infty} \overline{d^{(p)}} =: \bar{d}_+ \tag{12}$$

exists. Let us show that this limit $\bar{d}_+$ satisfies (9), in other words, that for any $\varepsilon > 0$ there exists $n_\varepsilon > 0$ such that

$$n^{-1}\Big|\sum_{k=s}^{s+n}(d_k - \bar{d}_+)\Big| < \varepsilon \qquad (\forall s > 0,\ \forall n > n_\varepsilon). \tag{13}$$

By definition (12), there exists $p_\varepsilon \ge 1$ such that $|\overline{d^{(p_\varepsilon)}} - \bar{d}_+| < \varepsilon/4, 2^{-p_\varepsilon} < \varepsilon/4$ and therefore

$$n^{-1}\Big|\sum_{k=s}^{s+n}(d_k - \bar{d}_+)\Big| \ \le\ n^{-1}\Big|\sum_{k=s}^{s+n}(d_k - d_t^{(p_\varepsilon)})\Big| + n^{-1}\Big|\sum_{k=s}^{s+n}(d_t^{(p_\varepsilon)} - \overline{d^{(p_\varepsilon)}})\Big| + |\overline{d^{(p_\varepsilon)}} - \bar{d}_+|.$$

Here, the last term is less than $\varepsilon/4$. and $\sup_{s\in\mathbb{Z}} n^{-1}\Big|\sum_{k=s}^{s+n}(d_t^{(p_\varepsilon)} - \overline{d^{(p_\varepsilon)}})\Big| < \varepsilon/4$ for $n > n'_\varepsilon$ and some n'_ε by periodicity. Let $\tau_s := n^{-1}\Big|\sum_{k=s}^{s+n}(d_k - d_t^{(p_\varepsilon)})\Big|$, then $\tau_s < 2^{-p_\varepsilon} < \varepsilon/4$ for $s > k_{p_\varepsilon}$, while for $0 < s \le k_{p_\varepsilon}$, we have $\tau_s \le n^{-1}\sum_{k=s}^{k_{p_\varepsilon}}(|d_k| + |d_t^{(p_\varepsilon)}|) + (\varepsilon/4)$, where the last sum does not exceed $(2\|\mathbf{d}\| + 2^{-p_\varepsilon})k_{p_\varepsilon}/n$, $\|\mathbf{d}\| := \sup_{t\in\mathbb{Z}} |d_t|$, and therefore this sum is less than $\varepsilon/4$ provided $n > n''_\varepsilon := 4(2\|\mathbf{d}\| + 1)k_{p_\varepsilon}/\varepsilon$. Hence (13) holds with $n_\varepsilon := n'_\varepsilon \vee n''_\varepsilon$. □

Remark 1.1 If $\mathbf{d} = (d_t, t \in \mathbb{Z})$ is bounded and the limit $d_+ = \lim_{t\to\infty} d_t$ exists, then $\mathbf{d} \in \mathcal{AP}(+\infty)$ and $\bar{d}_+ = d_+$ (the approximating periodic sequence $\mathbf{d}^\varepsilon$ in this case is the constant sequence $(d_t^\varepsilon = d_+, t \in \mathbb{Z})$, for each $\varepsilon > 0$).

Remark 1.2 (i) The inverse inclusions in Proposition 1.1 are not true. Indeed, let $d_t = 1$ if $t = 2^k$, $k = 1, 2, ..., d_t = 0$ eslewhere on $\mathbb{Z}$. Then $\mathbf{d} = (d_t, t \in \mathbb{Z}) \in \mathcal{A}(+\infty)$ but $\mathbf{d} \notin \mathcal{AP}(+\infty)$.

To check the first relation, write $n^{-1}\sum_{k=s}^{n+s} d_k = n^{-1}(L(n+s) - L(s-1)), s > 0$, where $L(t) = \#\{1 \le k \le t : k = 2^i\ (\exists i = 1, 2, ...))\}$. Then $L(t) = i$ for $2^i \le t < 2^{i+1}$ and therefore

$$\frac{log t}{log 2} \le L(t) < \frac{log t}{log 2} + 1 = \frac{log 2t}{log 2},\ t = 1, 2, ...,$$

implying $n^{-1}\sum_{k=s}^{n+s} d_k \le (nlog2)^{-1}(log(2(n+s)) - log(s-1)) \le n^{-1}(1+log(n+1)/log2) = o(1)$ uniformly in $s \ge 2$. Therefore $\mathbf{d} \in \mathcal{A}(+\infty)$ with $\bar{d}_+ = 0$ as mean value at $+\infty$.

To check the second relation, assume *ad absurdum*, that there exist $\tilde{k} > 0$ and periodic sequence $(\tilde{d}_t, t \in \mathbb{Z})$ such that $\sup_{t>\tilde{k}}|d_t - \tilde{d}_t| < 1/2$. Then $\sup_{t\in\mathbb{Z}}\tilde{d}_t > 1/2$ and $\tilde{d}_t < 1/2$ for all $2^i < t < 2^{i+1}$ and any $i > 0$ large enough, which is a contradiction since 2^i exceeds the period starting with some $i > i_0 > 0$.

(ii) Let $d_t = sgn(t), t \in \mathbb{Z}$. Then $(d_t, t \in \mathbb{Z}) \in \mathcal{AP}(+\infty)$ with mean value $\bar{d}_+ = 1$ at $+\infty$ but $(d_t, t \in \mathbb{Z}) \notin \mathcal{A}$. Indeed, $(2n)^{-1}\sum_{k=s}^{s+2n} d_t \to 1$ uniformly in $s \in \mathbb{Z}$ does not exist.

Remark 1.3 Each of the classes $\mathcal{A}P, \mathcal{A}P(+\infty), \mathcal{A}P(-\infty)$ is closed under algebraic operations, shifts and uniform limits. Moreover, these classes are also closed under compositions with continuous functions. In particular, if $(d_t, t \in \mathbb{Z}) \in \mathcal{A}P(+\infty)$, then $(|d_t|^\alpha, t \in \mathbb{Z}) \in \mathcal{A}P(+\infty)$ for any $\alpha \ge 0$.

Definition 1.3 *Let $W = (W(\tau),\ \tau \ge 0)$ be a stochastic process. We say that*

(i) W has asymptotically stationary increments if as T goes to $+\infty$

$$(W(T+\tau) - W(T),\ \tau \ge 0)\ \ \to_{\mathrm{FDD}} (\widetilde{W}(\tau),\ \tau \ge 0)$$

where $\widetilde{W}$ is a nontrivial stochastic process and $\to_{\mathrm{FDD}}$ for weak convergence of finite dimensional distributions only.

(ii) W has asymptotically vanishing increments if the conevrgence (i) holds with $\widetilde{W}(\tau) \equiv 0$.

Let $(\widetilde{Z}(x), x \in \mathbb{R}),\ \widetilde{Z}(0) = 0$ be an α-stable process with homogeneous and independent increments, $1 < \alpha \le 2, \mathrm{E}\widetilde{Z}(x) = 0$ (see e.g. Sato [55], Samorodnitsky and Taqqu (1996) for definition). In particular, for $\alpha = 2$, $\widetilde{Z}(x)$ is a Brownian motion. Introduce the following stochastic processes

$$J_d(\tau) := \int_0^\tau \widetilde{Z}(\,\mathrm{d}x) \int_x^\tau (y-x)^{d-1}\,\mathrm{d}y, \tag{14}$$

$$U_{d_+,d_-}(\tau) := \int_{-\infty}^0 \widetilde{Z}(\,\mathrm{d}x) \int_0^\tau y^{d_+-d_-}(y-x)^{d_--1}\,\mathrm{d}y, \tag{15}$$

$$V_{d_+,d_-}(\tau) := \int_{-\infty}^0 (-x)^{d_--d_+}\widetilde{Z}(\,\mathrm{d}x) \int_0^\tau (y-x)^{d_+-1}\,\mathrm{d}y, \tag{16}$$

Note that J_d is independent of U_{d_+,d_-} and V_{d_+,d_-}, that $U_{d,d} \equiv V_{d,d}$, and that

$$J_d(\tau) + U_{d,d}(\tau) = W_d(\tau) \quad \tau \geq 0, \tag{17}$$

is a fractional α-stable process with stationary increments, with self-similarity index $H = d + (1/\alpha)$. In particular, for $\alpha = 2$, the process in W_d in (17) is a fractional Brownian motion with Hurst parameter $H = d + 1/2 \in (1/2, 1)$. The processes $J_d, U_{d_+,d_-}, V_{d_+,d_-}$ were introduced in PSV [53] for $\alpha = 2$. The following proposition generalizes the corresponding result in PSV [53] for $1 < \alpha < 2$.

Proposition 1.2 [see [16]] *Let $d, d_+, d_- \in (0, 1 - (1/\alpha))$, $\alpha \in (1, 2]$. Then:*

(i) The processes $J_d, U_{d_+,d_-}, V_{d_+,d_-}$ are well-defined. They are self-similar with respective indices $d + (1/\alpha), d_+ + (1/\alpha)$ and $d_- + (1/\alpha)$ and have a.s. continuous trajectories. Moreover, finite dimensional distributions of $J_d, U_{d_+,d_-}, V_{d_+,d_-}$ are α-stable.

(ii) The processes U_{d_+,d_-} and V_{d_+,d_-} have asymptotically vanishing increments, while J_d has asymptotically stationary increments tending to those of a fractional stable process W_d in (17).

(iii) Trajectories of U_{d_+,d_-} and V_{d_+,d_-} are a.s. infinitely differentiable on $(0, \infty)$.

Proof. We restrict the proof to the case $1 < \alpha < 2$, the case $\alpha = 2$ was proved in PSV [53]. Note all processes in (14-11) can be written as stochastic integrals of the form $\int f(x;\tau)\widetilde{Z}(dx)$ with a corresponding integrand $f(\cdot;\tau))$, $\int = \int_{\mathbb{R}}$. It is well-known that $\int f(x)\widetilde{Z}(dx)$ is well-defined and has α-stable distribution if $f \in L^\alpha(\mathbb{R})$, moreover, for any $\varepsilon > 0$ there exists a constant $C = C_\varepsilon > 0$ such that

$$E\Big|\int f(x)\widetilde{Z}(dx)\Big|^{\alpha-\varepsilon} \leq C(\max(\|f\|_{\alpha-\varepsilon}, \|f\|_{\alpha+\varepsilon}))^{\alpha-\varepsilon}, \tag{18}$$

where $\|\cdot\|_\alpha$ is the norm in $L^\alpha(\mathbb{R})$; see e.g. Surgailis [58]. Using (18) (where $\varepsilon > 0$ should be chosen small enough), all facts in (i)-(ii) can be proved similary as in the case $\alpha = 2$ in PSV [53]. To prove (iii) for U_{d_+,d_-} in (15), note the integrand $f(x;\tau) = \int_0^\tau y^{d_+-d_-}(y-x)^{d_--1}dy\mathbb{I}_{]-\infty,0]}(x)$ is infinitely differentiable with respect to $\tau > 0$ and $\|f_\tau^{(n)}(\cdot;\tau)\|_{\alpha\pm\varepsilon} < C$ is uniformly bounded for $\varepsilon > 0$ sufficiently small on any compact interval $[\tau_1, \tau_2] \subset (0, \infty)$ (the constant C depends on $\tau_1 > 0$, $n \geq 0$, $\varepsilon > 0$). This implies that $U_{d_+,d_-}(\tau)$ is n-times differentiable in $L^{\alpha-\varepsilon}(\Omega)$ on $\tau \in (0, \infty)$ and the derivative $U^{(n)}_{d_+,d_-}(\tau) = \int_{-\infty}^0 f_\tau^{(n)}(x;\tau)\widetilde{Z}(dx)$; moreover, one can easily check that $U^{(n)}_{d_+,d_-}(\tau)$ is a.s. continuous on $(0, \infty)$ and therefore it coincided with the nth pathwise derivative of $U_{d_+,d_-}(\tau)$. The proof of (iii) for V_{d_+,d_-} in (11) is analogous. Proposition 1.2 is proved. □

1.2 Main results

Recall the definition of time-varying fractionally integrated filters $A(\mathbf{d})G, B(\mathbf{d})G$ in (3)-(5), (8). In order to study partial sums limits of the integrated processes $Y_t = A(\mathbf{d})G\varepsilon_t, X_t = A(\mathbf{d})G\varepsilon_t$, we introduce the following conditions on the sequence $\mathbf{d}$ and the short memory filter G.

Assumption A1. Let $\mathcal{M}$ be a class of sequences $\mathbf{d}$ averageable at $+\infty$ and $-\infty$ and closed under algebraic operations, shifts, and uniform limits.

Examples of such $\mathcal{M}$ are: (1) $\mathcal{M} = \mathcal{AP}$, (2) $\mathcal{M} = \{\mathbf{d} : \lim_{t\to\pm\infty} d_t = d_\pm \in \mathbb{R}\ exist\}$, (3) $\mathcal{M} = \mathcal{AP}(+\infty) \cap \mathcal{AP}(-\infty)$.

Assumption A2. Assume $\mathbf{d} \in \mathcal{M}$, $d_t \notin \mathbb{Z}_- := \{0, -1, -2, \ldots\}$ for any $t \in \mathbb{Z}$, $\bar{d}_\pm \in (0, 1 - 1/\alpha)$. Moreover, let there exist C and $0 < \delta < 1$ such that for all $s < t$

$$\Big|(t-s)^{-1} \sum_{i=s+1}^{t} (d_i - \bar{d}_+)\Big| \;\le\; C|t-s|^{-\delta} \qquad (0 \le s < t), \tag{19}$$

$$\Big|(t-s)^{-1} \sum_{i=s+1}^{t} (d_i - \bar{d}_-)\Big| \;\le\; C|t-s|^{-\delta} \qquad (s < t \le 0). \tag{20}$$

Assumption A3. Assume that $\bar{g} := \sum_{j=0}^{\infty} g_j \ne 0$ and there exist some $C, \delta_1 > 0$ such that

$$|g_j| \;\le\; Cj^{-1-\delta_1} \qquad (j \ge 1). \tag{21}$$

Let be given a sequence $\mathbf{d} = (d_t, t \in \mathbb{Z}) \in \mathcal{M}$ having mean values $\bar{d}_\pm$ at $\pm\infty$ as in (9)-(10). Define a new sequence $(\bar{d}_t, t \in \mathbb{Z})$ having a single jump at $t = 0$ as

$$\bar{d}_t := \begin{cases} \bar{d}_+, & \text{if } t \ge 0, \\ \bar{d}_-, & \text{if } t < 0. \end{cases}$$

Denote

$$q_A(t) \;:=\; \prod_{k<t} \Big(1 + \frac{d_k - \bar{d}_t}{\bar{d}_t + t - k - 1}\Big), \qquad q_B(t) \;:=\; \prod_{k\ge t} \Big(1 + \frac{d_k - \bar{d}_t}{\bar{d}_t + k - t + 1}\Big) \tag{22}$$

Introduce also

$$Q_B(s) := \sum_{i=0}^{\infty} g_i q_B(s+i), \quad s \in \mathbb{Z}. \tag{23}$$

Definition 1.4 *Write* $\varepsilon \in D(\alpha)$ $(1 < \alpha \le 2)$ *if*

(i) $\alpha = 2$ *and* $\mathrm{E}\varepsilon = 0, \mathrm{E}\varepsilon^2 < \infty$,

(ii) $1 < \alpha < 2, \mathrm{E}\varepsilon = 0$ *and there exist some constants* $c_i \ge 0, c_1 + c_2 \ne 0$ *such that*

$$\mathbb{P}(\varepsilon > x) \sim c_1 x^{-\alpha} \quad (x \to \infty), \qquad \mathbb{P}(\varepsilon \le x) \sim c_2 |x|^{-\alpha} \quad (x \to -\infty).$$

Let $\varepsilon_t, t \in \mathbb{Z}$ be a sequence of i.i.d. rv's, with zero mean, whose generic distribution $\varepsilon \in \mathcal{D}(\alpha)$, $1 < \alpha \le 2$. The last assumption implies that the ε_t's belong to the domain attraction of α-stable law (Ibrahimov and Linnik [36], Theorem 2.6.7), in other words

$$n^{-1/\alpha} \sum_{t=1}^{n} \varepsilon_t \ \Rightarrow \ Z, \tag{24}$$

where $\Rightarrow$ denotes convergence in distribution and Z is an α-stable r.v. with the characteristic function

$$\mathrm{E}e^{\mathrm{i}\theta Z} = \begin{cases} e^{-\omega_\alpha}(\theta; c_1, c_2), & \text{if } 1 < \alpha < 2, \\ e^{-\sigma^2\theta^2/2}, & \text{if } \alpha = 2 \end{cases} \tag{25}$$

where $\sigma^2 := \mathrm{E}\varepsilon^2$ and

$$\omega_\alpha(\theta; c_1, c_2) := \frac{|\theta|^\alpha \Gamma(2-\alpha)}{1-\alpha} \Big((c_1 + c_2)\cos(\frac{\pi\alpha}{2}) - \mathrm{i}(c_1 - c_2)\mathrm{sign}(\theta)\sin(\frac{\pi\alpha}{2}) \Big). \tag{26}$$

The following Lemma 1.1 describes limit behavior of weighted sums of ε's. In this lemma, we do not use the special form of weights $Q_B(t)$ in (23). The notation in (27)-(28) is convenient in the formulation of Theorem 1.2.

Lemma 1.1 [see [16]] *Let* $\varepsilon_t, t \in \mathbb{Z}$ *be i.i.d. rv's,* $\varepsilon \in D(\alpha), 1 < \alpha \le 2$.

(i) Let $1 < \alpha < 2$ *and* $(Q_B(t)) \in \mathcal{AP}(+\infty)$. *Then*

$$n^{-1/\alpha} \sum_{t=1}^{n} Q_B(t)\varepsilon_t \ \Rightarrow \ Z_+, \tag{27}$$

where Z_+ *is* α*-stable r.v. whose characteristic function is given in (54-56);*

(ii) Let $1 < \alpha < 2$ *and* $(Q_B(t)) \in \mathcal{AP}(-\infty)$. *Then*

$$n^{-1/\alpha} \sum_{t=-n}^{-1} Q_B(t)\varepsilon_t \ \Rightarrow \ Z_-, \tag{28}$$

where Z_- is α-stable r.v. whose characteristic function is given in (54-56);

(iii) Let $\alpha = 2$ and let $(Q^2_B(t))$ be averageable at $+\infty$. Then (27) holds, with $Z_+ \sim N(0, \sigma^2_+)$, $\sigma^2_+ = \sigma^2 \overline{Q^2_{B+}}$.

(iv) Let $\alpha = 2$ and let $(Q^2_B(t))$ be averageable at $-\infty$. Then (28) holds, with $Z_- \sim N(0, \sigma^2_-)$, $\sigma^2_- = \sigma^2 \overline{Q^2_{B-}}$.

Let $(Z(x), x \in \mathbb{R})$, $(Z_+(x), x \in \mathbb{R})$, $(Z_-(x), x \in \mathbb{R})$ be α-stable processes, $Z(0) = Z_+(0) = Z_-(0) := 0$, whose distribution is completely determined by the distribution at time $x = 1$:

$$Z(1) =_{\text{law}} Z, \qquad Z_+(1) =_{\text{law}} Z_+, \qquad Z_-(1) =_{\text{law}} Z_-,$$

where Z, Z_+, Z_- are defined in (24), (27), (28), respectively, and $=_{\text{law}}$ stands for equality of distributions. We shall also assume that $(Z_+(x), x \in \mathbb{R})$ and $(Z_-(x), x \in \mathbb{R})$ are mutually independent.

Let α-stable self-similar processes

$$J_d,\ J^+_d, J^-_d,\ U_{d_+,d_-}, U^+_{d_+,d_-}, U^-_{d_+,d_-},\ V_{d_+,d_-}, V^+_{d_+,d_-}, V^-_{d_+,d_-},\ W_d, W^+_d, W^-_d$$

be defined as in (14)-(17), with the random measure $\widetilde{Z}(\,\mathrm{d}x)$ replaced by $Z(\,\mathrm{d}x) = \mathrm{d}Z(x)$, $Z_+(\,\mathrm{d}x) = \mathrm{d}Z_+(x)$, $Z_-(\,\mathrm{d}x) = \mathrm{d}Z_-(x)$, respectively.

Let $\to_{D[0,1]}$ denote weak convergence of random elements in the Skorohod space $D[0,1]$ endowed with the sup-topology. Introduce (asymptotic) constants $c_A, c^{\pm}_B$ by

$$c_A := \frac{\overline{q_{A+}}\bar{g}}{\Gamma(\bar{d}_+)}, \quad c^+_B := \frac{1}{\Gamma(\bar{d}_+)}, \quad c^-_B := \frac{\bar{d}_+}{\bar{d}_-\Gamma(\bar{d}_-)},$$

where $\overline{q_{A\pm}}$ is the mean values at $\pm\infty$ of $q_A = (q_A(t))$. Our main results are the following theorems.

Theorem 1.1 [see [16]] *Let $Y_t = A(\mathbf{d})G\varepsilon_t$ as defined in (7), where $(\varepsilon_t, t \in \mathbb{Z})$ are i.i.d. rv's, $\varepsilon \in D(\alpha)$, $1 < \alpha \le 2$. Assume that* $\mathbf{d}$, $\mathcal{M}$ *and G satisfy Assumptions A1-A3, respectively, and that*

$$\bar{d}_\pm \in (0, 1 - (1/\alpha)). \tag{29}$$

Then

$$N^{-\bar{d}_+-(1/\alpha)} \sum_{t=1}^{[N\tau]} Y_t \quad \to_{D[0,1]} \quad c_A \left(J_{\bar{d}_+}(\tau) + U_{\bar{d}_+,\bar{d}_-}(\tau) \right). \tag{30}$$

Theorem 1.2 [see [16]] *Let $X_t = B(\mathbf{d})G\varepsilon_t$ as defined in (6), where $(\varepsilon_t, t \in \mathbb{Z}), \mathbf{d}, G$ satisfy the same conditions as in Theorem 1.1, including (29).*

(i) Let $\bar{d}_+ > \bar{d}_-$. Moreover, in the case $1 < \alpha < 2$, assume $\mathbf{d} \in \mathcal{A}P(+\infty)$. Then

$$N^{-\bar{d}_+-(1/\alpha)} \sum_{t=1}^{[N\tau]} X_t \quad \to_{D[0,1]} c_B^+ J_{\bar{d}_+}^+(\tau). \tag{31}$$

(ii) Let $\bar{d}_+ < \bar{d}_-$. Moreover, in the case $1 < \alpha < 2$ assume $\mathbf{d} \in \mathcal{A}P(-\infty)$. Then

$$N^{-\bar{d}_--(1/\alpha)} \sum_{t=1}^{[N\tau]} X_t \quad \to_{D[0,1]} c_B^- V_{\bar{d}_+,\bar{d}_-}^-(\tau).$$

(iii) Let $\bar{d}_+ = \bar{d}_- =: \bar{d}$. Moreover, assume that for $1 < \alpha < 2$, the sequence $\mathbf{d}$ is almost periodic at $+\infty$ and $-\infty$ (i.e., $\mathbf{d} \in \mathcal{A}P(+\infty) \cap \mathcal{A}P(-\infty)$). Then

$$N^{-\bar{d}-(1/\alpha)} \sum_{t=1}^{[N\tau]} X_t \quad \to_{D[0,1]} \quad c_B W_{\bar{d}}^+(\tau),$$

where $c_B := c_B^+ = c_B^-$ for $\bar{d}_+ = \bar{d}_- = \bar{d}$.

Theorems 1.1 and 1.2 essentially follow from Lemma 1.2 below combined with Lemma 1.1, see the proofs in Sect. 1.3. Lemma 1.2 relates the asymptotic behavior of coefficients of time-varying filters $A(\mathbf{d}), B(\mathbf{d})$ and their "short memory perturbations" $A(\mathbf{d})G, B(\mathbf{d})G$, to the asymptotic behavior of FARIMA coefficients $\psi_j(-\bar{d}_\pm)$. The proofs of Lemmas 1.1 and 1.2 are given in Sect. 1.3.

Lemma 1.2 [see [16]] *Let $\mathbf{d}$, $\mathcal{M}$ and G satisfy Assumptions (A1)-(A3). Then there exist $C, \delta_2 > 0$ independent of s, t and such that*

$$(a \star g)_{t-s}(t) \quad = \quad \begin{cases} \bar{g} q_A(t) \psi_{t-s}(-\bar{d}_+) + \Theta_A(t,s), & 0 \le s \le t, \\ \bar{g} q_A(t) \psi_{t-s}(-\bar{d}_-) \frac{\psi_t(-\bar{d}_+)}{\psi_t(-\bar{d}_-)} + \Theta_A(t,s), & s < 0 \le t, \end{cases} \tag{32}$$

$$(b \star g)_{t-s}(t) \quad = \quad \begin{cases} \frac{d_{t-1}}{\bar{d}_+} Q_B(s) \psi_{t-s}(-\bar{d}_+) + \Theta_B(t,s), & 0 \le s \le t, \\ \frac{d_{t-1} \psi_t(-\bar{d}_-)}{\bar{d}_- \psi_t(-\bar{d}_+)} Q_B(s) \psi_{t-s}(-\bar{d}_+) + \Theta_B(t,s), & s < 0 \le t, \end{cases} \tag{33}$$

where

$$|\Theta_A(t,s)| \le C(|t-s|^{-\delta_2} + |s|^{-\delta_2} + |t|^{-\delta_2}) \begin{cases} (t-s)^{\bar{d}_+-1}, & 0 \le s < t, \\ t^{\bar{d}_+-\bar{d}_-}(t-s)^{\bar{d}_--1}, & s < 0 \le t, \end{cases} \tag{34}$$

$$|\Theta_B(t,s)| \le C(|t-s|^{-\delta_2} + |s|^{-\delta_2} + |t|^{-\delta_2}) \begin{cases} (t-s)^{\bar{d}_+-1}, & 0 \le s < t, \\ |s|^{(\bar{d}_--\bar{d}_+)\vee 0}(t-s)^{\bar{d}_+-1}, & s < 0 \le t. \end{cases} \tag{35}$$

The sequences $q_A = (q_A(t)), q_B = (q_B(t)), Q_B = (Q_B(t))$ in (22)-(23) are well-defined and belong to the class $\mathcal{M}$. Moreover, if $\mathbf{d} \in \mathcal{AP}(+\infty)$ (respectively, $\mathbf{d} \in \mathcal{AP}(-\infty)$), then $Q_B \in \mathcal{AP}(+\infty)$ (respectively, $Q_B \in \mathcal{AP}(-\infty)$).

1.3 Proofs

Before turning to the formal proofs of Theorems 1.1-1.2 and Lemmas 1.1, 1.2, let us clarify the meaning of the introduced infinite products in (22). Let first $0 \le s < t$. Then from (4), (2) one has

$$\begin{aligned} a_{t-s}(t) &= \psi_{t-s}(-\bar{d}_+)\frac{a_{t-s}(t)}{\psi_{t-s}(-\bar{d}_+)} \\ &= \psi_{t-s}(-\bar{d}_+)\prod_{s\le k<t}\Big(1+\frac{d_k-\bar{d}_+}{\bar{d}_+ + t-k-1}\Big) \\ &= \psi_{t-s}(-\bar{d}_+)q_A(t)\theta_A(t,s), \end{aligned} \tag{36}$$

where

$$\begin{aligned} \theta_A(t,s) &:= \prod_{0\le k<s}\Big(1+\frac{d_k-\bar{d}_+}{\bar{d}_+ + t-k-1}\Big)^{-1}\prod_{p<0}\Big(1+\frac{d_p-\bar{d}_-}{\bar{d}_- + t-p-1}\Big)^{-1} \\ &= \prod_{k<s}\Big(1+\frac{d_k-\bar{d}_k}{\bar{d}_k + t-k-1}\Big)^{-1} \end{aligned} \tag{37}$$

tends to 1 as $t \to \infty$ and $t - s \to \infty$; see Lemma 1.1 below. The resulting relation $a_{t-s}(t) \sim \psi_{t-s}(-\bar{d}_+)q_A(t)$ in (36) gives the behavior of the fractionally integrated filter $a_{t-s}(t)$ as $t \to \infty, t-s \to \infty$. Next, let $s \le 0 < t$. Then

$$\begin{aligned} a_{t-s}(t) &= \psi_{t-s}(-\bar{d}_-)\frac{\psi_t(-\bar{d}_+)}{\psi_t(-\bar{d}_-)}\frac{a_{t-s}(t)\psi_t(-\bar{d}_-)}{\psi_{t-s}(-\bar{d}_-)\psi_t(-\bar{d}_+)} \\ &= \psi_{t-s}(-\bar{d}_-)\frac{\psi_t(-\bar{d}_+)}{\psi_t(-\bar{d}_-)}\prod_{s\le p<0}\Big(1+\frac{d_p-\bar{d}_-}{\bar{d}_- + t-p-1}\Big)\prod_{0\le k<t}\Big(1+\frac{d_k-\bar{d}_+}{\bar{d}_+ + t-k-1}\Big) \\ &= \psi_{t-s}(-\bar{d}_-)\frac{\psi_t(-\bar{d}_+)}{\psi_t(-\bar{d}_-)}q_A(t)\theta_A(t,s), \end{aligned}$$

where

$$\theta_A(t,s) := \prod_{k<s}\Big(1+\frac{d_k-\bar{d}_-}{\bar{d}_- + t-k-1}\Big)^{-1} = \prod_{k<s}\Big(1+\frac{d_k-\bar{d}_k}{\bar{d}_k + t-k-1}\Big)^{-1} \tag{38}$$

tends to 1 and therefore $a_{t-s}(t) \sim \psi_{t-s}(-\bar{d}_-)\frac{\psi_t(-\bar{d}_+)}{\psi_t(-\bar{d}_-)}q_A(t)$ as $s \to -\infty$ and $t > 0$; see Lemma 1.4. Lemma 1.4 also gives related asymptotics of the filter coefficients $b_{t-s}(t)$ in terms of the FARIMA $(0, \bar{d}_\pm, 0)$ coefficients and the function $q_B(t)$ in (22).

In the sequel, C will stand for a generic constant C which may change from line to line.

Lemma 1.3 [see [16]] *Let* $\mathbf{d}$, $\mathcal{M}$ *satisfy Assumption (A1)-(A2). Then the infinite products in (22) converge uniformly in* $t \in \mathbb{Z}$. *Both sequences* $q_A = (q_A(t))$ *and* $q_B = (q_B(t))$ *belong to* $\mathcal{M}$.

Proof follows that in PSV([51], Lemma 2.2), with appropriate modifications. To prove the uniform convergence of $q_A(t)$, it is enough to show that, as $n, m \to \infty$

$$\sup_{t \in \mathbb{Z}} |q_A(t-n,t) - q_A(t-m,t)| \to 0,$$

where

$$q_A(s,t) \;:=\; \prod_{s \le k < t} \Big(1 + \frac{d_k - \bar{d}_k}{\bar{d}_k + t - k - 1}\Big) \;=:\; \prod_{s \le k < t} (1 + \beta_k(t)).$$

The sequence $(d_t, t \in \mathbb{Z})$ is bounded, thus there exists n_0 such that $|\beta_{t-p}(t)| < 1/2$ for all $p \ge n_0, t \in \mathbb{Z}$. Since, the uniform convergence of $q_A(t-n,t)$ is equivalent to the uniform convergence of $q_A(t-n,t)/q_A(t-n_0,t)$. Therefore, we can suppose that $n_0 = 1$. Then $q_A(t-n,t) > 0 \; \forall n \ge 1$ and

$$|q_A(t-n,t) - q_A(t-m,t)| \le q_A(t-n,t) \left| \prod_{k=t-m}^{t-n-1} (1 + \beta_k(t)) - 1 \right| \qquad (n < m). \tag{39}$$

For $|x| < 1/2$, we have $\mathrm{e}^{x - x^2} \le 1 + x \le \mathrm{e}^x$. Thus we obtain $q_A(t-n,t) \le \exp\big\{\sum_{k=t-n}^{t-1} \beta_k(t)\big\}$ and

$$\exp\Big\{ \sum_{k=t-m}^{t-n-1} \beta_k(t) - \sum_{k=t-m}^{t-n-1} \beta_k^2(t) \Big\} \;\le\; \prod_{k=t-m}^{t-n-1} (1 + \beta_k(t)) \le \exp\Big\{ \sum_{k=t-m}^{t-n-1} \beta_k(t) \Big\}. \tag{40}$$

We claim that there exists a constant C such that inequality

$$\sum_{k=t-m}^{t-n-1} |\beta_k(t)| \;\le\; C n^{-\delta}, \tag{41}$$

holds for any $t \in \mathbb{Z}$ and any $m > n \ge 1$. Whence, $\sum_{k=t-n}^{t-1} |\beta_k(t)| < C$ and $q_A(t-n,t) \le \exp\big\{\sum_{k=t-n}^{t-1} \beta_k(t)\big\} < C$ are bounded uniformly in $t \in \mathbb{Z}, n \ge 1$. From (39), (41) we also obtain

$$|q_A(t-n,t) - q_A(t-m,t)| \le C(|\mathrm{e}^{-Cn^{-\delta}} - 1| + |\mathrm{e}^{Cn^{-\delta}} - 1|) \le C n^{-\delta} \quad (n < m),$$

proving the statement of the lemma about the uniform convergence of the products $q_A(t)$ in (22). As the class $\mathcal{M}$ is closed under algebraic operations, shifts and uniform limits, therefore $q_{A,n} = (q_A(t-n,t))$ and $q_A = (q_A(t))$ belong to $\mathcal{M}$. The proof for $q_B = (q_B(t))$ follows similarly.

It remains to prove the claim (41). We shall separately consider three cases: Case (1): $0 \le t-m < t-n-1$, Case (2): $t-m < 0 \le t-n-1$, and Case (3): $t-m < t-n-1 \le 0$.

Consider first Case (1). Then $\beta_k(t) = \frac{d_k - \bar{d}_+}{\bar{d}_+ + t - k - 1}$ $(t-m \le k < t)$ by the definition of $\bar{d}_+$. Using summation by parts,

$$\sum_{k=t-m}^{t-n-1} \beta_k(t) \;=\; \frac{D_+(t-m,t-n)}{\bar{d}_+ + m - 1} \;+ \sum_{k=t-m}^{t-n-2} \frac{D_+(k+1,t-n)}{(\bar{d}_+ + t - k - 1)(\bar{d}_+ + t - k)}, \tag{42}$$

with $D_+(k,t-n) := \sum_{i=k}^{t-n-1}(d_i - \bar{d}_+)$. The hypothesis (19) implies that $|D_+(k,t-n)| \le C|k-(t-n)|^{1-\delta}$ for $0 \le t-m \le k < t-n$, where C does not depend on k,t,n,m. In particular, $|D_+(t-m,t-n)| \le C|m-n|^{1-\delta}$ and

$$\Big|\frac{D_+(t-m,t-n)}{\bar{d}_+ + m - 1}\Big| \;\le\; C|m-n|^{1-\delta}/m \;\le\; Cn^{-\delta},$$

for $m > n$. Similarly,

$$\sum_{k=t-m}^{t-n-2} \frac{|D_+(k+1,t-n)|}{(\bar{d}_+ + t - k - 1)(\bar{d}_+ + t - k)} \;\le\; C\sum_{k=t-m}^{t-n-2} \frac{|k-(t-n)|^{1-\delta}}{(t-k)^2} \;\le\; C\sum_{\ell=1}^{\infty} \frac{\ell^{1-\delta}}{(n+\ell)^2} \le Cn^{-\delta}$$

proving (41).

Next, consider Case (2). Split $\sum_{k=t-m}^{t-n-1} \beta_k(t) = \sum_{k=0}^{t-n-1} \beta_k(t) + \sum_{k=t-m}^{-1} \beta_k(t) =: \Sigma_1 + \Sigma_2$. Then $|\Sigma_1| \le Cn^{-\delta}$ according to (41) above. Consider Σ_2. Note $\beta_k(t) = \frac{d_k - \bar{d}_-}{\bar{d}_- + t - k - 1}$ $(t-m \le k < 0)$. Similarly as (42),

$$\Sigma_2 = \sum_{k=t-m}^{-1} \beta_k(t) = \frac{D_-(t-m)}{\bar{d}_- + m - 1} + \sum_{k=t-m}^{-2} \frac{D_-(k+1)}{(\bar{d}_- + t - k - 1)(\bar{d}_- + t - k)}$$

with $D_-(k) := \sum_{i=k}^{-1}(d_i - \bar{d}_-)$. The hypothesis (20) implies that $|D_-(k)| \le C|k|^{1-\delta}$ for $k < 0$. Therefore $|D_-(t-m)/(\bar{d}_- + m - 1)| \le C(m-t)^{1-\delta}/m \le Cm^{-\delta} \le Cn^{-\delta}$ and $\sum_{k=t-m}^{-2} |D_-(k+1)/(\bar{d}_- + t - k - 1)(\bar{d}_- + t - k)| \le C\sum_{\ell=1}^{\infty} \ell^{1-\delta}/(t+\ell)^2 \le Ct^{-\delta} \le Cn^{-\delta}$ as $t > n$. This proves (41) in Case (2).

Finally, consider Case (3). Similarly as (42), with $\beta_k(t) = \frac{d_k - \bar{d}_-}{\bar{d}_- + t - k - 1}$ we obtain

$$\sum_{k=t-m}^{t-n-1} \beta_k(t) \;=\; \frac{D_-(t-m,t-n)}{\bar{d}_- + m - 1} \;+ \sum_{k=t-m}^{t-n-2} \frac{D_-(k+1,t-n)}{(\bar{d}_- + t - k - 1)(\bar{d}_- + t - k)},$$

with $D_-(k,t-n) := \sum_{i=k}^{t-n-1}(d_i - \bar d_-)$ satisfying $|D_-(k,t-n)| \le C|k-(t-n)|^{1-\delta}$ in view of the hypothesis (20). The remaining details of the proof are the same as Cases (1) and (2) above. This proves the claim (41) and the lemma, too. □

Lemma 1.4 [see [16]] *Under the hypotheses of Lemma 1.3,*

$$a_{t-s}(t) \;=\; q_A(t)\begin{cases}\psi_{t-s}(-\bar d_+)\theta_A(t,s), & \text{if } 0 \le s < t,\\ \psi_{t-s}(-\bar d_-)\frac{\psi_t(-\bar d_+)}{\psi_t(-\bar d_-)}\theta_A(t,s), & \text{if } s < 0 < t,\end{cases} \tag{43}$$

where $q_A(t), \theta_A(t,s)$ are defined in (22), (37), (38), respectively. Moreover,

$$|\theta_A(t,s) - 1| \le C|t-s|^{-\delta} \qquad (s<t) \tag{44}$$

for some constant C independent of t,s. Also,

$$b_{t-s}(t) \;=\; q_B(s)d_{t-1}\psi_{t-s}(-\bar d_+)\theta_B(t,s)\begin{cases}(1/\bar d_+), & \text{if } 0 \le s < t,\\ (1/\bar d_-)\frac{\psi_{1-s}(-\bar d_-)}{\psi_{1-s}(-\bar d_+)}, & \text{if } s < 0 < t,\end{cases} \tag{45}$$

where $q_B(t)$ is defined in (22) and

$$\theta_B(t,s) := \prod_{k=t-1}^{\infty}\Big(1 + \frac{d_k - \bar d_k}{k-s+1+\bar d - k}\Big)^{-1} \tag{46}$$

satisfies

$$|\theta_B(t,s) - 1| \le C|s-t|^{-\delta} \qquad (s<t), \tag{47}$$

with some constant $C<\infty$ independent of s,t.

Proof. Let us prove (44). Let $0 \le s < t$. According to (37) and using the same notation as in the proof of the previous lemma, $\theta_A(t,s) = \prod_{k=-\infty}^{s-1}(1+\beta_k(t))^{-1}$. Using (40)-(41), it is easy to infer that

$$\theta_A(t,s) \le \exp\Big\{\sum_{k=0}^{s-1}|\beta_k(t)| + \sum_{p=-\infty}^{-1}|\beta_p(t)|\Big\} \le \mathrm{e}^{C(t-s)^{-\delta}}$$

and, similarly, $\theta_A(t,s) \ge \mathrm{e}^{-C(t-s)^{-\delta}}$, implying (44) for $0 \le s < t$. In the case $s < 0 < t$, use (38). Then, similarly as above, $\theta_A(t,s) = \prod_{p=\infty}^{s-1}\Big(1+\beta_p(t)\Big)^{-1} \le \exp\Big\{\sum_{p=-\infty}^{s-1}|\beta_p(t)|\Big\} \le \mathrm{e}^{C(t-s)^{-\delta}}$, $\theta_A(t,s) \ge \mathrm{e}^{-C(t-s)^{-\delta}}$, thereby proving (43)-(44). The proof of (45)-(47) follows similarly. □

Proof of Lemma 1.2. Consider (32). Let $0 \le s < t$. According to (43),

$$(a \star g)_{t-s}(t) - \bar g q_A(t)\psi_{t-s}(-\bar d_+) = q_A(t)\Big(\widetilde\Theta_1(t,s) + \widetilde\Theta_2(t,s)\Big),$$

where

$$\widetilde{\Theta}_1(t,s) \;:=\; \sum_{i=0}^{t-s} g_i \psi_{t-s-i}(-\bar{d}_+) - \bar{g}\psi_{t-s}(-\bar{d}_+),$$

$$\widetilde{\Theta}_2(t,s) \;:=\; \sum_{i=0}^{t-s} g_i \psi_{t-s-i}(-\bar{d}_+)(\theta_A(t,s+i)-1).$$

Using Lemma 1.4, the terms $\widetilde{\Theta}_i(t,s), i = 1,2$ are estimated exactly as in PSV([53], proof Lemma 5.1) and satisfy (34). Hence (32), (34) follow in the case $0 \le s < t$, noting that $q_A(t)$ are bounded uniformly in t, see Lemma 1.3. For $s < 0 < t$, (32), (34) follow similarly, by using our Lemma 1.4 and the estimates in PSV([53], proof of Lemma 5.1).

Next, consider (33), (35). Again, the case $0 \le s < t$ is similar to PSV ([53], proof of Lemma 5.1). Let $s < 0 < t$. According to (45),

$$(b \star g)_{t-s}(t) - \frac{d_{t-1}\psi_{1-s}(-\bar{d}_-)}{\bar{d}_-\psi_{1-s}(-\bar{d}_+)}\psi_{t-s}(-\bar{d}_+)Q_B(s) \;=\; \frac{d_{t-1}}{\bar{d}_-}\sum_{i=3}^{5}\widetilde{\Theta}_i(t,s) + \widetilde{\Theta}_6(t,s),$$

where

$$\widetilde{\Theta}_3(t,s) \;:=\; \sum_{i=0}^{-s-1} g_i q_B(s+i)\Big(\psi_{t-s-i}(-\bar{d}_+)\frac{\psi_{1-s-i}(-\bar{d}_-)}{\psi_{1-s-i}(-\bar{d}_+)} - \psi_{t-s}(-\bar{d}_+)\frac{\psi_{1-s}(-\bar{d}_-)}{\psi_{1-s}(-\bar{d}_+)}\Big),$$

$$\widetilde{\Theta}_4(t,s) \;:=\; \sum_{i=0}^{-s-1} g_i q_B(s+i)\psi_{t-s-i}(-\bar{d}_+)\frac{\psi_{1-s-i}(-\bar{d}_-)}{\psi_{1-s-i}(-\bar{d}_+)}(\theta_B(t,s+i)-1),$$

$$\widetilde{\Theta}_5(t,s) \;:=\; -\psi_{t-s}(-\bar{d}_+)\frac{\psi_{1-s}(-\bar{d}_-)}{\psi_{1-s}(-\bar{d}_+)}\sum_{i=-s+}^{\infty} g_i q_B(s+i),$$

$$\widetilde{\Theta}_6(t,s) \;:=\; (d_{t-1}/\bar{d}_+)\sum_{i=-s}^{t-s} g_i q_B(s+i)\psi_{t-s-i}(-\bar{d}_+)\theta_B(t,s+1).$$

Noting that $|g_i q_B(s+i)| \le C i^{-1-\delta_1}$ in view of (21) and boundedness q_B, the proof of the fact that $\widetilde{\Theta}_i(t,s), i = 3,...,5$ satisfy (35) (with factor $|s|^{(\bar{d}_- - \bar{d}_+)\vee 0}$ replaced by $|s|^{\bar{d}_- - \bar{d}_+}$), is completely analogous as in PSV([53], proof of Lemma 5.1). Consider $|\widetilde{\Theta}_6(t,s)| \le C\sum_{i=|s|}^{t+|s|} i^{-1-\delta_1}(t+|s|+1-i)^{\bar{d}_+-1}$. By splitting the last sum into two sums $\sum_1 := \sum_{|s|\le i<(t+|s|)/2}$, $\sum_2 := \sum_{(t+|s|)/2\le i\le t+|s|}$ (the first sum may be empty), we obtain $\sum_1 \le C(t+|s|)^{\bar{d}_+-1}\sum_{i=|s|}^{\infty} i^{-1-\delta_1} \le C(t-s)^{\bar{d}_+-1}|s|^{-\delta_1}$, $\sum_2 \le C(t+|s|)^{-1-\delta_1}\sum_{i=1}^{(t+|s|)/2} i^{\bar{d}_+-1} \le C(t-s)^{\bar{d}_+-1-\delta_1}$, thus proving the bound (35) for $\widetilde{\Theta}_6(t,s)$ and the representation (33), (35) as well.

Relation $q_A, q_B \in \mathcal{M}$ was proved in Lemma 1.3. Whence and from the fact that $\mathcal{M}$ is closed under translations, linear operations and uniform limits, it follows that $Q_B \in \mathcal{M}$.

Let $\mathbf{d} \in \mathcal{AP}(+\infty)$. Then $q_B \in \mathcal{AP}(+\infty)$, as the class $\mathcal{AP}(+\infty)$ is closed under shifts, products and uniform limits; see Remark 1.2. Consequently, $Q_B \in \mathcal{AP}(+\infty)$ as the series in (23) converges uniformly in $s \in \mathbb{Z}$. The proof for $\mathbf{d} \in \mathcal{AP}(-\infty)$ is analogous. Lemma 1.2 is proved. □

Proof of Lemma 1.1. Parts (iii)-(iv) follow by the classical Lindeberg central limit theorem. Let us prove part (i); the proof of part (ii) is analogous. As $Q_B \in \mathcal{AP}(+\infty)$, for any $\delta > 0$ there exist a $k_\delta > 0$ and a periodic sequence $(\ell_{t,\delta}, t \in \mathbb{Z})$ such that

$$\sup_{t > k_\delta} |Q_B(t) - \ell_{t,\delta}| < \delta. \tag{48}$$

Write

$$\sum_{t=1}^{n} Q_B(t)\varepsilon_t \;=\; \sum_{t=1}^{n} \ell_{t,\delta}\varepsilon_t + \sum_{t=1}^{n}(Q_B(t) - \ell_{t,\delta})\varepsilon_t \;=:\; S_{n,\delta} + R_{n,\delta}.$$

We need to show

$$\mathrm{E}|R_{n,\delta}|^{\alpha'} \le \gamma(\delta) n^{\alpha'/\alpha} \tag{49}$$

for some $\alpha' < \alpha$, all $n \ge 1$ and some $\gamma(\delta)$ independent of n and tending to 0 as $\delta \to 0$; and, moreover, that

$$n^{-1/\alpha} S_{n,\delta} \;\Rightarrow\; Z_\delta \ (\forall \delta > 0, n \to \infty) \tag{50}$$

$$Z_\delta \;\Rightarrow\; Z_+ \ (\delta \to 0), \tag{51}$$

where Z_+ is α-stable r.v. defined in (54-56) below.

To show (49), we use the inequality

$$\mathrm{E}\Big|\sum_{i=1}^{n} c_i\varepsilon_i\Big|^{\alpha-\varepsilon} \le C\Big\{n^{-\varepsilon/\alpha}\sum_{i=1}^{n}|c_i|^{\alpha-\varepsilon} + \Big(n^{\varepsilon/\alpha}\sum_{i=1}^{n}|c_i|^{\alpha+\varepsilon}\Big)^{(\alpha-\varepsilon)/(\alpha+\varepsilon)}\Big\}, \tag{52}$$

see Astrauskas [3], which is true for any numbers $c_1, ..., c_n$ and any $\varepsilon > 0$ such that $\alpha + \varepsilon \le 2$. Inequality (52) follows by writing $\varepsilon_i = (\varepsilon'_{i,n} - \mathrm{E}\varepsilon'_{i,n}) + (\varepsilon''_{i,n} - \mathrm{E}\varepsilon''_{i,n})$, $\varepsilon'_{i,n} := \varepsilon_s \mathbb{I}(|\varepsilon_s| \le n^{1/\alpha})$, $\varepsilon''_{s,n} := \varepsilon_s \mathbb{I}(|\varepsilon_s| > n^{1/\alpha})$, and using $\mathrm{E}|\varepsilon'_{i,n} - \mathrm{E}\varepsilon'_{i,n}|^{\alpha+\varepsilon} \le Cn^{\varepsilon/\alpha}$, $\mathrm{E}|\varepsilon''_{i,n} - \mathrm{E}\varepsilon''_{i,n}|^{\alpha-\varepsilon} \le Cn^{-\varepsilon/\alpha}$, together with independence of ε_i, $i \ge 1$. Relation (49) follows from (52), (48) and boundedness of $(Q_b(t))$, with $\gamma(\delta) = C|\delta|^{\alpha'} \to 0$ $(\delta \to 0)$.

Let $(\ell_{t,\delta}, t \in \mathbb{Z})$ be periodic with period $m < \infty$. Then (50) holds with Z_δ is α-stable having the characteristic function $Ee^{i\theta Z_\delta} = exp\{-\omega_\alpha(\theta; c_{1,\delta}, c_{2,\delta})\}$, where $\omega_\alpha(\theta; c_1, c_2)$ is given in (25-26) and $c_{1,\delta} \geq 0$, $c_{2,\delta} \geq 0$, $c_{1,\delta} + c_{2,\delta} > 0$ are uniquely determined by equations

$$\overline{|\ell_\delta|^\alpha}(c_1 + c_2) = c_{1,\delta} + c_{2,\delta}, \quad \overline{|\ell_\delta|^\alpha \mathrm{sgn}(\ell_\delta)}(c_2 - c_1) = c_{2,\delta} - c_{1,\delta}, \tag{53}$$

where $\overline{|\ell_\delta|^\alpha}$, $\overline{|\ell_\delta|^\alpha \mathrm{sgn}(\ell_\delta)}$ are mean values of the periodic sequences $(|\ell_{t,\delta}|^\alpha)$, $(|\ell_{t,\delta}|^\alpha \mathrm{sgn}(\ell_{t,\delta}))$, respectively. Clearly, $(|Q_B(t)|^\alpha)$ and $(|Q_B(t)|^\alpha \mathrm{sgn}(Q_B(t)))$ can be aproximated at $+\infty$, in the sence of (48), by $(|\ell_{t,\delta}|^\alpha)$, $(|\ell_{t,\delta}|^\alpha \mathrm{sgn}(\ell_{t,\delta}))$, respectively, and therefore one can pass to the limit $\delta \to 0$ in (53) and in $Ee^{i\theta Z_\delta}$, thereby proving (51) and Lemma 1.1, with

$$\begin{aligned} E\exp\{i\theta Z_\pm\} &= \exp\{-\omega_\alpha(\theta; c_{1,\pm}, c_{2,\pm})\}, &(54)\\ c_{1,\pm} &:= \overline{|Q_B|^\alpha \mathbb{I}_{\{Q_B>0\}}}_\pm c_1 + \overline{|Q_B|^\alpha \mathbb{I}_{\{Q_B<0\}}}_\pm c_2, &(55)\\ c_{2,\pm} &:= \overline{|Q_B|^\alpha \mathbb{I}_{\{Q_B>0\}}}_\pm c_2 + \overline{|Q_B|^\alpha \mathbb{I}_{\{Q_B<0\}}}_\pm c_1. &(56) \end{aligned}$$

□

Proof of Theorem 1.1. We shall use the scheme of discrete stochastic integrals as in PSV [51, 53] or Astrauskas [3], with appropriate modifications. Accordingly, relation (30) is written as

$$\int f_N(\tau, x) Z_N(\mathrm{d}x) \ \to_{D[0,1]} \int f(\tau, x) Z(\mathrm{d}x) \tag{57}$$

where $Z(\mathrm{d}x)$ is the α-stable random measure in the stochastic integral representations (14)-(11) of the limit process in (30), and Z_N is a discrete random measure defined by

$$Z_N(x', x''] := N^{-1/\alpha} \sum_{x'N < s \leq x''N} \varepsilon_s.$$

Let $1 < \alpha < 2$ (the case $\alpha = 2$ is simpler and omitted). Fix a sufficiently small $\varepsilon > 0, 1 \leq \alpha - \varepsilon < \alpha < \alpha + \varepsilon \leq 2$, and consider the Banach space $L^{\alpha,\varepsilon}(\mathbb{R})$ of all measurable real functions $f = f(x), x \in \mathbb{R}$ with finite norm $\|f\|_{\alpha,\delta} := \max(\|f\|_{\alpha-\varepsilon}, \|f\|_{\alpha+\varepsilon})$. The α-stable stochastic integral $\int f(x)\,\mathrm{d}Z(x) \equiv \int f\,\mathrm{d}Z$ is well-defined for any $f \in L^{\alpha,\varepsilon}(\mathbb{R})$ and satisfies

$$\mathrm{E}\Big|\int f\,\mathrm{d}Z\Big|^{\alpha-\varepsilon} \leq C\|f\|_{\alpha,\delta}^{\alpha-\varepsilon},$$

see (18). Let $L_N^{\alpha,\varepsilon}(\mathbb{R}) \subset L^{\alpha,\varepsilon}(\mathbb{R})$ consist of functions f taking constant values f_s on intervals $\Delta_s = (s/N, (s+1)/N], s \in \mathbb{Z}$. The discrete stochastic integral $\int_{\mathbb{R}} f(x) Z_N(\mathrm{d}x) \equiv \int f\,\mathrm{d}Z_N$ is defined for any $f \in L_N^{\alpha,\varepsilon}(\mathbb{R})$ by $\int f\,\mathrm{d}Z_N := \sum_s f_s Z_N(\Delta_s)$ for each function $f \in L^{\alpha,\varepsilon}(\mathbb{R})$, and satisfies a similar bound

$$\mathrm{E}\Big|\int f\,\mathrm{d}Z_N\Big|^{\alpha-\varepsilon} \leq C\|f\|_{\alpha,\delta}^{\alpha-\varepsilon}, \tag{58}$$

with some constant C independent of f (see (52), also Astrauskas [3]). Convergence in distribution

$$\int f_N \,\mathrm{d}Z_N \Rightarrow \int f \,\mathrm{d}Z \tag{59}$$

of a sequence of discrete stochastic integrals $\int f_N \,\mathrm{d}Z_N$ ($f_N \in L_N^{\alpha,\varepsilon}(\mathbb{R})$), $N = 1, 2, \dots$ to an α-stable stochastic integral $\int f \,\mathrm{d}Z$ ($f \in L^{\alpha,\varepsilon}(\mathbb{R})$), follows from

$$(Z_N(x_1', x_1''], \dots, Z_N(x_m', x_m''])\ \Rightarrow (Z(x_1', x''], \dots, Z(x_m', x_m'']) \tag{60}$$

for any $m < \infty$ and any disjoint intervals $(x_i', x_i''], i = 1, \dots, m$; and

$$\|f_N - f\|_{\alpha,\varepsilon} \to 0; \tag{61}$$

see Astrauskas [3], also PSV [51, 53].

Consider the convergence of finite dimensional distributions in (57). For simplicity, we shall restrict ourselves to the convergence of one-dimensional integrals at $\tau = 1$. The integrands $f_N(x) = f_N(1, x), f(x) = f(1, x)$ in (57) are given by

$$\begin{aligned} f_N(x) &:= N^{-\bar d_+} \sum_{t=1}^{N} (a \star g)_{t-s}(t), \qquad x \in ((s-1)/N, s/N], \quad s \in \mathbb{Z}, \\ f(x) &:= c_A \left(\int_0^1 (y-x)_+^{\bar d_+ - 1} \,\mathrm{d}y\ \mathbb{I}_{]0,1]}(x) + \int_0^1 y^{\bar d_+ - \bar d_-}(y-x)^{\bar d_- - 1} \,\mathrm{d}y\ \mathbb{I}_{]-\infty,0]}(x) \right), \end{aligned}$$

respectively (for $s > t$, we put $a_{t-s}(t) = b_{t-s}(t) := 0$). Convergence (60) is immediate by the central limit theorem (24) and independence of $\varepsilon_s, s \in \mathbb{Z}$. To prove (61), from Lemma 1.2, (32), (34), it suffices to show the convergences

$$\|f_N' - f\|_{\alpha,\varepsilon} \to 0, \qquad \|f_N''\|_{\alpha,\varepsilon} \to 0, \tag{62}$$

where

$$\begin{aligned} f_N'(x) &:= \bar g \overline{q_{A+}} N^{-\bar d_+} \mathbb{I}_{]\frac{s-1}{N}, \frac{s}{N}]}(x) \begin{cases} \sum_{t=1}^N \psi_{t-s}^+, & x > 0, \\ \sum_{t=1}^N \psi_{t-s}^- \psi_t^+ / \psi_t^-, & x \le 0, \end{cases} \\ f_N''(x) &:= \bar g N^{-\bar d_+} \mathbb{I}_{]\frac{s-1}{N}, \frac{s}{N}]}(x) \begin{cases} \sum_{t=1}^N (q_A(t) - \overline{q_{A+}}) \psi_{t-s}^+, & x > 0, \\ \sum_{t=1}^N (q_A(t) - \overline{q_{A+}}) \psi_{t-s}^- \psi_t^+ / \psi_t^-, & x \le 0, \end{cases} \end{aligned}$$

$s \in \mathbb{Z}$, and where $\psi_{t-s}^\pm := \psi_{t-s}(-\bar d_\pm)$ if $s \le t$, $:= 0$ otherwise. The proof of the first relation in (62) is analogous as in PSV([53], proof of Theorem 5.1), taking into account condition $\bar d_\pm \in (0, 1 - (1/\alpha))$ (29) of Theorem 1.1 and choosing $\varepsilon > 0$ small enough (the choice of $\varepsilon > 0$ depends on $\bar d_\pm$).

Consider the second relation of (62), or $\|f_N'' \mathbb{I}_{]0,1]}\|_{\alpha,\varepsilon} + \|f_N'' \mathbb{I}_{]-\infty,0]}\|_{\alpha,\varepsilon} \to 0$. For brevity, we shall restrict to the proof of $\|f_N'' \mathbb{I}_{]-\infty,0]}\|_{\alpha,\varepsilon} \to 0$, which is equivalent to

$$R_N\ :=\ \sum_{s=0}^{\infty} \Big| \sum_{t=1}^{N} (q_A(t) - \overline{q_{A+}}) \psi_{t+s}^- \psi_t^+ / \psi_t^- \Big|^{\alpha'}\ =\ o(N^{1+\alpha' \bar d_+}), \tag{63}$$

for α' taking values $\alpha' = \alpha + \varepsilon$ and $\alpha' = \alpha - \varepsilon$.

Let $G(t) := \sum_{u=1}^{t}(q_A(u) - \overline{q_{A+}})$, then $\gamma_t := |G(t)/t| \to 0$ $(t \to \infty)$ since $(q_A(t) - \overline{q_{A+}}, t \in \mathbb{Z})$ is averaging at $+\infty$ with mean value 0 at $+\infty$. Denote $\varphi_{t,s} := \psi^-_{t+s}\psi^+_t/\psi^-_t$. Using summation by parts, for $s \ge 0$ we obtain

$$\sum_{t=1}^{N}(q_A(t) - \overline{q_{A+}})\varphi_{t,s} \;=\; G(N)\varphi_{N,s} + \sum_{t=1}^{N-1} G(t)(\varphi_{t+1,s} - \varphi_{t,s}).$$

From definition (2) easily follow the bounds

$$|\psi_t(-d)| \le Ct^{d-1}, \qquad |\psi_{t+1}(-d) - \psi_t(-d))| \le Ct^{d-2}, \tag{64}$$

$$|\psi_t^{-1}(-d)| \le Ct^{1-d}, \qquad |\psi_{t+1}^{-1}(-d) - \psi_t^{-1}(-d))| \le Ct^{-d}, \tag{65}$$

implying

$$\begin{aligned} |\varphi_{N,s}| &\le C(N+s)^{\bar d_- - 1}N^{\bar d_+ - \bar d_-}, \\ |\varphi_{t+1,s} - \varphi_{t,s}| &\le C\Big((t+s)^{\bar d_- -2}t^{\bar d_+ - \bar d_-} + (t+s)^{\bar d_- -1}t^{\bar d_+ - \bar d_- -1}\Big) \\ &\le C(t+s)^{\bar d_- -1}t^{\bar d_+ - \bar d_- -1} \qquad (t, s \ge 1). \end{aligned}$$

We thus obtain

$$\begin{aligned} R_N &\le C\gamma_N^{\alpha'}N^{\alpha'(1+\bar d_+ - \bar d_-)}\sum_{s=0}^{\infty}(N+s)^{\alpha'(\bar d_- -1)} + C\sum_{s=0}^{\infty}\Big(\sum_{t=1}^{N}\gamma_t t^{\bar d_+ - \bar d_-}(t+s)^{\bar d_- -1}\Big)^{\alpha'} \\ &= o(N^{1+\alpha'\bar d_+}), \end{aligned}$$

where we used $\gamma_t = o(1)$ and $(1 - \bar d_-)\alpha' > 1$ (the last inequality is satisfied in view of $\bar d_- < 1 - (1/\alpha)$, since $\varepsilon > 0$ can be chosen arbitrarily small). This proves (63), (61) and the convergence of finite dimensional distributions in (57) and in Theorem 1.1.

The proof of tightness in the space $D[0,1]$ with the sup-topology follows by Kolmogorov's criterion. Namely, it suffices to show that there exist $C, \gamma > 0$ such that for any $N \ge 1$ and any $0 \le \tau < \tau + h \le 1$

$$E\Big|\sum\nolimits_{t=[N\tau]}^{[N(\tau+h)]} Y_t\Big|^{\alpha-\varepsilon} \;\le\; Ch^{1+\gamma}N^{(\bar d_+ + (1/\alpha))(\alpha-\varepsilon)}. \tag{66}$$

Using the representation of the sum in the l.h.s. of (66) as a discrete stochastic integral, together with (58) and Lemma 1.2, (32), (34), boundedness of $(q_A(t))$, the proof of (66) reduces to

$$\sum_{s=1}^{N}\Big(\sum_{t=[N\tau]}^{[N(\tau+h)]}|\psi^+_{t-s}|\Big)^{\alpha'} \;\le\; Ch^{1+\gamma}N^{1+\bar d_+\alpha'}, \tag{67}$$

$$\sum_{s=0}^{\infty}\Big(\sum_{t=[N\tau]}^{[N(\tau+h)]}|\psi^-_{t+s}\psi^+_t/\psi^-_t|\Big)^{\alpha'} \;\le\; Ch^{1+\gamma}N^{1+\bar d_+\alpha'}, \tag{68}$$

where we use the same notation as in (63).

Let us check (68); the proof of (67) follows similarly and is omitted. By using (64)-(65), integral approximation to the sums on the l.h.s. of (68) and a change of variables, (68) follows from

$$I(\tau,h) := \int_0^\infty \Big(\int_\tau^{\tau+h} (t+s)^{\bar d_- -1} t^{\bar d_+ - \bar d_-}\,\mathrm{d}t\Big)^{\alpha'} \mathrm{d}s \ \le\ Ch^{1+\gamma}.$$

If $\bar d_+ \ \ge\ \bar d_-$, then $I(\tau,h) \ \le\ \int_0^\infty \Big(\int_\tau^{\tau+h}(t+s)^{\bar d_- -1}\,\mathrm{d}t\Big)^{\alpha'} \mathrm{d}s \ \le\ \int_0^\infty \Big(\int_0^h(t+s)^{\bar d_- -1}\,\mathrm{d}t\Big)^{\alpha'} \mathrm{d}s = O(h^{1+\alpha'\bar d_-})$ and (68) holds with $\gamma = \alpha'\bar d_- > 0$. Let $\bar d_+ \ge \bar d_-$, then $I(\tau,h) \le I(0,h) = O(h^{1+\alpha'\bar d_+})$ and (68) again holds with $\gamma = \alpha'\bar d_+ > 0$. This proves (66) and concludes the proof of Theorem 1.1. □

Proof of Theorem 1.2. We shall prove part (i) only; the proof of (ii) and (iii) is similar and will be omitted. Similary as in the proof of Theorem 1.1, we restrict ourselves to the proof of one-dimensional convergence (at $\tau = 1$) in (31), for $1 < \alpha < 2$. With Lemma 1.2 (33) in mind, write $S_N := \sum_{i=1}^N = \sum_{i=1}^3 S_{i,N}$, where

$$\begin{aligned}
S_{1,N} &:= \sum_{s=1}^N \varepsilon_s Q_B(s) \sum_{t=s}^N \psi_{t-s}(-\bar d_+),\\
S_{2,N} &:= \sum_{s=1}^N \varepsilon_s Q_B(s) \sum_{t=s}^N \psi_{t-s}(-\bar d_+)\Big(\frac{d_{t-1}}{\bar d_+} - 1\Big),\\
S_{3,N} &:= \sum_{s=1}^N \varepsilon_s \sum_{t=s}^N \Theta_B(t,s),\\
S_{4,N} &:= \sum_{s\le 0} \varepsilon_s \sum_{t=1}^N (b\star g)_{t-s}(t).
\end{aligned}$$

It suffices to show

$$N^{-\bar d_+ -(1/\alpha)} S_{1,N} \ \Rightarrow\ c_B^+ J^+_{\bar d_+}(1), \tag{69}$$

$$E|S_{i,N}|^{\alpha-\varepsilon} \ =\ o(N^{(\alpha-\varepsilon)(\bar d_+ +(1/\alpha))}),\ i = 2,3,4, \tag{70}$$

for some $\varepsilon > 0$. Consider (70) for $i = 2$. Using (52) and boundedness of $Q_B(s)$, this follows from

$$N^{\pm\varepsilon/\alpha} \sum_{s=1}^N \Big| \sum_{t=s}^N \psi^+_{t-s}(d_{t-1} - \bar d_+)\Big|^{\alpha\pm\varepsilon} = o(N^{(\alpha\pm\varepsilon)(\bar d_+ +(1/\alpha))}).$$

The proof of the last relation uses the fact that $(d_{t-1} - \bar d_+,\ t \in \mathbb{Z})$ is averaging at $+\infty$, with zero mean value, and is similar to the proof of (63) above (see also

PSV [51],proof of Theorem 3.1). We omit the details for brevity. Relation (70) for $i = 3, 4$ follows similarly from Lemma 1.2 (33 - 35).

Relation (69) can be proved using the scheme of discrete stochastic integrals as in proof of Theorem 1.1. To that end, write $N^{-\bar{d}_+ - (1/\alpha)} S_{1,N} = \int h_N \mathrm{d} Z_{+,N}$, $c_B^+ J_{\bar{d}_+}^+(1) = \int h \mathrm{d} Z_+$, where Z_+ is defined in Lemma 1.1 (27), $h(x) := c_B^+ \int_x^1 (y - x)_+^{\bar{d}_+ - 1} \mathrm{d}y$, $h_N(x) := N^{-\bar{d}_+} \sum_{t=s}^N \psi_{t-s}(-\bar{d}_+)$ $(x \in ((s-1)/N, s/N], s = 1, 2, ..., N), :=$ 0 elsewhere, and the discrete random measure

$$Z_{+,N}(x', x''] := N^{-1/\alpha} \sum_{x'N < s \le x''N} Q_B(s) \varepsilon_s$$

coverges to Z_+ in the sense of (60) (see Lemma 1.1, (27)). The remaining details of the proof of (69) are analogous as in the proof of Theorem 1.1. Theorem 1.2 is proved. $\square$

Chapter 2

The Increment Ratio statistic under deterministic trends

2.1 Main result

The Increment Ratio (IR) statistic was introduced by Surgailis et al. [61]. It is defined for given observations $X_1, \dots, X_N$ as the sum of ratios of partial sums

$$IR := \frac{1}{N-3m} \sum_{k=0}^{N-3m-1} \frac{\left| \sum_{t=k+1}^{k+m} (X_{t+m} - X_t) \ + \ \sum_{t=k+m+1}^{k+2m} (X_{t+m} - X_t) \right|}{\left| \sum_{t=k+1}^{k+m} (X_{t+m} - X_t) \right| + \left| \sum_{t=k+m+1}^{k+2m} (X_{t+m} - X_t) \right|}, \quad (1)$$

with the convention $0/0 = 1$; here $m = 1, 2, \dots$ is bandwidth parameter (see [17] generalization of the IR statistics). The IR statistic can be used for testing non-parametric hypotheses for d-integrated $(-1/2 < d < 5/4)$ behavior of time series $(X_t, 1 \le t \le N)$, including short memory $(d = 0)$, (stationary) long memory $(0 < d < 1/2)$ and unit roots $(d = 1)$. If partial sums process of X_t's behaves asymptotically as an (integrated) fractional Brownian motion with parameter $H = d + 1/2$, the IR statistic converges (as $N, m, N/m \to \infty$) to the expectation

$$\Lambda(d) := \mathrm{E}\left[\frac{|Z_1 + Z_2|}{|Z_1| + |Z_2|}\right], \quad (2)$$

where (Z_1, Z_2) have a jointly Gaussian distribution with zero mean, unit variances, and the covariance

$$\varrho(d) := \mathrm{cov}(Z_1, Z_2) \ = \ \frac{-9^{d+.5} + 4^{d+1.5} - 7}{2(4 - 4^{d+.5})} \quad (3)$$

The function $\Lambda(d)$ in (2) is strictly increasing in the interval $(-1/2, 3/2)$ and is explicitly written in [61]. For Gaussian observations $\{X_t\}$, in [61], a rate of decay of the bias $\mathrm{E}IR - \Lambda(d)$ and a central limit theorem (see below) in the region

$-1/2 < d < 5/4$ are obtained. The corresponding IR test rejecting the null hypothesis $\mathrm{H}_0 : d = d_0$ in favor of $\mathrm{H}_1 : d \neq d_0$ has the critical region

$$|IR - \Lambda(d_0)| \;>\; z_{\alpha/2}\sigma(d_0)\sqrt{\frac{m}{N-3m}}, \tag{4}$$

where z_α is a standard normal quantile, and the function $\sigma(d)$ is numerically tabulated in Stoncelis and Vaičiulis [57] (see also the graph in [61]). A simulation study in [61] shows that the IR test for short memory ($d = 0$) against stationary long–memory alternatives ($0 < d < 1/2$) has good size and power properties and is robust against changes in mean, slowly varying trends, and nonstationarities.

We assume that the observed sample comes from the model

$$X_t \;=\; g_{N,t} + X_t^0 \qquad (1 \le t \le N), \tag{5}$$

where $g_{N,t}$ is a slowly varying deterministic trend, and $\{X_t^0\}$ is a stationary/stationary increment Gaussian process. We want to study the impact of the trend on the limit distribution of the IR statistic. In particular, we obtain conditions on the trend and the stationary component guaranteeing that the limit distribution of the IR statistic under the model (5) follows the same central limit theorem as in the absence of the trend.

Let us recall the main result of [61]. For brevity, we formulate it under slightly stronger assumptions than in [61].

Assumption A $\{X_t^0\}$ is a zero mean stationary Gaussian sequence with spectral density $f(x), x \in [-\pi, \pi]$ of the form

$$f(x) \;=\; |x|^{-2d}\left(c_0 + O(|x|^\beta)\right) \qquad (x \to 0), \tag{6}$$

where $c_0 > 0, 0 < \beta < 2d+1, d \in (-1/2, 1/2)$ are some constants. Moreover, $f(x)$ is differentiable on $(0, \pi)$ and $|f'(x)| \le C|x|^{-1-2d}$, where $C > 0$ is some positive constant.

Assumption B The differences $\{X_t^0 - X_{t-1}^0\}$ form a zero mean stationary Gaussian sequence whose spectral density satisfies

$$f(x) \;=\; |x|^{2-2d}\left(c_0 + O(|x|^\beta)\right) \qquad (x \to 0) \tag{7}$$

for some constants $c_0 > 0, 0 < \beta < 2d - 1, 1/2 < d < 5/4$. Moreover, $f(x)$ is differentiable on $(0, \pi)$ and $|f'(x)| \le C|x|^{1-2d}$, where $C > 0$ is some positive constant.

Let IR^0 denote the IR statistic in (1) with $X_t = X_t^0$.

Theorem 2.1 [see [61]] *Suppose that $\{X_t^0\}$ satisfies Assumption A or Assumption B. Then, as $N, m, N/m \to \infty$,*

$$\begin{aligned} \mathrm{E}IR^0 - \Lambda(d) &= O(m^{-\beta}), & (8)\\ \mathrm{E}\left(IR^0 - \Lambda(d)\right)^2 &= o(1), & (9)\\ (N/m)^{1/2}(IR^0 - \mathrm{E}IR^0) &\Rightarrow \mathcal{N}(0, \sigma^2(d)), & (10) \end{aligned}$$

where $\sigma(d) > 0$ is defined in [61], and $\Rightarrow$ denotes the convergence in distribution.

Introduce the following notation:

$$\begin{aligned} G_m(k) &:= V_m^{-1}\Big|\sum_{t=k+1}^{k+m}(g_N(t+m) - g_N(t))\Big|, \\ \overline{G_m^i} &:= \frac{1}{N-2m}\sum_{k=0}^{N-2m-1} G_m^i(k) \qquad (i = 1, 2), \\ V_m^2 &:= \mathrm{E}\Big(\sum_{t=1}^{m}(X_{t+m}^0 - X_t^0)\Big)^2. \end{aligned}$$

Under Assumptions A or B, for any $d \in (-1/2, 5/4), d \neq 1/2$

$$V_m^2 \sim c(d)m^{1+2d} \qquad (m \to \infty), \tag{11}$$

where $c(d) > 0$ is a constant which is explicitly written in [61], (Eqs. (2.20), (2.22)).

The main result of the Chapter is Theorem 2.2, which gives a bound of the bias of the IR statistic and a central limit theorem for the centered IR statistic for trended observations as in (5).

Theorem 2.2 [see [18]] *Suppose that observations X_t, $t = 1, \dots, N$, follow the model (5). Let N and $m = m(N)$ both tend to ∞ so that $m = o(N)$.*

(i) *Let $\{X_t^0\}$ satisfy Assumption A or Assumption B. Then*

$$\mathrm{E}IR - \Lambda(d) = O\left(\max\left(m^{-\beta}, \overline{G_m^2}\right)\right). \tag{12}$$

In addition, if $\overline{G_m^1} \to 0$, then

$$\mathrm{E}\left(IR - \Lambda(d)\right)^2 \to 0. \tag{13}$$

(ii) *Let* $\{X_t^0\}$ *satisfy Assumption A or Assumption B. If*

$$\overline{G_m^i} \; = \; o\left((m/N)^{1/2}\right) \qquad (i = 1, 2), \tag{14}$$

then

$$(N/m)^{1/2}\,(IR \; - \mathrm{E}IR) \;\Rightarrow\; \mathcal{N}\left(0, \sigma^2(d)\right), \tag{15}$$

where $\sigma^2(d)$ *is the same as in Theorem 2.1.*

Corollary 2.1 Let $\{X_t^0\}$ satisfy conditions of Theorem 2.1, $m^{-\beta} = o((m/N)^{1/2})$ and $g_N(t)$ satisfy (14). Then the IR test of the hypothesis $\mathrm{H}_0 : d = d_0,\;\; d_0 \in (-1/2, 5/4),\; d_0 \neq 1/2$ under the model (5) follows the same asymptotic confidence intervals in (4) as in the absence of trend.

2.2 Monte Carlo simulations

In this section, we examine the finite sample performance of the IR and V/S tests by means of a sample Monte Carlo study. We consider testing

$$H_0 : \; X_t \text{ satisfies Assumption A with } d = 0 \tag{16}$$

versus

$$H_1 : \; X_t \text{ satisfies Assumption A with } d \in (0, 1/2). \tag{17}$$

Applying the IR test, we reject H_0 if $IR - \Lambda(0) > z_\alpha \sigma(0)\sqrt{m/(N-3m)}$.

Now we describe the testing procedure proposed by Giraitis *et al.* [31]. The V/S statistic is defined as $V_N/\hat{s}^2_{N,q}$, where

$$V_N = \frac{1}{N^2}\Big\{\sum_{k=1}^{N}\Big(\sum_{j=1}^{k}(X_j - \bar{X}_N)\Big)^2 - \frac{1}{N}\Big(\sum_{k=1}^{N}\sum_{j=1}^{k}(X_j - \bar{X}_N)\Big)^2\Big\} \tag{18}$$

is the estimator of the variance of the partial sum process $S_k^* = \sum_{j=1}^{k}(X_j - \bar{X}_N)$, $k = 1, ..., N$, and $\hat{s}^2_{N,q}$ is the estimator of $\sigma^2 = \sum_{j=-\infty}^{\infty}\mathrm{cov}(X_0, X_j)$ defined by

$$\hat{s}^2_{N,q} = \hat{\gamma}_0 + 2\sum_{j=1}^{q}\Big(1 - \frac{j}{q+1}\Big)\hat{\gamma}_j, \quad \hat{\gamma}_j = \frac{1}{N}\sum_{k=1}^{N-j}(X_k - \bar{X}_N)(X_{k+j} - \bar{X}_N), \quad 0 \le j \le n.$$

The V/S test rejects the null hypothesis (16) of short memory in favor of long-memory alternative (17) if $V_N/\hat{s}^2_{N,q} > K_\alpha$, where K_α is α critical value of the distribution

$$F(x) = 1 + 2\sum_{k=1}^{\infty}(-1)^k e^{-2k^2\pi^2 x}.$$

The data generating process is assumed to be (5), where X_t^0 has one of the forms

$$WN: \ X_t^0 = \xi_t, \quad \xi_t \sim \text{i.i.d. } N(0,1),$$
$$AR: \ X_t^0 = 0.5X_{t-1}^0 + \xi_t, \quad \xi_t \sim \text{i.i.d. } N(0,1),$$
$$FI: \ X_t^0 = (1-L)^{-1/4}\xi_t, \quad \xi_t \sim \text{i.i.d. } N(0,1),$$

with L being the backward shift. The standart normal random variables are generated by the acceptance-complement method [24]. The realizations of the first-order AR process are generated recursively from the autoregressive equation with initial value $X_0^0 = 0$. The FI(1/4) processes are simulated using the truncated moving-average expansion [6]. For Monte Carlo simulations, we choose the deterministic trends g_N (discussed in Introduction Examples 1-3):

$$T_1: g_N(t) = \frac{1}{4}(t+N)^{1/4},$$
$$T_2: g_N(t) = \frac{1}{4}\sin(\frac{2\pi t}{N}),$$
$$T_3: g_N(t) = \frac{1}{10}I\Big\{\frac{N}{4} < t \le \frac{N}{2}\Big\} + \frac{2}{10}I\Big\{\frac{N}{2} < t \le \frac{3N}{4}\Big\} + \frac{3}{10}I\Big\{\frac{3N}{4} < t \le N\Big\},$$

where $I\{A\}$ is the indicator of a set A.

We have simulated 10000 replications of each series of length $N = 1000$. Tables 1-3 show the percentage of replications in which the rejection of the short-memory null hypothesis ($d = 0$) was observed. The choice of bandwidths m and q in the range $[N^{1/3}] \le m, q \le [N^{1/2}]$, as a reasonable compromise between size and power distortions for the V/S and IR tests, were suggested in [31] and [61], respectively.

Table 1. Frequency of rejection of the null hypothesis of short memory for "pure stochastic" processes. Test size 5%.

	WN	AR	FI
IR, m=10	4.7	19.0	55.4
IR, m=30	4.8	7.6	29.2
V/S, q=10	4.2	7.0	62.0
V/S, q=30	3.6	4.4	38.8

Table 2. Frequency of rejection of the null hypothesis of short memory for short-memory processes perturbed by deterministic trends. Test size 5%.

	$WN+T_1$	$WN+T_2$	$WN+T_3$	$AR+T_1$	$AR+T_2$	$AR+T_3$
IR, m=10	4.7	5.0	4.7	19.1	19.1	19.0
IR, m=30	4.8	7.1	5.5	7.6	8.3	7.6
V/S, q=10	4.2	99.9	71.0	14.3	74.5	26.4
V/S, q=30	3.6	99.8	61.1	9.5	65.2	18.1

Table 3. Frequency of rejection of the null hypothesis of short memory for long-memory processes perturbed by deterministic trends. Test size 5%.

	$FI+T_1$	$FI+T_2$	$FI+T_3$
IR, m=10	55.4	55.7	55.8
IR, m=30	29.3	31.0	29.4
V/S, q=10	65.6	81.7	69.0
V/S, q=30	42.2	66.6	46.9

Table 1 (see also Tables 1 and 6 in [61]) indicates that the IR and V/S tests have similar good sizes when $X_1, ..., X_N$ is a sample from i.i.d. Gaussian sequence. The results in Table 1 (see also Tables 1 and 5 in [61]) also indicate that, in the absence of a trend, the V/S test has better size and power than the IR test when $X_1, ..., X_N$ is the observed sample from a stationary weakly/strongly dependent Gaussian sequence.

Finally, from Tables 2-3 (see also Tables 2 and 3 in [61]) we may conclude that the IR test is far more robust to deterministic trends than the V/S test. We want to note once more, that this property is ambivalent. It is useful for testing hypothesis about the unknown memory parameter d. On the other hand, the robustness of the IR statistic to the trends restricts its usage for testing hypotheses about trends.

2.3 Proof of the Theorem 2.2

Define

$$\psi(x,y) := \frac{|x+y|}{|x|+|y|}, \qquad x,y \in \mathbb{R}. \tag{19}$$

Lemma 2.1 [see [18]] *Suppose that* (U,V) *is a zero mean Gaussian vector with* $\mathrm{E}U^2 = \mathrm{E}V^2 = 1$ *and* $\varrho = \mathrm{Cov}\,(U,V)\,,\ |\varrho| < 1$. *Then, for any reals* a *and* b,

$$|\mathrm{E}\,[\psi(a+U,b+V) - \psi(U,V)]| \le C_1(\varrho)\left(a^2+b^2\right), \tag{20}$$
$$\mathrm{E}\,[\psi(a+U,b+V) - \psi(U,V)]^2 \le C_2(\varrho)\left(a^2+b^2\right), \tag{21}$$
$$|\mathrm{E}\,[U\psi(a+U,b+V)]| \le C_3(\varrho)\left(|a|+|b|\right), \tag{22}$$
$$|\mathrm{E}\,[V\psi(a+U,b+V)]| \le C_3(\varrho)\left(|a|+|b|\right), \tag{23}$$

where the constants $C_i(\varrho) > 0$, $i = 1,2,3$ *depend only on* $|\varrho| < 1$.

Proof. Let us prove (20). For $a^2+b^2 > 1$, the inequality in (20) is trivial, since the l.h.s. in (20) does not exceed 1. Let $a^2+b^2 \le 1$. Then

$$\mathrm{E}\,[\psi(a+U,b+V) - \psi(U,V)] = \mathrm{E}\left[\psi(U,V)\left(\mathrm{e}^{-h(U,V,a,b)} - 1\right)\right] \tag{24}$$
$$= W_1(a,b) + W_2(a,b), \tag{25}$$

where

$$\begin{aligned}
h(x,y,a,b) &:= \left[2(1-\varrho^2)\right]^{-1}\left(2x(\varrho b-a)+2y(\varrho a-b)+(a^2-2\varrho ab+b^2)\right),\\
W_1(a,b) &:= -\mathrm{E}\left[\psi(U,V)h(U,V,a,b)\right],\\
W_2(a,b) &:= \mathrm{E}\left[\psi(U,V)\left(\mathrm{e}^{-h(U,V,a,b)}-1+h(U,V,a,b)\right)\right].
\end{aligned}$$

Some of these expectations can be explicitly calculated:

$$\begin{aligned}
&\mathrm{E}\left[\psi(U,V)\right]=2P_\varrho+Q_\varrho, &(26)\\
&\mathrm{E}\left[U\psi(U,V)\right]=\mathrm{E}\left[V\psi(U,V)\right]=0, &(27)\\
&\mathrm{E}\left[UV\psi(U,V)\right]=2\varrho P_\varrho+(1+\varrho)Q_\varrho, &(28)\\
&\mathrm{E}\left[U^2\psi(U,V)\right]=\mathrm{E}\left[V^2\psi(U,V)\right] \;=\; 2P_\varrho+(1+\varrho)Q_\varrho, &(29)
\end{aligned}$$

where

$$\begin{aligned}
P_\varrho &:= (1/\pi)\arctan\sqrt{(1+\varrho)/(1-\varrho)},\\
Q_\varrho &:= (1/\pi)\sqrt{(1+\varrho)/(1-\varrho)}\log\sqrt{(1+\varrho)/(1-\varrho)}.
\end{aligned}$$

The proof of (26) is given in [61], the remaining relations (27)-(29) can be proved similarly. Using (26) and (27), we obtain

$$|W_1(a,b)| \;=\; \frac{a^2-2\varrho ab+b^2}{2(1-\varrho^2)}\mathrm{E}\left[\psi(U,V)\right] \;\le\; \frac{a^2+b^2}{1-\varrho^2}. \tag{30}$$

Next, we have

$$\begin{aligned}
|W_2(a,b)| \;&\le\; (1/2)\mathrm{E}\left[\psi(U,V)h^2(U,V,a,b)I\left(h(U,V,a,b)\ge 0\right)\right]\\
&\quad +(1/2)\mathrm{E}\left[\psi(U,V)h^2(U,V,a,b)\mathrm{e}^{-h(U,V,a,b)}I\left(h(U,V,a,b)<0\right)\right]\\
&=:\; W_2'(a,b)+W_2''(a,b),
\end{aligned}$$

where we used inequality

$$|\mathrm{e}^z-1-z| = \left|\int_0^z (e^u-1)\,\mathrm{d}u\right| \le \frac{z^2}{2}\max\left(\mathrm{e}^z,1\right), \quad z\in\mathbb{R}.$$

Using (26)-(28), we get

$$W_2'(a,b) \;\le\; (1/2)\mathrm{E}\left[h^2(U,V,a,b)\right] \;\le\; \frac{9(a^2+b^2)}{4(1-\varrho^2)^2}. \tag{31}$$

Next, we have

$$\begin{aligned}
W_2''(a,b) \;&\le\; (1/2)\mathrm{E}\left[h^2(U,V,a,b)\mathrm{e}^{-h(U,V,a,b)}\right]\\
&=\; (1/2)\mathrm{E}\left[h^2(U+a,V+b,a,b)\right] \le \frac{7(a^2+b^2)}{2(1-\varrho^2)^2}. \qquad (32)
\end{aligned}$$

Now (20) follows from (25)-(32), with $C_1(\varrho) = 7/(1-\varrho^2)^2$.

Since the proofs of the inequalities in (22) and (23) are similar, we consider the inequality (22) only. For $|a|+|b| > 1$, (22) is trivial, since the l.h.s. does not exceed $E|U| \le 1$. Let $|a|+|b| \le 1$. We have

$$\mathrm{E}\left[U\psi(U+a,V+b)\right] = -a\mathrm{E}\left[\psi(U,V)\mathrm{e}^{-h(U,V,a,b)}\right] + \mathrm{E}\left[U\psi(U,V)\mathrm{e}^{-h(U,V,a,b)}\right] \tag{33}$$

The absolute value of the first term on the r.h.s. in (33) does not exceed $|a|\mathrm{E}\left[\mathrm{e}^{-h(U,V,a,b)}\right] \le (|a|+|b|)$. Next, $\mathrm{E}\left[U\psi(U,V)\mathrm{e}^{-h(U,V,a,b)}\right] = \mathrm{E}\left[U\psi(U,V)\left(\mathrm{e}^{-h(U,V,a,b)}-1\right)\right]$ by (27). The last expectation is equal to the sum

$$\begin{aligned}&-\mathrm{E}\left[Uh(U,V,a,b)\psi(U,V)\right] + \mathrm{E}\left[U\psi(U,V)\left(\mathrm{e}^{-h(U,V,a,b)} - 1 + h(U,V,a,b)\right)\right]\\ &=: \tilde{W}_1(a,b) + \tilde{W}_2(a,b).\end{aligned}$$

Equalities (26)-(29) imply

$$\left|\tilde{W}_1(a,b)\right| = |2aP_\varrho + (a+b)Q_\varrho| \;\le\; 2(|P_\varrho| + |Q_\varrho|)(|a|+|b|).$$

The term $\tilde{W}_2(a,b)$ is estimated similarly to $W_2(a,b)$ above. Write

$$\begin{aligned}\left|\tilde{W}_2(a,b)\right| &\le \frac{1}{2}\mathrm{E}\left[|U|h^2(U,V,a,b)\right] + \frac{1}{2}\mathrm{E}\left[|U+a|h^2(a+U,b+V,a,b)\right]\\ &\le \frac{1}{2}\left(\mathrm{E}U^2\right)^{1/2}\left(\mathrm{E}h^4(U,V,a,b)\right)^{1/2}\\ &+ \frac{1}{2}\left(\mathrm{E}(U+a)^2\right)^{1/2}\left(\mathrm{E}h^4(U+a,V+b,a,b)\right)^{1/2}.\end{aligned}$$

One can easily verify that

$$\begin{aligned}\mathrm{E}h^4(U,V,a,b) &\le \frac{69\,(|a|+|b|)^4}{(1-\varrho^2)^4},\\ \mathrm{E}h^4(U+a,V+b,a,b) &\le \frac{69\,(|a|+|b|)^4}{(1-\varrho^2)^4}.\end{aligned}$$

Hence $\left|\tilde{W}_2(a,b)\right| \le (11/((1-\varrho^2)^2))(|a|+|b|)$, thereby proving (22).

Finally, let us to prove (21). For $2(|a|+|b|) > 1$, the desired inequality holds with $C_2(r) = 1/(\pi(1-\varrho^2))$. Consider the case $2(|a|+|b|) \le 1$. Let $S_\nu = \{(x,y) : x^2+y^2 \le \nu\}$, where $\nu := \{2(|a|+|b|)\}^{1/2}$. Write

$$\begin{aligned}\mathrm{E}\left[\psi(a+U,b+V) - \psi(U,V)\right]^2 &= \mathrm{E}\left[\{\psi(a+U,b+V)-\psi(U,V)\}\,I\{(U,V)\in S_\nu\}\right]^2\\ &+ \mathrm{E}\left[\{\psi(a+U,b+V)-\psi(U,V)\}\,I\{(U,V)\notin S_\nu\}\right]^2\\ &=: J_1(a,b) + J_2(a,b).\end{aligned}$$

The bound of $J_1(a,b)$ is trivial:

$$J_1(a,b) \le \frac{\mathrm{mes}\,(S_\nu)}{2\pi(1-\varrho^2)} = \frac{|a|+|b|}{1-\varrho^2},$$

where $\mathrm{mes}(S_\nu)$ denotes the area of S_ν. To evaluate $J_2(a,b)$, we use the inequality

$$|\psi(a+x,b+y)-\psi(x,y)| \le \frac{4(|a|+|b|)}{\max\{|x|,|y|\}}, \quad (x,y)\in\mathbb{R}^2\setminus S_\nu, \tag{34}$$

which will be proved later. Then

$$\begin{aligned} J_2(a,b) &\le \frac{8(|a|+|b|)^2}{\pi(1-\varrho^2)}\int_0^{2\pi}\frac{\mathrm{d}\varphi}{\max\{\sin^2\varphi,\cos^2\varphi\}}\int_\nu^\infty R^{-1}e^{-\frac{R^2(1-\varrho\sin(2\varphi))}{2(1-\varrho^2)}}\,\mathrm{d}R \\ &\le \frac{4(|a|+|b|)^2}{\pi(1-\varrho^2)}\int_0^{2\pi}\frac{\ln\left(1+(1/\tilde{\nu})\right)\mathrm{d}\varphi}{\max\{\sin^2\varphi,\cos^2\varphi\}}, \end{aligned}$$

where $\tilde{\nu} := (1/2)\nu^2(1-\varrho\sin(2\varphi)/(1-\varrho^2)$. Hence, using inequality $\ln(1+x)<x$ for any $x>0$, one obtains $J_2(a,b)\le 32(|a|+|b|)/(\pi(1-|\varrho|))$, and thus (21) holds with $C_2(\varrho)=12/(1-|\varrho|)$.

It remains to prove (34). Assume for simplicity $a>0$ and $b>0$. The left-hand side of (34) is equal to zero in the region

$$(x,y)\in\{(0,+\infty)\times(0,+\infty)\cup(-\infty,-a)\times(-\infty,-b)\cup(-a,0)\times(-b,0)\}.$$

If $(x,y)\in(-\infty,0)\times(0,+\infty)\setminus S_\nu$, it is convenient to split the last region into four disjoint parts. Let first $(x,y)\in(-a,0)\times(0,+\infty)\setminus S_\nu$. Then $|\psi(a+x,b+y)-\psi(x,y)| = (-2x)/(y-x) \le 2(a+b)/y$. Next, let the inequalities $x<-a$ and $y+x>0$ be satisfied. Then

$$|\psi(a+x,b+y)-\psi(x,y)| = \frac{2(a(y+b)-b(x+a))}{((y+b)-(x+a))\,(y-x)} \le \frac{2\max\{a,b\}}{y} \le \frac{2(a+b)}{y}.$$

If $x+a<0$, $-(a+b)<y+x<0$ and $x^2+y^2>\nu$, then $y-a+b>0$. Using this, we get

$$\begin{aligned} |\psi(a+x,b+y)-\psi(x,y)| &= \frac{2\,|(y^2-x^2)+(yb-xa)|}{((y+b)-(x+a))(y-x)} \\ &\le \frac{2\,|y+x|}{(y+b)-(x+a)} + \frac{2\,|\max\{a,b\}|}{(y+b)-(x+a)} \le \frac{4(a+b)}{-x}. \end{aligned}$$

In the case $x<-(a+b)$ and $0<y<-x-a-b$, we have

$$|\psi(a+x,b+y)-\psi(x,y)| = \frac{2a(y+b)-2b(x+a)}{((y+b)-(x+a))(y-x)} \le \frac{2(a+b)}{-x}.$$

Using a similar argument, one can prove (34) for $(x, y) \in (0, +\infty) \times (-\infty, 0) \setminus S_\nu$. Assume now that $-b < y < 0$, $-\infty < x < -y - a - b$ and $(x, y) \notin S_\nu$. Then

$$|\psi(a + x, b + y) - \psi(x, y)| = \frac{2(y + b)}{(y + b) - (x + a)} \leq \frac{2(a + b)}{-x},$$

where one can check the last inequality directly. Finally, in the case $-a < x < 0$, $y < -x - a - b$, and $(x, y) \notin S_\nu$, we have $|\psi(a + x, b + y) - \psi(x, y)| \leq -2(a + b)/y$. Lemma 2.1 is proved. □

Proof of Theorem 2.2. Note that, for fixed $m \in \mathbb{N}$, random variables

$$Y_m(j) := V_m^{-1} \sum_{t=j+1}^{j+m} (X^0_{t+m} - X^0_t), \quad j \in \mathbb{Z}, \tag{35}$$

form a stationary Gaussian sequence with zero mean and unit variance.

(i). Let us prove (12). With (8) in mind, it suffices to show that

$$\left|\mathrm{E}IR - \mathrm{E}IR^0\right| = O\left(\overline{G^2_m}\right). \tag{36}$$

Applying inequality (20) with $U = Y_m(j)$, $V = Y_m(j)$, $a = G_m(j)$, and $b = G_m(j + m)$, one obtains

$$\begin{aligned}
\left|\mathrm{E}IR - \mathrm{E}IR^0\right| \;&\leq\; \frac{1}{N - 3m} \sum_{j=0}^{N-3m-1} \left|\mathrm{E}\psi\left(Y_m(j) + G_m(j), Y_m(j + m) + G_m(j + m)\right)\right. \\
&\qquad\qquad \left. -\mathrm{E}\psi\left(Y_m(j), Y_m(j + m)\right)\right| \\
&\leq\; \frac{C_1(\varrho_m)}{N - 3m} \sum_{j=0}^{N-3m-1} \left(G^2_m(j) + G^2_m(j + m)\right) \leq C\overline{G^2_m},
\end{aligned}$$

for some constant $C < \infty$ independent of m for all m large enough. Here, $\varrho_m := \mathrm{E}\left[Y_m(j)Y_m(j + m)\right]$ does not depend on j and $\lim_{m\to\infty} \varrho_m = \varrho(d) \in (-1, 1)$ depends only on d (see [61]).

The proof of (13) follows from

$$\mathrm{E}\left(IR - \mathrm{E}IR^0\right)^2 = O\left(\overline{G^1_m}\right) \tag{37}$$

and (9). Similarly as in the proof of (36) above, we apply inequality (21). The left-hand side of (37) does not exceed

$$\begin{aligned}
&\frac{1}{N - 3m} \sum_{j=0}^{N-3m-1} \mathrm{E}\Big[\psi\left(Y_m(j) + G_m(j), Y_m(j + m) + G_m(j + m)\right) - \\
&- \;\; \psi\left(Y_m(j), Y_m(j + m)\right)\Big]^2 \leq \frac{C_2(\varrho_m)}{N - 3m} \sum_{j=0}^{N-3m-1} \left(|G_m(j)| + |G_m(j + m)|\right) \;\leq\; C\overline{G^1_m}.
\end{aligned}$$

(ii) We follow the proofs in [61] and [7]. Rewrite IR and IR^0 as

$$IR = \frac{1}{N-3m}\sum_{j=0}^{N-3m-1}\eta_m(j), \qquad IR^0 = \frac{1}{N-3m}\sum_{j=0}^{N-3m-1}\eta^0_m(j),$$

where

$$\begin{aligned}\eta_m(j) &:= \psi\left(Y_m(j)+G_m(j), Y_m(j+m)+G_m(j+m)\right),\\ \eta^0_m(j) &:= \psi\left(Y_m(j), Y_m(j+m)\right).\end{aligned}$$

Clearly, relation (15) follows from (10) and

$$\mathrm{var}\left(\sum_{j=0}^{N}(\eta_m(j)-\eta^0_m(j))\right) = o(Nm). \tag{38}$$

Introduce random variables

$$\xi_{1m}(j) := Y_m(j), \quad \xi_{2m}(j) := (1-\varrho_m^2)^{-1/2}(Y_m(j+m)-\varrho_m Y_m(j)). \tag{39}$$

For any given $j = 0,1,2,\ldots, m \in \mathbb{N}$, random vector $(\xi_{1m}(j), \xi_{2m}(j))$ has a standard Gaussian distribution in $\mathbb{R}^2$, i.e. $\xi_{1m}(j)$ and $\xi_{2m}(j)$ are independent standard normal random variables. Define

$$g^0_m(x,y) := \psi\left(x, \varrho_m x + (1-\varrho_m^2)^{-1/2}y\right), \tag{40}$$
$$g_{j,m}(x,y) := \psi\left(x+G_m(j), \varrho_m x + (1-\varrho_m^2)^{-1/2}y + G_m(j+m)\right). \tag{41}$$

Then

$$\eta_m(j) = g_{j,m}(\xi_{1m}(j), \xi_{2m}(j)), \qquad \eta^0_m(j) = g^0_m(\xi_{1m}(j), \xi_{2m}(j)).$$

Let $H_n(x)$, $n = 0,1,2,\ldots$ denote the standard Hermite polynomials with leading coefficient 1. Write the Hermite expansion:

$$\eta_m(j) = \sum_{k,\ell\ \geq 0}\frac{c^{(m)}_{k,\ell}(j)}{k!\ell!}H_k(\xi_{0m}(j))H_\ell(\xi_{1m}(j)),$$

where

$$c^{(m)}_{k,\ell}(j) := E[g_{j,m}(\xi_{1m}(j), \xi_{2m}(j))H_k(\xi_{1m}(j))H_\ell(\xi_{2m}(j))].$$

Write

$$\eta_m(j) - \mathrm{E}\eta_m(j) = \eta'_m(j) + \eta''_m(j),$$

where

$$\eta'_m(j) := c_{1,0}^{(m)}(j)\xi_{1m}(j) + c_{0,1}^{(m)}(j)\xi_{2m}(j),$$
$$\eta''_m(j) := \sum_{k+\ell\ \geq 2} \frac{c_{k,\ell}^{(m)}(j)}{k!\ell!} H_k(\xi_{1m}(j))H_\ell(\xi_{2m}(j)).$$

Relation (38) follows from

$$\mathrm{E}\left(\sum_{j=1}^{N} \eta'_m(j)\right)^2 = o(mN) \tag{42}$$

and

$$\Sigma := \mathrm{E}\left(\sum_{j=1}^{N} \left(\eta''_m(j) - \eta^0_m(j) + \mathrm{E}\eta^0_m(j)\right)\right)^2 = o(mN). \tag{43}$$

Let us prove (43). We shall use the following inequality, due to Arcones [1], in the particular case of Gaussian vectors in $\mathbb{R}^2$.

Namely, for any integer $r \geq 1$, any standard Gaussian vectors (ξ_1, ξ_2), (ξ'_1, ξ'_2), any functions $F_i(x_1, x_2)$, $i = 1, 2$, $(x_1, x_2) \in \mathbb{R}^2$ with $\|F_i\|^2 := \mathrm{E}F_i^2(\xi_1, \xi_2) < \infty$ both having Hermite rank equal or greater than r, the following inequality holds:

$$|\mathrm{cov}\,(F_1(\xi_1, \xi_2), F_2(\xi'_1, \xi'_2))| \leq \|F_1\|\|F_2\|\bar{\varrho}^r, \tag{44}$$

where

$$\bar{\varrho} := \max\{|\mathrm{E}\xi_u\xi'_v| : u, v = 0, 1\}$$

is the maximal correlation coefficient between (ξ_1, ξ_2) and (ξ'_1, ξ'_2).

From definitions (40)-(41), we have

$$\eta''_m(j) - \eta^0_m(j) + \mathrm{E}\eta^0_m(j) = F_{j,m}\left(\xi_{1m}(j), \xi_{2m}(j)\right),$$

where

$$F_{j,m}(x_1, x_2) := g_{j,m}(x_1, x_2) - c_{0,1}^{(m)}(j)x_1 - c_{1,0}^{(m)}(j)x_2$$
$$- \mathrm{E}g_{j,m}(\xi_{1m}(j), \xi_{2m}(j)) - g^0_m(x_1, x_2) + \mathrm{E}g^0_m(\xi_{1m}(j), \xi_{2m}(j)).$$

Note that for any $j = 0, 1, 2, \ldots$ and $m \in \mathbb{N}$, the Hermite rank of $F_{j,m}$ is not less than 2. From inequality (44) we have

$$|\mathrm{cov}\,(F_{j,m}(\xi_{1m}(j), \xi_{2m}(j)), F_{\ell,m}(\xi_{1m}(\ell), \xi_{2m}(\ell)))| \leq \|F_{j,m}\|\|F_{\ell,m}\|\left(\bar{\varrho}_m(j,\ell)\right)^2 \tag{45}$$

and, therefore,

$$\Sigma \leq \sum_{k,l=1}^{N} \|F_{j,m}\|\|F_{\ell,m}\|\left(\bar{\varrho}_m(j,\ell)\right)^2, \tag{46}$$

where

$$\bar{\varrho}_m(j,\ell) \;:=\; \max\left\{|\mathrm{E}\xi_{u,m}(j),\xi_{v,m}(\ell)|:\; u,v=1,2\right\}.$$

The quantity $\bar{\varrho}_m(j,\ell)=\bar{\varrho}_m(j-l)$ depends on $j-\ell$ and is estimated in Lemma 5.1 of [61]; in particular,

$$\sum_{j\in\mathbb{Z}}(\bar{\varrho}_m(j))^2 \;\le\; Cm \tag{47}$$

for some constant $C=C(d)>0$ independent of m. Using the fact that $\|F_{j,m}\|\le 4$ is bounded, from (46) and (47) one obtains

$$\begin{aligned}
\Sigma \;&\le\; 4N\sum_{k=1}^{N}\|F_{j,m}\|\sum_{j\in\mathbb{Z}}\left(\bar{\varrho}_m(j,\ell)\right)^2\\
&\le\; 4mN\left(N^{-1}\sum_{k=1}^{N}\|F_{j,m}\|\right)\\
&\le\; 4mN\left(N^{-1}\sum_{k=1}^{N}\|F_{j,m}\|^2\right)^{1/2}.
\end{aligned}\tag{48}$$

To finish the proof of (43), we use the fact that for each $j=0,1,\dots$ and $m\in\mathbb{N}$,

$$\|F_{j,m}\|^2\le C\left\{(|G_m(j)|+|G_m(j+m)|)+(|G_m(j)|+|G_m(j+m)|)^2\right\}. \tag{49}$$

Then the right-hand side of (48) does not exceed $CmN(\overline{G^1_m}+\overline{G^2_m})=o(mN)$, and (43) follows. To show (49), by definition of $F_{j,m}$ one has

$$\|F_{j,m}\|^2 \;\le\; 6\|g_{j,m}-g^0_m\|^2\;+3\left\{\left(c^{(m)}_{0,1}(j)\right)^2+\left(c^{(m)}_{1,0}(j)\right)^2\right\}, \tag{50}$$

where

$$\|g_{j,m}-g^0_m\|^2\le C\left(|G_m(j)|+|G_m(j+m)|\right), \tag{51}$$

$$\left|c^{(m)}_{0,1}(j)\right|+\left|c^{(m)}_{1,0}(j)\right|\le C\left(|G_m(j)|+|G_m(j+m)|\right), \tag{52}$$

see (40)-(41) and Lemma 2.1. Putting together (50), (51), and (52) gives (49).

Relation (42) follows from

$$\mathrm{E}\left(\sum_{j=0}^{N-3m-1}c^{(m)}_{1,0}(j)\xi_{1m}(j)\right)^2=o(mN),\;\;\mathrm{E}\left(\sum_{j=0}^{N-3m-1}c^{(m)}_{0,1}(j)\xi_{2m}(j)\right)^2=o(mN) \tag{53}$$

as $N,m,N/m\to\infty$. Since the proofs of the relations in (53) are similar, we will consider the left relation only. Applying (52), (47), and the Cauchy inequality, we obtain

$$
\begin{aligned}
\mathrm{E}\left(\sum_{j=0}^{N-3m-1} c_{1,0}^{(m)}(j)\xi_{1m}(j)\right)^2 &= \sum_{j,\ell=0}^{N-3m-1} c_{1,0}^{(m)}(j)c_{1,0}^{(m)}(\ell)\mathrm{E}\left(\xi_{1m}(j)\xi_{1m}(\ell)\right) \\
&\leq \left\{\sum_{j=0}^{N-3m-1}\left(c_{1,0}^{(m)}(j)\right)^2\right\}\left\{\sum_{j,\ell=0}^{N-3m-1}\left(\bar{\varrho}_m(j-\ell)\right)^2\right\}^{1/2} \\
&\leq CN\overline{G_m^2}\left\{N\sum_{j\in\mathbb{Z}}\left(\bar{\varrho}_m(j)\right)^2\right\}^{1/2} \\
&\leq CN\overline{G_m^2}\{Nm\}^{1/2} \;=\; o(mN),
\end{aligned}
$$

according to (14). This proves (53) and also Theorem 2.2. □

Chapter 3

Asymptotic independence of distant partial sums of linear process

3.1 Main results

Let $(\xi_t, t \in \mathbb{Z})$ be a (weak) white noise, with zero mean $E\xi_t = 0$, unit variance $E\xi_t^2 = 1$, $E\xi_t\xi_s = 0$ $(t \neq s)$. Let

$$X_t = \sum_{i=0}^{\infty} b_i \xi_{t-i}, \qquad t \in \mathbb{Z} \tag{1}$$

be a moving average (linear) process, where $b_i, i = 0, 1, \ldots$ are nonrandom weights satisfying $\sum_{i=0}^{\infty} b_i^2 < \infty$. We shall assume that b_i satisfy one of the following conditions:

(i) $b_i = L(i)i^{d-1}$, for some $0 < d < 1/2$ and a slowly varying at infinity function $L(\cdot)$;

(ii) $\sum_{i=0}^{\infty} |b_i| < \infty$, $\sum_{i=0}^{\infty} b_i \neq 0$;

(iii) $b_i = L(i)i^{d-1}$, for some $-1/2 < d < 0$ and a function $L(\cdot)$ slowly varying at infinity; moreover, $\sum_{i=0}^{\infty} b_i = 0$.

Recall that a measurable function $L(x)$, $x \geq 0$ is called slowly varying at infinity if $L(\cdot)$ is bounded on each compact interval, $L(x) > 0$ $(x > x_0)$ for some x_0 and for aech $x > 0$

$$\lim_{\lambda \to \infty} \frac{L(\lambda x)}{L(\lambda)} = 1. \tag{2}$$

We shall denote $(b_i) \in \gamma(d)$ $(0 < |d| < 1/2)$ if (b_i) satisfy (i) or (iii) with corresponding d, the notation $(b_i) \in \gamma(0)$ being equivalent to condition (ii). If $(b_i) \in \gamma(d)$ $(0 < d < 1/2)$, the process X_t in (1) is called *long memory with memory parameter* $d \in (0, 1/2)$. We also say that X_t in (1) has *short memory* if $(b_i) \in \gamma(0)$ and X_t in (1) has *negative memory with parameter* $d \in (-1/2, 0)$ if $(b_i) \in \gamma(d)$ $(-1/2 < d < 0)$. See e.g. [32], [34] concerning these definitions.

Note that the process X_t (1) is covariance stationary: the covariance

$$r(t) = EX_s X_{s+t} = \sum_{i=0}^{\infty} b_i b_{i+t} \ (t \geq 0)$$

does not depend on $s \in \mathbb{Z}$. The variance

$$A_n^2 := E\Big(\sum_{t=1}^{n} X_{t+m}\Big)^2 \ = \ \sum_{t,s=1}^{n} r(t-s) \tag{3}$$

does not depend on m and grows as n^{2d+1} with $n \to \infty$. More precisely, if $(b_i) \in \gamma(d)$ then

$$A_n^2 \sim \begin{cases} c^2(d) L^2(n) n^{2d+1}, & \text{if } 0 < |d| < 1/2, \\ c^2(0) n, & \text{if } d = 0, \end{cases} \tag{4}$$

where $c(0) = \sum_{i=0}^{\infty} b_i$ and

$$c(d) = \frac{\Gamma(d + (1/2))}{\{\Gamma(2d+1)\sin(\pi d)\}^{1/2}},$$

is a constant depending only on d. Here and below, $\sim$ stands for asymptotic equivalence: $A_n \sim B_n$ $(n \to \infty)$ if and only if $\lim_{n\to\infty} A_n/B_n = 1$.

Consider the normalized partial sums process

$$U_n(\tau) := A_n^{-1} \sum_{t=1}^{[n\tau]} X_t, \quad \tau \in [0,1]. \tag{5}$$

Under condition $(b_i) \in \gamma(d)$ and some additional weak dependence conditions on the white noise sequence (ξ_t), the partial sums process in (5) converges to a fractional Brownian motion with Hurst parameter $H = d + 1/2$:

$$U_n(\tau) \to_{D[0,1]} B_H(\tau) \tag{6}$$

Recall that $B_H(\tau), \tau \geq 0$ (a fractional Brownian motion with parameter $0 < H < 1$) is a Gaussian process with zero mean and the covariance

$$E\left[B_H(\tau) B_H(\tau')\right] \ = \ (1/2)\left(|\tau|^{2H} + |\tau'|^{2H} - |\tau - \tau'|^{2H}\right), \qquad \tau, \tau' \geq 0.$$

Notation $\to_{D[0,1]}$ stands for weak convergence of random elements in the Skorohod space $D[0,1]$ of cadlag functions $[0,1] \to \mathbb{R}^2$, with uniform topology.

Denote the vector-valued process

$$\mathbf{U}(\tau) \ := \ (U_n^{(1)}(\tau), U_n^{(2)}(\tau)), \qquad \tau \in [0,1], \tag{7}$$

where $U_n^{(1)}(\tau) := U_n(\tau)$ is given in (5) and $U_n^{(2)}(\tau)$ is the "shifted" partial sums process

$$U_n^{(2)}(\tau) := A_n^{-1} \sum_{t=1}^{[n\tau]} X_{t+m}, \qquad \tau \in [0,1],$$

when the "shift" $m \to \infty$ grows to infinity faster than n: $m/n \to \infty$.

Let us formulate the main result. We shall assume the following condition:

$$n^{-1/2}\Big(\sum_{t=1}^{[n\tau]} \xi_t, \sum_{t=1}^{[n\tau]} \xi_{t+m}\Big) \ \to_{\mathrm{FDD}} \ \mathbf{W}(\tau), \tag{8}$$

where $\mathbf{W}(\tau) = (W^{(1)}(\tau), W^{(2)}(\tau))$, $\tau \geq 0$ is a standard Brownian motion with values in $\mathbb{R}^2$ and unit covariance matrix (in other words, the components $W^{(1)}(\tau)$ and $W^{(2)}(\tau)$ are independent copies of a standard Brownian motion $W(\tau), \tau \geq 0$ with $EW(\tau)W(\tau') = \min(\tau, \tau')$), and $\to_{\mathrm{FDD}}$ stands for weak convergence of finite dimensional distributions. The assumption (8) on the white noise is very weak and is satisfied e.g. by any stationary ergodic martingale difference sequence with strong mixing (see Remark 3.1 below). Our main result is the following theorems.

Theorem 3.1 [see [17]] *Let X_t be a linear process as in (1), where $(b_i) \in \gamma(d)$, for some $d \in (-1/2, 1/2)$, and where (ξ_t) is a weak white noise satisfying condition (8). Then, as $n, m, m/n \to \infty$, the bivariate partial sums process $\mathbf{U}_n(\tau)$ in (7) converges, in sense of weak convergence of finite dimensional distributions, to a bivariate fractional Brownian motion $\mathbf{B}_H(\tau) = (B_H^{(1)}(\tau), B_H^{(2)}(\tau))$ with Hurst parameter $H = d + 1/2$ and mutually independent components:*

$$\mathbf{U}_n(\tau) \ \to_{\mathrm{FDD}} \ \mathbf{B}_H(\tau). \tag{9}$$

Recall that a sequence $\{\xi_t, \ t \in \mathbb{Z}\}$ of random variables is called a martingale difference sequence if $E|\xi_t| < \infty$ and $E(\xi_{t+1}|\xi_s, \ s \leq t) = 0$, a.s. for every $t \in \mathbb{Z}$. The next theorem discusses the convergence of the bivariate process $\mathbf{U}_n(\tau)$ in the Skorochod space $D[0,1]$.

Theorem 3.2 [see [17]] *Let X_t be a linear process as in Theorem 3.1, where (ξ_t) is a weak white noise satisfying condition (8). In addition, assume that $(\xi_t)$is a stationary martingale difference sequence and one of the following conditions (i)-(ii) holds:*

(i) $(b_i) \in \gamma(d)$, for some $d \in (-1/2, 0)$, and $E|\xi_0|^\alpha < \infty$ for some $\alpha > 2/(1+2d)$;

(ii) $(b_i) \in \gamma(0)$, and $E|\xi_0|^\alpha < \infty$, for some $\alpha > 2$;

(iii) $(b_i) \in \gamma(d)$, *for some* $d \in (0, 1/2)$, *and* $E|\xi_0|^2 < \infty$.

Then, as $n, m, m/n \to \infty$,

$$\mathbf{U}_n(\tau) \ \to_{D[0,1]} \ \mathbf{B}_H(\tau), \tag{10}$$

where $\mathbf{B}_H(\tau)$ *is the same as in Theorem 3.1.*

Remark 3.1 *Let us note that the assumption (8) holds for strongly mixing stationary martingale difference sequence* (ξ_t), *with zero mean and unit variance. Indeed, any such sequence is ergodic and satisfies the Donsker invariance principle, see ([13], Thm. 23.1). In particular, for any* $m \in \mathbb{Z}$, *as* $n \to \infty$ *we have marginal convergences*

$$S_n^{(1)}(\tau) \to_{\mathrm{FDD}} \mathbf{W}^{(1)}(\tau), \quad S_n^{(2)}(\tau) \to_{\mathrm{FDD}} \mathbf{W}^{(2)}(\tau), \tag{11}$$

where

$$S_n^{(1)}(\tau) := n^{-1/2} \sum_{t=1}^{[n\tau]} \xi_t, \quad S_n^{(2)}(\tau) := n^{-1/2} \sum_{t=1}^{[n\tau]} \xi_{t+m},$$

and $\mathbf{W}^{(1)}(\tau)$, $\mathbf{W}^{(2)}(\tau)$ *are standard Brownian motions. Moreover, for any* $0 \le \tau_1 < \ldots < \tau_p \le 1, p \le 1$, *the sequence of random vectors* $((S_n^{(1)}(\tau_k), S_n^{(2)}(\tau_k)), k = 1, \ldots, p) \in \mathbb{R}^{2p}$, $n, m = 1, 2, \ldots$ *is compact, or tight, in the topology of weak convergence of probability measures on* $\mathbb{R}^{2p}$. *Therefore (8) follows from (11) and the fact that* $S_n^{(1)}(\tau)$ *and* $S_n^{(2)}(\tau)$ *are asymptotically independent as* $n, m/n \to \infty$, *i. e.,*

$$E \exp\left\{\mathrm{i}x S_n^{(1)}(\tau) + \mathrm{i}y S_n^{(2)}(\tau)\right\} - E \exp\left\{\mathrm{i}x S_n^{(1)}(\tau)\right\} E \exp\left\{\mathrm{i}y S_n^{(2)}(\tau)\right\} \to 0, \tag{12}$$

for any $\tau \in [0,1], (x,y) \in \mathbb{R}^2$. *Under a strong mixing condition on* (ξ_t), *relation (12) follows from ([36], Thm. 17.2.1), as the left hand side of (12) does not exceed* $4\alpha(m)$, *where* $\alpha(m) \to 0$ $(m \to \infty)$ *is the strong mixing coefficient. Thus, in the case of a strongly mixing stationary martingale difference sequence* (ξ_t), *relation (8) holds provided* $n, m \to \infty$; *condition* $m/n \to \infty$ *is not needed here. An open question is whether (8) is satisfied if one assumes only ergodicity instead of strong mixing condition, as in the classical martingale central limit theorem.*

See [47] for a result on CLT for partial sums of linear processes with strong mixing innovations.

3.2 Discrete stochastic integrals

We will use the so-called "scheme of discrete stochastic integrals" introduced in [59]. We refer the interested reader to review [60].

Given $n, m = 1, 2, \ldots$ and a weak white noise (ξ_t), let define a vector-valued random set function $\mathbf{W}_n(B) = (W_n^{(1)}(B), W_n^{(2)}(B))$ on intervals $B = (x_1, x_2] \subset \mathbb{R}$ of the real line, as follows:

$$\mathbf{W}_n(B) := \frac{1}{n^{1/2}}\Big(\sum_{t/n \in B, t \le n+m/2} \xi_t, \sum_{t/n \in B, m+t > n+m/2} \xi_{m+t}\Big). \tag{13}$$

Note for any disjoint intervals $B = (x_1, x_2],\ B' = (x_1', x_2'],\ B \cap B' = \emptyset$,

$$\begin{aligned} W_n^{(1)}(B) + W_n^{(1)}(B') &= \frac{1}{n^{1/2}}\Big(\sum_{t/n \in B, t \le n+m/2} \xi_t + \sum_{t/n \in B', t \le n+m/2} \xi_t \Big) \\ &= \frac{1}{n^{1/2}} \sum_{t/n \in B \cup B', t \le n+m/2} \xi_t \\ &= W_n^{(1)}(B \cup B') \end{aligned}$$

and, similarly, $W_n^{(2)}(B) + W_n^{(2)}(B') = W_n^{(2)}(B \cup B')$. Therefore, the stochastic set function $\mathbf{W}_n$ satisfies the finite additivity property on the algebra formed by finite intervals of the real line. In the sequel, we call $\mathbf{W}_n$ a random measure.

We shall need the following properties of the random measure in (13).

Proposition 3.1 [see [17]]

(i) For any intervals $B, B' \subset \mathbb{R}$

$$E[W_n^{(i)}(B) W_n^{(i)}(B')] = 0 \qquad (B \cap B' = \emptyset,\ i = 1, 2), \tag{14}$$

$$E[W_n^{(1)}(B) W_n^{(2)}(B)] = 0. \tag{15}$$

Moreover, for any interval $B = (x_1, x_2]$ with $|B| := x_2 - x_1 < \infty$,

$$EW_n^{(i)}(B) = 0, \qquad E(W_n^{(i)}(B))^2 \le |B| \qquad (i = 1, 2). \tag{16}$$

(ii) Assume condition (8). For any $L \ge 1$ and any mutually disjoint intervals $B_j = (x_{1j}, x_{2j}],\ j = 1, 2, \ldots, L,$ as $n, m, n/m \to \infty$,

$$\begin{aligned} &\Big(W_n^{(1)}(B_j), W_n^{(2)}(B_j),\ j = 1, 2, \ldots, L\Big) \\ &\to_{\mathrm{FDD}} \Big(W^{(1)}(B_j), W^{(2)}(B_j),\ j = 1, 2, \ldots, L\Big), \end{aligned} \tag{17}$$

where $W^{(i)}(B) = \int_B W^{(i)}(dx), i = 1, 2$ are independent standard Gaussian noises.

Proof. Part (i) follows by orthogonality of (ξ_t) and the definition (13). Part (ii) is an easy consequence of (8). $\square$

Introduce the Hilbert space $\mathbf{L}^2(\mathbb{R})$ of vector valued functions $\mathbf{f} = (f^{(1)}, f^{(2)}) : \mathbb{R} \to \mathbb{R}^2$ with the norm

$$\|\mathbf{f}\| := \Big(\int_{\mathbb{R}} \big((f^{(1)}(x))^2 + (f^{(2)}(x))^2\big)\, dx \Big)^{1/2}. \tag{18}$$

A function $\mathbf{f} \in \mathbf{L}^2(\mathbb{R})$ will be called *simple* if it takes a finite number of constant nonzero values $\mathbf{f}^{\Delta} = (f^{1,\Delta}, f^{2,\Delta})$ on intervals $\Delta = (i_1/n, (i_2+1)/n]$, $i_1 \le i_2$. The class of all simple functions will be denoted by $\mathbf{L}_n(\mathbb{R})$. For any $n \ge 1$ and any simple function $\mathbf{f} \in \mathbf{L}_n(\mathbb{R})$ we define the discrete stochastic integral with respect to the random measure $\mathbf{W}_n$ in (13) as

$$I(\mathbf{f}, \mathbf{W}_n) := \sum_{\Delta} \big(f^{1,\Delta} W_n^{(1)}(\Delta) + f^{2,\Delta} W_n^{(2)}(\Delta) \big). \tag{19}$$

Note $EI(\mathbf{f}, \mathbf{W}_n) = 0$. Moreover, according to Proposition 3.1 (i),

$$\begin{aligned} E(I(\mathbf{f}, \mathbf{W}_n))^2 &\le 2 \sum_{i=1}^{2} E \left(\sum_{\Delta} f^{i,\Delta} W_n^{(i)}(\Delta) \right)^2 \\ &\le 2 \sum_{i=1}^{2} \sum_{\Delta} (f^{i,\Delta})^2 |\Delta| \;=\; 2\|\mathbf{f}\|^2. \end{aligned} \tag{20}$$

We also introduce the stochastic integral of a function $\mathbf{f} \in \mathbf{L}^2(\mathbb{R})$ with respect to a vector-valued standard Brownian motion $\mathbf{W}$:

$$I(\mathbf{f}, \mathbf{W}) := \int \big(f^{(1)}(x) W^{(1)}(dx) + f^{(2)}(x) W^{(2)}(dx) \big). \tag{21}$$

It is well-known that $I(\mathbf{f}, \mathbf{W}) \sim N(0, \|\mathbf{f}\|^2)$; in particular,

$$EI(\mathbf{f}, \mathbf{W}) = 0, \qquad E(I(\mathbf{f}, \mathbf{W}))^2 = \|\mathbf{f}\|^2.$$

Write $\Rightarrow$ for the convergence in distribution.

Lemma 3.1 [see [17]] *Let* $\mathbf{f}_n \in \mathbf{L}_n(\mathbb{R})$ $(n = 1, 2, \ldots)$ *be a sequence of simple functions convergent to a function* $\mathbf{f} \in \mathbf{L}^2(\mathbb{R})$*:*

$$\|\mathbf{f}_n - \mathbf{f}\| \;\to\; 0 \qquad (n \;\to \infty). \tag{22}$$

Let (ξ_t) *satisfy condition (8). Then, as* $n, m, m/n \to \infty$,

$$I(\mathbf{f}_n, \mathbf{W}_n) \;\Rightarrow\; I(\mathbf{f}, \mathbf{W}). \tag{23}$$

Proof. Let us consider the case where n takes values $n = 2^s$, $s = 1, 2, \ldots$, i.e., the partitions $\{(i_1/n, (i_2+1)/n],\ i_1 < i_2,\ i_1, i_2 \in \mathbb{R}\}$ of $\mathbb{R}$ are monotone. This implies the monotonicity $\mathbf{L}^2_{n'}(\mathbb{R}) \subset \mathbf{L}^2_{n''}(\mathbb{R})$ $(n' < n'')$ of spaces of simple functions. For non-monotone partitions, the proof below requires slight modifications. To prove (23), it is suffices to show that for every $u \in \mathbb{R}$

$$E\mathrm{e}^{iuI(\mathbf{f}_n, \mathbf{W}_n)} \to E\mathrm{e}^{iuI(\mathbf{f}, \mathbf{W})}.$$

From (22) it follows that, for every $\varepsilon > 0$, there exist an integer $\tilde{n} \geq 1$ and simple function $f_{\tilde{n}} \in L_{\tilde{n}}(\mathbb{R})$ with compact support such that

$$\|\mathbf{f}_n - \mathbf{f}_{\tilde{n}}\| + \|\mathbf{f}_{\tilde{n}} - \mathbf{f}\| < \varepsilon \tag{24}$$

for all $n \geq \tilde{n}$.

We have

$$\begin{aligned} \left|E\mathrm{e}^{iuI(\mathbf{f}_n, \mathbf{W}_n)} - E\mathrm{e}^{iuI(\mathbf{f}, \mathbf{W})}\right| &\leq \left|E\mathrm{e}^{iuI(\mathbf{f}_n, \mathbf{W}_n)} - E\mathrm{e}^{iuI(\mathbf{f}_{\tilde{n}}, \mathbf{W}_n)}\right| \\ &+ \left|E\mathrm{e}^{iuI(\mathbf{f}_{\tilde{n}}, \mathbf{W}_n)} - E\mathrm{e}^{iuI(\mathbf{f}_{\tilde{n}}, \mathbf{W})}\right| \\ &+ \left|E\mathrm{e}^{iuI(\mathbf{f}_{\tilde{n}}, \mathbf{W})} - E\mathrm{e}^{iuI(\mathbf{f}, \mathbf{W})}\right| \\ &:= V_1 + V_2 + V_3. \end{aligned}$$

We need to show

$$\lim_{n\to\infty} V_i = 0, \quad i = 1, 2, 3.$$

Using (20), we obtain

$$\begin{aligned} V_1 &\leq E\Big\{ \left|\mathrm{e}^{iuI(\mathbf{f}_{\tilde{n}}, \mathbf{W}_n)}\right| \left|\mathrm{e}^{iuI(\mathbf{f}_n - \mathbf{f}_{\tilde{n}}, \mathbf{W}_n)} - 1\right| \Big\} \\ &\leq \Big\{ E\left|\mathrm{e}^{iuI(\mathbf{f}_{\tilde{n}}, \mathbf{W}_n)}\right|^2 \Big\}^{1/2} \Big\{ E\left|\mathrm{e}^{iuI(\mathbf{f}_n - \mathbf{f}_{\tilde{n}}, \mathbf{W}_n)} - 1\right|^2 \Big\}^{1/2} \\ &\leq |u| \Big\{ E\left|I(\mathbf{f}_n - \mathbf{f}_{\tilde{n}}, \mathbf{W}_n)\right|^2 \Big\}^{1/2} \\ &\leq 2|u| \cdot \|\mathbf{f}_n - \mathbf{f}_{\tilde{n}}\| < 2\varepsilon|u| \end{aligned}$$

for every $n \geq \tilde{n}$. A similar inequality holds for V_3. The sum $V_1 + V_3$ can be made arbitrary small for all sufficiently large n by an arbitrary choice of $\tilde{n}$ and ε. The convergence $\lim_{n\to\infty} V_2 = 0$ for every $1 \leq \tilde{n} < \infty$ follows from Proposition 3.1 (ii) and Cramér-Wold principle. □

3.3 Proof of Theorem 3.1

We need to show that for any $\tau_i \in [0, 1], \mathbf{a}_i \in \mathbb{R}^2, i = 1, 2, \ldots, k, k = 1, 2, \ldots$

$$\sum_{i=1}^{k} \langle \mathbf{a}_i, \mathbf{U}_n(\tau_i) \rangle \ \Rightarrow\ \sum_{i=1}^{k} \langle \mathbf{a}_i, \mathbf{B}_H(\tau_i) \rangle, \tag{25}$$

where $\langle \cdot, \cdot \rangle$ is the scalar product in $\mathbb{R}^2$. For simplicity of notation, we shall restrict the proof of (25) to one-dimensional convergence at $\tau = 1$, i.e. we shall prove that

$$a^{(1)} U_n^{(1)}(1) + a^{(2)} U_n^{(2)}(1) \ \Rightarrow \ a^{(1)} B_H^{(1)}(1) + a^{(2)} B_H^{(2)}(1), \tag{26}$$

for any reals $a^{(1)}, a^{(2)} \in \mathbb{R}$. To that end, we shall write both sides of (26) as stochastic integrals, the left hand side as a stochastic integral with respect to the discrete random measure of (25) and the right hand side as a stochastic integral with respect to a Gaussian white noise $\mathbf{W}$ in (21), and then apply Lemma 3.1. The cases (I): $0 < d < 1/2$, (II): $d = 0$ and (III): $-1/2 < d < 0$ will be considered separately. Put $b_i := 0$ $(i = -1, -2, \ldots)$.

Case (I): $0 < d < 1/2$. Write

$$\langle \mathbf{a}, \mathbf{B}_H(1) \rangle = I(\mathbf{f}, \mathbf{W}), \tag{27}$$

where

$$\begin{aligned} f^{(i)}(x) \ &:= \ a^{(i)} \chi(d) \int_0^1 (y-x)_+^{d-1} \, \mathrm{d}y \\ &= \ \frac{a^{(i)} \chi(d)}{d} \Big\{ (1-x)^d I\{x < 1\} - (-x)^d I\{x < 0\} \Big\}, \ x \in \mathbb{R}, \end{aligned} \tag{28}$$

where $(y-x)_+^{d-1} := (y-x)^{d-1}$ if $y > x$, $:= 0$ otherwise, and where $\chi(d) > 0$ is defined by

$$\chi(d)^{-2} = \int_{\mathbb{R}} \Big(\int_0^1 (y-x)_+^{d-1} \, \mathrm{d}y \Big)^2 \, \mathrm{d}x.$$

See e.g. [62] for the stochastic integral representation of B_H and the explicit form of the constant $\chi(d)$. Next, write

$$\langle \mathbf{a}, \mathbf{U}_n(1) \rangle = \Sigma_1 + \Sigma_2,$$

where

$$\begin{aligned} \Sigma_1 \ &= A_n^{-1} \textstyle\sum_{s \le n+m/2} \sum_{t=1}^n (a^{(1)} b_{t-s} + a^{(2)} b_{t-s+m}) \xi_s \\ \Sigma_2 \ &= A_n^{-1} \textstyle\sum_{s > n+m/2} \sum_{t=1}^n (a^{(1)} b_{t-s} + a^{(2)} b_{t-s+m}) \xi_s. \end{aligned}$$

Using

$$\begin{aligned} \xi_s / n^{1/2} I(s \le n + m/2) \ &= W_n^{(1)}((s-1)/n, s/n), \\ \xi_{s+m} / n^{1/2} I(s > n - m/2) \ &= W_n^{(2)}((s-1)/n, s/n), \end{aligned}$$

we obtain

$$\langle \mathbf{a}, \mathbf{U}_n(1) \rangle = I(\mathbf{f}_n, \mathbf{W}_n), \tag{29}$$

where

$$f_n^{(1)}(x) \quad := \quad n^{1/2}A_n^{-1}\sum_{t=1}^{n}\left(a^{(1)}b_{t-s}+a^{(2)}b_{t+m-s}I(s\le n+m/2)\right), \tag{30}$$

$$f_n^{(2)}(x) \quad := \quad n^{1/2}A_n^{-1}a^{(2)}\sum_{t=1}^{n}b_{t-s}I(s>n-m/2) \tag{31}$$

for $x\in((s-1)/n,s/n], s\in\mathbb{Z}$. It is easy to show that

$$\int_{\mathbb{R}}(f_n^{(i)}(x)-f^{(i)}(x))^2\,\mathrm{d}x\to 0 \quad (n\to\infty), \tag{32}$$

see [59] for details. Hence $\|\mathbf{f}_n-\mathbf{f}\|\to 0$ as $n\to\infty$. From (27-29) and Lemma 3.1 it follows the convergence $\langle \mathbf{a},\mathbf{B}_H(1)\rangle \Rightarrow \langle \mathbf{a},\mathbf{B}_H(1)\rangle$, or the finite dimensional convergence (25) in Case (I).

Case (II): $d=0$. Recall from (4) that $A_n\sim c(0)n^{1/2}$, where the constant

$$c(0)=\bar b:=\sum_{t=0}^{\infty}b_t\neq 0.$$

Consider the representation (29), with $\mathbf{f}_n$ given in (30-31), which is true in this case, too. Let $x\in\mathbb{R}$ be a real number and let $s\in\mathbb{Z}$ be defined by $x\in((s-1)/n,s/n]$. Then as $n\to\infty$,

$$\sum_{t=1}^{n}b_{t-s}\to\begin{cases}\bar b, & \text{if } x\ge 0,\\ 0, & \text{if } x<0.\end{cases}$$

Then it is easy to show that for each $x\in\mathbb{R}$,

$$f_n^{(1)}(x)\to f^{(1)}(x), \quad f_n^{(2)}(x)\to f^{(2)}(x) \tag{33}$$

as $n,m,m/n\to\infty$, where the limit functions are proportional to the indicator function:

$$f^{(i)}(x):=a^{(i)}I(0\le x\le 1), \quad i=1,2, \quad x\in\mathbb{R}. \tag{34}$$

It is also easy to show that the convergence (33) extends to the convergence in $\mathbf{L}^2(\mathbb{R})$.

In particular,

$$\begin{aligned}\int_{-\infty}^{0}|f_n^{(1)}(x)|^2\,\mathrm{d}x &= A_n^{-2}\sum_{s=-\infty}^{0}\left(\sum_{t=1}^{n}\left(a^{(1)}b_{t-s}+a^{(2)}b_{t+m-s}I(s\le n+m/2)\right)\right)^2\\ &\le Cn^{-1}\left\{\sum_{s\ge 0}\Big(\sum_{t=1}^{n}|b_{t+s}|\Big)^2+\sum_{s\ge m/2-n}\Big(\sum_{t=1}^{n}|b_{t+s+m}|\Big)^2\right\}\end{aligned}$$

for some constant C depending on $a^{(i)}, i = 1, 2$ but independent of m, n. Here,

$$n^{-1} \sum_{s \ge 0} \Big(\sum_{t=1}^{n} |b_{t+s}| \Big)^2 \le n^{-1} \sum_{t=1}^{n} \sum_{s=0}^{\infty} |b_{t+s}| \sum_{t'=1}^{\infty} |b_{t'+s}| \le n^{-1} |\bar{b}| \sum_{t=1}^{n} \sum_{u=t}^{\infty} |b_u| \ \to \ 0$$

and, similarly,

$$n^{-1} \sum_{s \ge m/2 - n} \Big(\sum_{t=1}^{n} |b_{t+s+m}| \Big)^2 \to 0.$$

Thus, condition (22) of Lemma 3.1 is satisfied, for $\mathbf{f}_n$ given in (30-31) and $\mathbf{f}$ given in (34). Then Lemma 2.1 yields $\langle \mathbf{a}, \mathbf{B}_H(1) \rangle \Rightarrow \ I(\mathbf{f}, \mathbf{W})$, where

$$I(\mathbf{f}, \mathbf{W}) = a^{(1)} W^{(1)}(1) + a^{(2)} W^{(2)}(1) = a^{(1)} B^{(1)}_{1/2}(1) + a^{(2)} B^{(2)}_{1/2}(1).$$

Case (III): $-1/2 < d < 0$. Again, consider the representation (29), with $\mathbf{f}_n$ given in (30-31). The limit rv, $\langle \mathbf{a}, \mathbf{B}_H(1) \rangle$ can be written as

$$\langle \mathbf{a}, \mathbf{B}_H(1) \rangle = I(\mathbf{f}, \mathbf{W}) = \int \left(f^{(1)}(x) W^{(1)}(\,\mathrm{d}x) + f^{(1)}(x) W^{(1)}(\,\mathrm{d}x) \right),$$

where

$$f^{(i)}(x) := a^{(i)} \chi(d) \begin{cases} -\int_1^\infty (y-x)^{d-1}\,\mathrm{d}y, & 0 < x < 1, \\ \int_0^1 (y-x)^{d-1}\,\mathrm{d}y, & x < 0, \\ 0, & x \ge 1, \end{cases} \tag{35}$$

$i = 1, 2$, where $\chi(d) > 0$ is defined by

$$\chi(d)^{-2} = \int_{\mathbb{R}} \Big(\int_1^\infty (y-x)_+^{d-1}\,\mathrm{d}y \Big)^2 \mathrm{d}x,$$

(see [65], Proposition 9.1). In order to apply Lemma 3.1, we need to verify (22), or

$$\int_{\mathbb{R}} (f_n^{(i)}(x) - f^{(i)}(x))^2\,\mathrm{d}x \ \to \ 0 \qquad (i = 1, 2). \tag{36}$$

We shall verify the last relation for $i = 1$ only, as the case $i = 2$ is analogous. For simplicity, we shall assume $a^{(1)} = a^{(2)} = 1$. According to (35), it suffices to show

$$\int_0^1 \Big(f_n^{(1)}(x) + \chi(d) \int_1^\infty (y-x)^{d-1}\,\mathrm{d}y \Big)^2 \mathrm{d}x \ \to \ 0, \tag{37}$$

$$\int_{-\infty}^0 \Big(f_n^{(1)}(x) - \chi(d) \int_0^1 (y-x)^{d-1}\,\mathrm{d}y \Big)^2 \mathrm{d}x \ \to \ 0, \tag{38}$$

$$\int_1^\infty (f_n^{(1)}(x))^2\,\mathrm{d}x \ \to \ 0, \tag{39}$$

as $n, m, m/n \to \infty$. Write

$$f_n^{(1)}(x) = g_{1n}(x) + g_{2n}(x),$$

where

$$\begin{aligned} g_{1n}(x) &:= n^{1/2} A_n^{-1} \sum_{t=1}^{n} b_{t-s}, \\ g_{2n}(x) &:= n^{1/2} A_n^{-1} \sum_{t=1}^{n} b_{t+m-s} I(s \le n + m/2), \end{aligned} \tag{40}$$

for $x \in ((s-1)/n, s/n]$. Then (37) follows from

$$\begin{aligned} \int_0^1 \Big(g_{1n}(x) + \chi(d) \int_1^\infty (y-x)^{d-1}\, \mathrm{d}y \Big)^2 \mathrm{d}x &\to 0, \\ \int_0^1 (g_{2n}(x))^2\, \mathrm{d}x &\to 0. \end{aligned} \tag{41}$$

Using condition $\sum_{i=0}^\infty b_i = 0$, we can rewrite

$$g_{1n}(x) = -\alpha_n \sum_{i > n-s} \frac{L(i) i^{d-1}}{c(d) L(n) n^d} \equiv \alpha_n \tilde{g}_{n1}(x), \qquad x \in ((s-1)/n, s/n] \subset (0,1],$$

where $\alpha_n := c(d) L(n) n^{d+(1/2)}/A_n \to 1$, see (4). Therefore it suffices to show (41) for $\tilde{g}_{1n}(x)$ instead of $g_{1n}(x)$. Note for each $x < 1$,

$$\tilde{g}_n(x) = -\frac{1}{c(d)} \int_{1-\frac{[nx]}{n}}^\infty \frac{L(n\frac{[ny]}{n})}{L(n)} \Big(\frac{[ny]}{n} \Big)^{d-1} \mathrm{d}y \to -\frac{1}{c(d)} \int_1^\infty (y-x)^{d-1}\, \mathrm{d}y$$

and the first relation in (41) follows by a standard argument involving the dominated convergence theorem, provided the asymptotic constants are related by $\chi(d) = c^{-1}(d)$.

Let us prove the second relation in (41). We have

$$\int_0^1 (g_{2n}(x))^2\, \mathrm{d}x = A_n^{-2} \sum_{s=1}^{n} (\sum_{t=1}^{n} b_{t+m-s})^2$$

and the required relation follows from

$$S_{n,m} := \sum_{s=1}^{n} \Big(\sum_{t=1}^{n} b_{t+m-s} \Big)^2 = o(L^2(n) n^{2d+1}). \tag{42}$$

Assume $m > 2n$ and $L(u) > 0$ $(u > n)$ without loss of generality. Then the inner sum in (42) does not exceed

$$
\begin{aligned}
\sum_{u>m-s} L(u)u^{d-1} &= L(n)n^{-\delta} \sum_{u>m-s} \frac{L(n(u/n))}{L(n)(u/n)^{\delta}} u^{d-1+\delta} \\
&\le L(n)n^{-\delta} \sum_{u>m-s} u^{d-1+\delta} \Big(\sup_{x\ge 1} \frac{L(nx)}{L(n)x^{\delta}} \Big) \\
&\le CL(n)n^{-\delta}(m-s)^{d+\delta}, \qquad (43)
\end{aligned}
$$

where $0 < \delta < -d$ and where we used the following well-known property of slowly varying functions: for any $x_0 > 0, \delta > 0$, there exists a constant $C = C(x_0, \delta) < \infty$ such that for all $n \ge 1$

$$
\sup_{x\in[x_0,\infty)} \frac{1}{x^{\delta}} \Big| \frac{L(nx)}{L(n)} \Big| \; < \; C.
$$

Consequently,

$$
S_{n,m} \;\le\; CL^2(n)n^{1+2d} \Big(\frac{n}{m-n} \Big)^{-2(d+\delta)} \;=\; o(L^2(n)n^{2d+1})
$$

as $d + \delta < 0$ and $n/(m-n) \to 0$. This proves (42).

Relation (38) follows from

$$
\int_{-\infty}^{0} \Big(g_{1n}(x) - \chi(d) \int_0^1 (y-x)^{d-1}\,\mathrm{d}y \Big)^2 \mathrm{d}x \;\to\; 0, \qquad \int_{-\infty}^{0} (g_{2n}(x))^2\,\mathrm{d}x \;\to\; 0, \quad (44)
$$

where g_{1n}, g_{2n} are defined in (40). Similarly as above, the first relation in (44) follows by the dominated convergence theorem, by writing $g_{1n}(x) = \alpha_n \tilde{g}_{1n}(x)$, with

$$
\alpha_n = c(d)L(n)n^{d+(1/2)}/A_n \to 1
$$

and

$$
\begin{aligned}
\tilde{g}_{1n}(x) &:= \frac{1}{(d)} \sum_{t=1}^{n} \frac{L(t-s)(t-s)^{d-1}}{L(n)n^d} \\
&= \frac{1}{c(d)} \int_0^1 \frac{L(n\frac{[ny]-[nx]}{n})}{L(n)} \Big(\frac{[ny]-[nx]}{n} \Big)^{d-1} \mathrm{d}y \\
&\to \frac{1}{c(d)} \int_0^1 (y-x)^{d-1}\,\mathrm{d}y
\end{aligned}
$$

for each $x < 0$, $x \in ((s-1)/n, s/n]$. This proves the first relation in (44). The second relation in (44) follows from

$$
R_{n,m} := \sum_{s\le 0} \Big(\sum_{t=1}^{n} b_{t+m-s} \Big)^2 \;=\; o(L^2(n)n^{2d+1}). \qquad (45)
$$

Indeed, assuming $L(u) > 0 (u > m)$, we have

$$\begin{aligned}\sum_{t=1}^{n} b_{t+m-s} &= \sum_{t=1}^{n} L(t-s+m)(t-s+m)^{d-1} \\ &\le CL(n)n^{-\delta} \sum_{t=1}^{n} (t+m-s)^{d+\delta-1} \\ &\le CL(n)n^{1-\delta}(m-s)^{d+\delta-1}. \end{aligned} \tag{46}$$

Then (45) is immediate by

$$\begin{aligned} L^2(n)n^{2-2\delta} \sum_{s\ge 0} (m+s)^{2(d+\delta-1)} &\le CL^2(n)n^{2-2\delta} m^{2(d+\delta-1)+1} \\ = CL^2(n)n^{1+2d}\Big(\frac{n}{m}\Big)^{1-2(d+\delta)} &= o(L^2(n)n^{2d+1}), \end{aligned}$$

as $1-2(d+\delta) > 0$ for $\delta > 0$ small enough.

Finally, let us prove (39), which follows from

$$P_{n,m} := \sum_{s=n+1}^{n+(m/2)} \Big(\sum_{t=1}^{n} b_{t+m-s}\Big)^2 = o(L^2(n)n^{2d+1}). \tag{47}$$

Using (46) and taking $m > 4n$ (which implies $(m/2) - n > m/4$) we obtain

$$\begin{aligned} P_{n,m} &\le CL^2(n)n^{2-2\delta} \sum_{s=n+1}^{n+(m/2)} (m-s)^{2(d+\delta-1)} \le CL^2(n)n^{2-2\delta} \sum_{u>(m/2)-n} u^{2(d+\delta-1)} \\ &\le CL^2(n)n^{2-2\delta}(m/4)^{2(d+\delta-1)+1} = CL^2(n)n^{1+2d}(n/m)^{1-2(d+\delta)} \\ &= o(L^2(n)n^{2d+1}), \end{aligned}$$

proving (47) and (39). The proof of Theorem 3.1 is complete. □

3.4 Proof of Theorem 3.2

We need to establish tightness of the random function sequence $\mathbf{U}_n(\tau)$. We shall use the Kolmogorov criterion: there exist $\delta > 0, C > 0$ such that for any $0 \le \tau_0 < \tau_1 < \tau_2 \le 1$ and any $n \ge 1$

$$E\,|\mathbf{U}_n(\tau_1) - \mathbf{U}_n(\tau_0)|\,|\mathbf{U}_n(\tau_2) - \mathbf{U}_n(\tau_1)| \le C|\tau_2 - \tau_0|^{1+\delta}. \tag{48}$$

It suffices to show (48) for $\tau_2 - \tau_0 \ge 1/n$, as $0 \le \tau_2 - \tau_0 < 1/n$ implies

$$|\mathbf{U}_n(\tau_1) - \mathbf{U}_n(\tau_0)|\,|\mathbf{U}_n(\tau_2) - \mathbf{U}_n(\tau_1)| = 0.$$

The inequality $|[n\tau'] - [n\tau'']| < 2n|\tau' - \tau''|$ holds for $\tau', \tau'' \in \mathbb{R}$. Thus we need to consider the case $\tau_i = k_i/n, i = 1, 2$, where $0 \le k_1 < k_2 \le n$ are integers. By applying Hölder inequality for $\alpha \geq 2$ (it will be specified later), we obtain that (48) follows from

$$E\,|U_n(k_2/n) - U_n(k_1/n)|^{\beta} \;\leq\; C|(k_2 - k_1)/n|^{1+\delta}. \tag{49}$$

The process X_t being covariance stationary, (49) reduces to the inequality

$$E|U_n(k/n)|^{\beta} \;\leq\; C\Big|\frac{k}{n}\Big|^{1+\delta}, \tag{50}$$

where $k := k_2 - k_1$.

To estimate moments of weighted sums in martingale differences, we use the following Lemma 3.2, which is a consequence of Burkholder's inequality (see also [66], Lemma2).

Lemma 3.2 [see [17]] *Let (η_j) be martingale difference sequence such that $E|\eta_j|^{\beta} < \infty$ for every $j \in \mathbb{Z}$ and some $\beta \geq 2$. Then exists a constant $C(\beta) < \infty$ depending on β only, such that*

$$E\Big|\sum \eta_j\Big|^{\beta} \leq C(\beta)\Big(\sum E^{2/\beta}|\eta_j|^{\beta}\Big)^{\beta/2}. \tag{51}$$

Proof. Let $\mathcal{F}_j := \sigma\{\eta_s, s \leq j\}$. We shall use the well-known Burkholder inequality (see e.g. [20]):

$$E^{1/\beta}\Big|\sum \eta_j\Big|^{\beta} \leq C(\beta)\left\{\Big(\sum E|\eta_j|^{\beta}\Big)^{1/\beta} + E^{1/\beta}\Big(\sum E(\eta_j^2|\mathcal{F}_{j-1})\Big)^{\beta/2}\right\}. \tag{52}$$

By Minkowski inequality,

$$E^{2/\beta}\Big(\sum E(\eta_j^2|\mathcal{F}_{j-1})\Big)^{\beta/2} \leq \sum E^{2/\beta}\left(E(\eta_j^2|\mathcal{F}_{j-1})\right)^{\beta/2} \leq \sum E^{2/\beta}|\eta_j|^{\beta},$$

where the last step follows from Hölder inequality

$$E\left(E(\eta_j^2|\mathcal{F}_{j-1})\right)^{\beta/2} \leq E|\eta_j|^{\beta},\ (\beta \geq 2).$$

Lemma 3.2 is proved. □

To prove the bound (49), write

$$U_n(k/n) = A_n^{-1}\sum_{j\leq k} c_{j,k}\xi_j, \qquad c_{j,k} \;:=\; \sum_{s=1\vee j}^{k} b_{s-j}.$$

By Lemma 3.2, for a given $\alpha \geq 2$,

$$E|U_n(k/n)|^{\alpha} \;\leq \frac{CE|\xi_0|^{\alpha}}{A_n^{\alpha}}\Big(\sum_{j\leq k} c_{j,k}^2\Big)^{\alpha/2} \leq CE|\xi_0|^{\alpha}\frac{A_k^{\alpha}}{A_n^{\alpha}}, \tag{53}$$

where the constant $C < \infty$ does not depend on n, k.

Consider first the case (iii), or $(b_i) \in \gamma(d)$ $(0 < d < 1/2)$, $\alpha = 2$. Using the asymptotics (4) and the property (44) of slowly varying functions, for any $\varepsilon > 0$ one can find a constant $C(\varepsilon) < \infty$ such that

$$\frac{A_k^2}{A_n^2} \le C(\varepsilon)\Big(\frac{k}{n}\Big)^{1+2d-\varepsilon} \tag{54}$$

holds for any $k, n \ge 1$. Then (50) holds with $\alpha = 2$ and $\delta = 2d - \varepsilon > 0$ provided ε was chosen so that $0 < \varepsilon < 2d$.

Next, consider the case (i), or $(b_i) \in \gamma(d)$ $(-1/2 < d < 0)$, $\alpha > 2/(1+2d)$. Then (54) yields

$$\Big(\frac{A_k^2}{A_n^2}\Big)^{\alpha/2} \le C\Big(\frac{k}{n}\Big)^{\alpha(1+2d-\varepsilon)/2}$$

for some $C = C(\varepsilon, \alpha) < \infty$, implying (50) by (53) with $\delta = (\alpha(1+2d-\varepsilon))/2 - 1 > 0$, provided $\varepsilon > 0$ is chosen small enough. The last case (ii), or $(b_i) \in \gamma(0)$, $\alpha > 2$, follows similarly from (50-54). Theorem 3.2 is proved. $\square$

3.5 Application: convergence of increment-type statistics

Let $f(x, y)$ be a real nonnegative function, defined for all $(x, y) \in \mathbb{R}^2$ and continuous at almost every (a.e.) point $(x, y) \in \mathbb{R}^2$, such that

$$f(x, y) \le C_f(1 + |x| + |y|)^{1-\delta}, \quad (x, y) \in \mathbb{R}^2 \tag{55}$$

for some constants $C_f < \infty$ and $\delta > 0$. Let

$$R_f := \frac{1}{N - 3n} \sum_{k=0}^{N-3n-1} f\Big(A_n^{-1} \sum_{t=k+1}^{k+n} (X_{t+n} - X_t),\ A_n^{-1} \sum_{t=k+n+1}^{k+2n} (X_{t+n} - X_t)\Big), \tag{56}$$

where $1 \le n \le N/3$ and $X_1, \dots, X_N$ is a sample from a second order stationary process $(X_t, t \in \mathbb{Z})$; A_n defined in (3). Note that the sum in (56) can be rewritten in terms of increments of partial sums process U_n of (5):

$$R_f = \frac{1}{N - 3n} \sum_{k=0}^{N-3n-1} f(\Delta^2 U_n(k/n), \Delta^2 U_n(1 + k/n)), \tag{57}$$

where

$$\Delta U_n(\tau) \ := \ U_n(\tau + 1) - U_n(\tau),$$

$$\Delta^2 U_n(\tau) \;:=\; \Delta(\Delta U_n(\tau)) = U_n(\tau+2) + U_n(\tau) - 2U_n(\tau+1)$$

are the first and the second order increments, respectively. We call (56) an *increment-type statistic.* If the function f is scaling invariant, i.e.

$$f(\lambda x, \lambda y) = f(x,y), \qquad \forall (x,y) \in \mathbb{R}^2,\ \forall \lambda > 0.$$

the statistic R_f (56) is scale-free, in other words it does not change when $(X_t,\ 1 \le t \le N)$ are replaced by linear transformations $(aX_t + b,\ 1 \le t \le N)$, with arbitrary $a \ne 0,\ b \in \mathbb{R}$. A particular case of the statistic R_f corresponding to the function $f(x,y) = (|x+y|)/(|x|+|y|)$ was studied in [61].

Theorem 3.3 [see [17]] *Let X_t be a linear process as in (1), where $(b_i) \in \gamma(d)$, for some $d \in (-1/2, 1/2)$, and where (ξ_t) is a weak white noise satisfying condition (8). Then as $N \to \infty$, $n \to \infty$, $N/n \to \infty$,*

$$E(R_f - \Lambda_f(H))^2 \to 0, \tag{58}$$

where the function $\Lambda_f(H)$ is defined by

$$\Lambda_f(H) := Ef(\Delta^2 B_H(0), \Delta^2 B_H(1)).$$

Here $(\Delta^2 B_H(0), \Delta^2 B_H(1))$ is a Gaussian vector with zero mean and the covariances

$$\begin{aligned} E(\Delta^2 B_H(0))^2 \;=\; E(\Delta^2 B_H(1))^2 &\;=\; 4 - 2^{2d+1}, \\ E[\Delta^2 B_H(0)\Delta^2 B_H(1)] &\;=\; -(1/2)3^{2d+1} + 2^{2d+3} - (7/2). \end{aligned}$$

Proof. By (55) and stationarity of (X_t)

$$\begin{aligned} & Ef(\Delta^2 U_n(k/n), \Delta^2 U_n(1 + (k/n))) = Ef(\Delta^2 U_n(0), \Delta^2 U_n(1)) \\ \le\; & \Big\{ E\Big(f(\Delta^2 U_n(0), \Delta^2 U_n(1)) \Big)^{1/(1-\delta)} \Big\}^{1-\delta} \le C(1 + 2E|\Delta^2 U_n(0)|)^{1-\delta} < C, \end{aligned}$$

where the constant $C < \infty$ does not depend on n, k. Thus,

$$\begin{aligned} ER_f \;&=\; \frac{1}{N-3n} \sum_{k=0}^{N-3n-1} Ef(\Delta^2 U_n(0), \Delta^2 U_n(1)) \\ &=\; Ef(\Delta^2 U_n(0), \Delta^2 U_n(1)). \end{aligned}$$

From (9) if follows that

$$(\Delta^2 U_n(0), \Delta^2 U_n(1)) \to_{\mathrm{FDD}} (\Delta^2 B_H(0), \Delta^2 B_H(1)) \quad (n \to \infty),$$

implying $f(\Delta^2 U_n(0), \Delta^2 U_n(1)) \Rightarrow f(\Delta^2 B_H(0), \Delta^2 B_H(1))$ and

$$Ef(\Delta^2 U_n(0), \Delta^2 U_n(1)) \to (\Delta^2 B_H(0), \Delta^2 B_H(1)), \tag{59}$$

as $N \geq n \to \infty$.

Next, let us prove

$$E(\eta_n(k)\eta_n(j)) \to \Big(Ef(\Delta^2 B_H(0), \Delta^2 B_H(1))\Big)^2, \tag{60}$$

as $n \to \infty$, $|j-k|/n \to \infty$. Here we defined

$$\eta_n(k) := f(\Delta^2 U_n(k/n), \Delta^2 U_n(1+(k/n))).$$

From Theorem 3.1 it follows that

$$\begin{aligned}&\Big(\Delta^2 U_n(k/n), \Delta^2 U_n(1+(k/n)), \Delta^2 U_n(j/n), \Delta^2 U_n(1+(j/n))\Big)\\ &\to_{\mathrm{FDD}} \Big(\Delta^2 B_H^{(1)}(0), \Delta^2 B_H^{(1)}(1), \Delta^2 B_H^{(2)}(0), \Delta^2 B_H^{(2)}(1)\Big)\end{aligned}$$

where $(B_H^{(1)}(\tau)$ and $(B_H^{(2)}(\tau)$ are independent fractional Brownian motions. Whence follows that the sequence of random variables $(\eta_n(k)\eta_n(j),\ n \in \mathbb{N})$ converges in distribution to

$$f(\Delta^2 B_H^{(1)}(0), \Delta^2 B_H^{(1)}(1)) f(\Delta^2 B_H^{(2)}(0), \Delta^2 B_H^{(2)}(1)),$$

by applying the continuous mapping theorem. Applying (55), we have

$$\begin{aligned}E(\eta_n(k))^{2/(1-\delta)} &\leq CE\Big(1 + \Delta^2 U_n(k/n) + \Delta^2 U_n(1+(k/n))\Big)^2\\ &\leq C(1 + E(\Delta^2 U_n(0))^2) < C.\end{aligned}$$

Then using Hölder inequality again,

$$E(\eta_n(k)\eta_n(j))^{1/(1-\delta)} < C.$$

To establish (60) we use well-known moment criterion: if the sequence (Y_n) converges in distribution to a random variable Y and $E|Y_n|^{\varrho}$ are bounded for some $\varrho > 0$, then $E|Y_n|^{\tilde\varrho} \to E|Y|^{\tilde\varrho}$ for any $\tilde\varrho < \varrho$ (see [29],p. 306). Relations (59-60) imply

$$\mathrm{cov}\Big(f(\Delta^2 U_n(k/n), \Delta^2 U_n(1+(k/n)), f(\Delta^2 U_n(j/n), \Delta^2 U_n(1+(j/n))\Big) \to 0, \tag{61}$$

as $n \to \infty, |j-k|/n \to \infty$. Denote l.h.s. of (61) by $\varrho_n(k,j)$. Relation (58) now follows from (61) and the inequalities

$$\begin{aligned}\varrho_n(k,j) &\leq \varrho_n^{1/2}(k,k)\varrho_n^{1/2}(j,j) = \varrho_n(0,0)\\ &\leq \Big\{E\Big(f(\Delta^2 U_n(0), \Delta^2 U_n(1))\Big)^{2/(1-\delta)}\Big\}^{1-\delta}\\ &\leq 3C_f^2(1 + E(\Delta^2 U_n(0))^2)^{1-\delta} < C,\end{aligned}$$

with $C < \infty$ independent of n, k, j, by writing

$$\mathrm{var}(R_f) = (N-3n)^{-2} \sum_{k,j=0}^{N-3n-1} \varrho_n(k,j)$$

and using the dominated convergence theorem. Theorem 3.3 is proved. □

Conclusions

Most of the studies in long memory assume that observations are stationary or have stationary increments, which is not realistic in the case of a long sample. The study of nonstationary long memory and (linear) models of time series with nonstationary long memory is important to theory and applications. The thesis studies the limit distribution of partial sums for certain linear time series models with nonstationary long memory, and the limit distribution of some statistics under trended long memory observations. In particularly, the results of Philippe, Surgailis, Viano [51] about time-varying fractionally integrated (tv-FI) filters with finite variance are extended the case of infinite variance processes and to more general "memory governing" sequences. Under the assumption that innovations are belong to the domain of attraction of an α-stable law ($1 < \alpha < 2$), we show that the partial sums process of infinite variance tv-FI filters converges to some α-stable self-similar process with nonstationary increments. The limit distribution of the Increment Ratio (IR) statistic for Gaussian observations superimposed on a slowly-varying deterministic trend is studied. The IR statistic was introduced in Surgailis, Teyssière, Vaičiulis [61] for testing nonparametric hypotheses about d-integrated ($-1/2 < d < 3/2$) behavior of the time series, which often can be confused with deterministic trends and change-points. Conditions on the trend are obtained so that the IR statistic follows the same asymptotic Gaussian confidence intervals as in the absence of trend. We also establish the consistency of the IR-type statistics for general linear models with long, short, or negative memory. The proof of the last fact is based on asymptotic independence of distant partial sums.

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Notation

C	– some positive constant, which may change from line to line
$\mathbb{R}$	– the set of all real numbers
$\mathbb{N}$	– the set of all natural numbers
$\mathbb{Z}$	– the set of all integer numbers
L	– the backward shift operator
I	– the unit operator
$L^2(\Omega)$	– the set of all real random variables, which are defined on set Ω and have second moment
$L^2(\mathbb{R})$	– the Hilbert space of vector valued functions $\mathbf{f} = (f^{(1)}, f^{(2)}) : \mathbb{R} \to \mathbb{R}^2$ with norm $\|\mathbf{f}\| := \Big(\int_{\mathbb{R}} ((f^{(1)}(x))^2 + (f^{(2)}(x))^2) dx \Big)^{1/2}$
$D[0,1]$	– the Skorohod space of real-valued functions on the interval [0,1] without second order discontinuities
$H_n(x)$	– Hermite polynomial
$L(i)$	– slowly varying at infinity function
$B_H(t)$	– fractional Brownian motion
$\Gamma(x)$	– the Gamma-function
$\mathrm{mes}(A)$	– the Lebesgue measure of set$A$
$[\cdot]$	– the integer part of number
$a_n \sim b_n (n \to \infty)$	– meaning, that $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$
$a_n = O(b_n)(n \to \infty)$	– meaning, that $\|a_n\| \leq C\|b_n\|,$ for some $C > 0, N \in \mathbb{N}$ and for all $n \geq N$
$a_n = o(b_n)(n \to \infty)$	– meaning, that $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$

$=_{\text{law}}$	– equality of distributions
$=_{\text{FDD}}$	– equality of finite dimensional distributions
$\Rightarrow$	– convergence in distribution
$\rightarrow_{\text{FDD}}$	– weak convergence of finite dimensional distributions
$\rightarrow_{D[0,1]}$	– weak convergence of random variables in the Skorohod space $D[0,1]$