VILNIUS UNIVERSITY

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MELLIN TRANSFORMS OF DIRICHLET L-FUNCTIONS

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Physical sciences, mathematics (01P)

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Notation

γ_0	Euler constant
p	prime number
B_k	k-th Bernoulli number
q	modulo of Dirichlet character χ
(k, l)	greatest common divisor of natural k and l
$\overline{k}$	the residue $\mod l$ defined by $k\overline{k} \equiv 1(\mod l)$
$\mathbb{N}$	set of all natural numbers
$\mathbb{N}_0$	$\mathbb{N} \cup \{0\}$
$\mathbb{Z}$	set of all integer numbers
$\mathbb{R}$	set of real numbers
$\mathbb{C}$	set of all complex numbers
i	imaginary unity: $i = \sqrt{-1}$
$s = \sigma + it$	complex variable
$\Re z$	real part of z
$\Im z$	imaginary part of z
χ	primitive Dirichlet character
$\overline{\chi}$	conjugate to Dirichlet character χ
χ_0	pricipal Dirichlet character
$SL(2, \mathbb{Z})$	full modular group
$(\mathbb{Z}/q\mathbb{Z})^*$	group of invertible residues $\mod q$
$\zeta(s)$	Riemann zeta-function $\zeta(s) = \sum\limits_{m=1}^{\infty} \frac{1}{m^s}$ for $\sigma > 1$
$L(s, \chi)$	Dirichlet L-function $L(s, \chi) = \sum\limits_{m=1}^{\infty} \frac{\chi(m)}{m^s}$ for $\sigma > 1$

$E(s;\frac{k}{l},\alpha)$	Estermann zeta-function $E(s;\frac{k}{l},\alpha)=\sum_{m=1}^{\infty}\frac{\sigma_\alpha(m)}{m^s}\exp\{2\pi im\frac{k}{l}\}$ for $\sigma>\max(1+\Re\alpha,1)$
$L(\lambda,\beta,s)$	Lerch zeta-function $L(\lambda,\beta,s)=\sum\limits_{m=0}^{\infty}\frac{e^{2\pi im\lambda}}{(m+\beta)^s}$ for $\sigma>1,\lambda\in\mathbb{R},0<\beta\leqslant 1$
$\zeta(s,\beta)$	Hurwitz zeta-function $\zeta(s,\beta)=\sum\limits_{m=1}^{\infty}\frac{1}{(m+\beta)^s}$ for $\sigma>1$
$d(m)$	divisor function $d(m)=\sum_{d\mid m}1$
$\sigma_\alpha(m)$	generalized divisor function $\sigma_\alpha(m)=\sum_{d\mid m}d^\alpha,\alpha\in\mathbb{C}$
$\Gamma(s)$	Euler gamma-function $\Gamma(s)=\int\limits_0^\infty \mathrm{e}^{-\mathrm{x}}\mathrm{x}^{\mathrm{s}-1}\mathrm{dx}$ for $\sigma>0$
$\varphi(m)$	Euler totient function
$\mu(m)$	Möbius function $\mu(m)=\begin{cases}1 & \text{if} \quad m=1,\\ (-1)^k & \text{if} \quad m=p_1p_2...p_k,\\ 0 & \text{otherwise}\end{cases}$
$a(q)$	$\sum_{p\mid q}\frac{\log p}{p-1}$
$c(q)$	$\sum_{a=1}^{q}\overline{\chi}(a)(q,a-1)$
$G(\chi)$	Gauss sum $G(\chi)=\sum_{l=1}^{q}\chi(l)e^{2\pi il/q}$
$\pi(x;a,q)$	number of primes $p\le x$, $p\equiv a \bmod q$
$\mathcal{Z}_1(s,\chi)$	modified Mellin transform of Dirichlet L-function
$\mathrm{Res}_{s=z}f(s)$	residue of the function f at the point z

The symbols O, $\sim$, and $\ll$ are used in their standard meaning.

Introduction

Dirichlet L-functions $L(s,\chi)$, $s = \sigma + it$, are a generalization of the Riemann zeta-function $\zeta(s)$. In the half-plane $\sigma > 1$, the function $L(s,\chi)$ is defined by the Dirichlet series

$$L(s,\chi) \stackrel{def}{=} \sum_{m=1}^{\infty} \frac{\chi(m)}{m^s},$$

where $\chi(m)$ is a Dirichlet character modulo $q \in \mathbb{N}$, i.e., $\chi(m)$ is a homomorphism of the group $(\mathbb{Z}/q\mathbb{Z})^*$ of invertible residues $\mod q$ into the multiplicative group of complex numbers modulo 1.

Actuality

The importance of Dirichlet L-functions is comparable with that of the famous Riemann zeta-function. They were introduced in 1837 by P.G.L. Dirichlet for investigations of prime numbers in arithmetic progressions. Using an analytic machinery of L-functions, he proved that each arithmetic progression $m \equiv a \mod q$, $(a,q) = 1$, contains infinitely many prime numbers. The accuracy of the asymptotic formula for the function

$$\pi(x;a,q) = \sum_{\substack{p \le x \\ p \equiv a \mod q}} 1$$

as $x \to \infty$ essentially depends on the properties of Dirichlet L-functions. Dirichlet L-functions also have a series of applications for solution of other difficult problems of analytic number theory. Therefore, many well known number theorists, among them R. Balasubramanian, E. Bombieri, B.Conrey, J.Friedlander, A.Fujii, P.X. Gallagher, A.A. Karatsuba, M.Katsurada, S.M. Gonek, S. Kanemitsu, E. Kowalski, A. Laurinčikas, Yu.V. Linnik, K. Matsumoto, H.L.Montgomery, Y. Motohashi, M.J. Narlicar, K. Ramachandra, P. Sarnak, K. Soundararajan, J, Steuding, K. Titchmarsh, A.I. Vinogradov, S.M.Voronin, V. Zhang and others, investigated the value distribution of Dirichlet L-functions.

In the theory of Dirichlet L-functions, an important role is played by the moments. In 1995, Y. Motohashi [29] observed that, for investigation of the moments of the Riemann zeta-function

$$\int_0^T \left| \zeta\left(\frac{1}{2}+it\right) \right|^{2k} dt, k \geq 0,$$

the modified Mellin transforms can be successfully applied. This was realized in a series of works by A. Ivič, M. Jutila, Y. Motohashi, and by M. Lukkarinen. She also began to study the Mellin transforms corresponding the average mean square of Dirichlet L-functions

$$\sum_{\chi (\bmod q)} \int_0^T \left| L\left(\frac{1}{2}+it, \chi\right) \right|^2 dt.$$

There exists a more difficult problem to study the individual mean square of L-functions

$$\int_0^T \left| L\left(\frac{1}{2}+it, \chi\right) \right|^2 dt.$$

Some results in this direction were obtained in [28]. For this problem, also the modified Mellin transform can be applied. Therefore, analytic properties of the modified Mellin transform

$$\mathcal{Z}_1(s,\chi) \stackrel{def}{=} \int_1^\infty \left| L\left(\frac{1}{2}+ix, \chi\right) \right|^2 x^{-s} dx$$

are needed.

Aims and problems

The aim of the thesis is to obtain a meromorphic continuation for the modified Mellin transform of $\left| L\left(\frac{1}{2}+it,\chi\right) \right|^2$.

The problems are the following.

1. Formulae for the Laplace transforms of $\left|L\left(\frac{1}{2}+it,\chi\right)\right|^2$ with primitive character χ and principal character χ_0.
2. Transformation formulae for some functions involving the divisor function.
3. A meromorphic continuation to the whole complex plane of Mellin transforms with principal and primitive characters modulo q, i.e., for the functions $Z_1(s,\chi_0)$ and $Z_1(s,\chi)$.

Methods

For the proof of results obtained, various methods of functions of a complex variable are applied. Among them, the methods of countour integration , residue theory and transformations are used.

Novelty

All results of the thesis are new. Earlier the modified Mellin transforms corresponding the individual moments of Dirichlet L-functions were not considered.

History of the problem and results

In the moment problem of zeta and L-functions, usually approximate functional equations are applied. For example, an asymptotic formula for the mean square of the Riemann zeta-function

$$\int\limits_0^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^2 dt$$

is easily obtained by using the approximate functional equation [33]

$$\zeta(s)=\sum_{m\le x}\frac{1}{m^s}+\chi(s)\sum_{m\le y}\frac{1}{m^{1-s}}+O\left(x^{-\sigma}\right)+O\left(|t|^{\frac12-\sigma}y^{\sigma-1}\right)$$

which is valid in the critical strip $0<\sigma<1$,

$$\chi(s)=2^{-s}\pi^{-s-1}\sin\frac{\pi s}{2}\Gamma(s),$$

and x, y and t are related by $2\pi xy=t$, $x,y>c>0$. Taking in this equation $x=\frac{t}{2\pi\sqrt{\log t}}$ and $y=\sqrt{\log t}$ gives, in view of the estimate $\chi\left(\frac12+it\right)=O(1)$, that

$$\zeta\left(\frac12+it\right)=\sum_{m\le x}\frac{1}{m^{\frac12+it}}+O\left(x^{-\sigma}\right)+O\left(\log^{\frac14}t\right).$$

Thus, it suffices to prove that

$$\int_0^T\left|\sum_{m\le x}\frac{1}{m^{\frac12+it}}\right|^2dt\sim T\log T$$

as $T\to\infty$, and this a simple exercise. However, in the case of the higher moments, approximate functional equations for powers $\zeta^k(s)$ becomes more complicated, and the problems arise for their using. Therefore, it is natural to use another approaches for investigations of the moments of zeta-functions.

In 1995, a Japanese mathematician Y. Motohashi proposed [29] for the investigation of moments of the Riemann zeta-function to apply Mellin transforms. We remind that the Mellin transform $F(s)$ of the function $f(x)$ is defined by

$$F(s)=F(s,f)\stackrel{def}{=}\int_0^\infty f(x)x^{s-1}dx$$

provided that the integral exists. If the function $f(x)$ is continuous and the function $f(x)x^{\sigma-1}$ is integrable over $(0,\infty)$, then it is well known the inverse formula

$$f(x)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}F(s)x^{-s}ds. \tag{1}$$

Y. Motohashi proposed [29] to use the so-called modified Mellin transforms of the powers of the functions $\zeta(s)$. Let, for $\sigma>\sigma_0(k)>1$,

$$\mathcal{Z}_k(s)\stackrel{def}{=}\int_1^\infty\left|\zeta\left(\frac12+ix\right)\right|^{2k}x^{-s}dx,\ k>0.$$

Then an application of formula (1) with $c>1$ leads to

$$\int_1^\infty f\left(\frac{x}{T}\right)\left|\zeta\left(\frac{1}{2}+ix\right)\right|^{2k}dx = \int_1^\infty \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left(F(s)\left(\frac{T}{x}\right)^s ds\right)\left|\zeta\left(\frac{1}{2}+ix\right)\right|^{2k}dx$$
$$= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}F(s)T^s\mathcal{Z}_k(s)ds.$$

Thus, if we have a sufficient information on the modified Mellin transform $\mathcal{Z}_k(s)$, we may to evaluate the integral

$$\int_1^\infty f\left(\frac{x}{T}\right)\left|\zeta\left(\frac{1}{2}+ix\right)\right|^{2k}dx.$$

Therefore, a suitable choice of the function $f(x)$ allows to estimate the moments

$$\int_0^T\left|\zeta\left(\frac{1}{2}+it\right)\right|^{2k}dt.$$

This example shows that the modified Mellin transforms of zeta-functions are powerful tool in analytic number theory.

Let $F_1(s)$ be the modified Mellin transform of $f(x)$, i.e.,

$$F_1(s) = F_1(s,f) \stackrel{def}{=} \int_1^\infty f(x)x^{-s}dx.$$

We note that sometimes the function $F_1(s)$ is more convenient than $F(s)$ because a convergence problem for the integral does not arise at point $x=0$. Moreover, there exists a simple relation between $F(s)$ and $F_1(s)$. Define

$$f_1(x) = \begin{cases} f\left(\frac{1}{x}\right) & \text{if} \quad 0<x\le 1, \\ 0 & \text{otherwise.}\end{cases}$$

Then it was observed in [12] that

$$F_1(s,f) = F\left(s, \frac{1}{x}f_1(x)\right).$$

Historically, the function $\mathcal{Z}_2(s)$ was the first one studied for needs of the fourth power moment of $\zeta(s)$. The function $\mathcal{Z}_2(s)$ was introduced in [29] by Y. Motohashi, see also

[30], Chapter 5. He proved that, in the half-plane $\sigma > 0$, the function $\mathcal{Z}_2(s)$ has the pole at $s = 1$ of order five, simple poles at $s = \frac{1}{2} \pm i\kappa_j$, $\kappa_j = \sqrt{\lambda_j - \frac{1}{4}}$, where $\left\{\lambda_j = \kappa_j^2 + \frac{1}{4}\right\} \cup \{0\}$ is the discrete spectrum of the non-Euclidean Laplacian

$$\triangle = -y^2 \left(\left(\frac{\partial}{\partial x} \right)^2 + \left(\frac{\partial}{\partial y} \right)^2 \right)$$

acting on the space of automorphic forms with respect to the full modular group $SL(2, \mathbb{Z})$, and poles at $s = \frac{\rho}{2}$, where ρ are a complex zeros of $\zeta(s)$. Using properties of the function $\mathcal{Z}_2(s)$, several new results concerning the fourth power moment were obtained. Let

$$\int_0^T \left| \zeta \left(\frac{1}{2} + it \right) \right|^4 dt = TP_4(\log T) + E_2(T),$$

where $P_4(y)$ is a polynomial of degree 4. In [29] and [30], it was obtained that $E_2(T) = \Omega_{\pm}(T^{\frac{1}{2}})$, where $f(x) = \Omega_+(g(x))$ means that exists a constant $c > 0$ such that $f(x) > cg(x)$ for a sequence $x = x_n$, $\lim x_n = \infty$, $f(x) = \Omega_-(g(x))$ means that then exists a constant $c > 0$ such that $f(x) < -cg(x)$ for a sequence $x = x_n$, $\lim x_n = \infty$, and $f(x) = \Omega_{\pm}(g(x))$ means that both $f(x) = \Omega_+(g(x))$ and $f(x) = \Omega_-(g(x))$ are true. Moreover, in [8], [9], [10], it was proved the estimates

$$E_2(T) \ll T^{\frac{2}{3}} \log^{C_1} T,$$

$$\int_0^T E_2^2(T) \ll T^2 \log^{C_2} T,$$

$$\int_0^T E_2(T) \ll T^{\frac{3}{2}}$$

with effective constants C_1 and C_2.

The study of the function $\mathcal{Z}_2(s)$ was continued in [15], where the analytic continuation and growth for $\mathcal{Z}_2(s)$ were discussed. It was obtained that $\mathcal{Z}_2(s)$, for $\rho > -\frac{1}{2}$, has only poles at the points $s = \frac{\rho}{2}$, where, as above, ρ are complex zeros of $\zeta(s)$. Moreover, it was proved that, for $\varepsilon - \frac{1}{2} \leq \sigma \leq 1 - \varepsilon$, there exists a constant $b = b(\sigma) > 0$ such that

$$\mathcal{Z}_2(s) \ll (1 + |t|)^b .$$

The analytic properties of the function $\mathcal{Z}_1(s)$ were began to study in [15]. There the meromorphic continuation for $\mathcal{Z}_1(s)$ to the half-plane $\sigma > -\frac{3}{4}$ had been obtained. More precisely, it was proved that $\mathcal{Z}_1(s)$ is regular for $\sigma > -\frac{3}{4}$, except for a double pole at $s = 1$, and

$$\mathcal{Z}_1(s) = \frac{1}{(s-1)^2} + \frac{2\gamma_0 + \log 2\pi}{s-1} + \dots.$$

In [16], M. Jutila, by another method, observed that $\mathcal{Z}_1(s)$, for $\sigma < 0$, has at most double poles at $s = -k$, $k \in \mathbb{N}$. Finally, M Lukkarinen proved [26] that the function $\mathcal{Z}_1(s)$ has simple poles at $s = -(2k-1), k \in \mathbb{N}$, and no other singularities. She also found that

$$\operatorname*{Res}_{\substack{s=-(2k-1)\\ k\in\mathbb{N}}} \mathcal{Z}_1(s) = \frac{i^{-2k}(1-2^{1-2k})B_{2k}}{2k},$$

where B_k denote the Bernoulli numbers. Also, in [15], the following results for $\mathcal{Z}_1(s)$ were obtained. For $0 \le \sigma \le 1, t \ge t_0 > 0$,

$$\mathcal{Z}_1(s) \ll_\varepsilon t^{1-\sigma+\varepsilon},$$

and

$$\int_1^T |\mathcal{Z}_1(\sigma+it)|^2\, dt \ll_\varepsilon \begin{cases} T^{3-4\sigma+\varepsilon} & \text{if} \quad 0 \le \sigma \le \frac{1}{2}, \\ T^{2-2\sigma+\varepsilon} & \text{if} \quad \frac{1}{2} < \sigma \le 1. \end{cases}$$

In [16], the above estimate for $\mathcal{Z}_1(s)$ was replaced by the following

$$\mathcal{Z}_1(\sigma+it) \ll \begin{cases} (|t|+1)^{1-4\sigma/3+\varepsilon} & \text{if} \quad 0 \le \sigma \le \frac{1}{2}, \\ (|t|+1)^{5/6-\sigma+\varepsilon} & \text{if} \quad \frac{1}{2} < \sigma \le 1. \end{cases}$$

It turned out that to remove a ε-factor is not easy. Finally, in [26] it was proved that, for $0 \le \sigma \le 1$ and $|t| \ge 2$,

$$\mathcal{Z}_1(\sigma+it) \ll |t|^{1-\sigma} \log^2 |t|.$$

which, for a small σ, is better than the above Jutila's estimate.

The meromorphic continuation and some estimates for the modified Mellin transform

$$\mathcal{Z}_1(s,\rho) = \int_1^\infty \left|\zeta(\rho+ix)\right|^2 x^{-s} dx$$

with fixed $\frac{1}{2} < \rho < 1$ were discussed in [20]-[24], and [11], [13]

M. Lukkarinen, in [26], shortly discussed the modified Mellin transform of Dirichlet L-functions

$$\hat{\mathcal{Z}}_1(s, \chi) = \sum_{\chi \bmod q} \int_1^{\infty} \left| L\left(\frac{1}{2} + ix, \chi\right) \right|^2 x^{-s} dx.$$

and indicated a way how to obtain meromorphic continuation for $\hat{\mathcal{Z}}_1(s, \chi)$ to the half-plane $\sigma > 0$.

In the thesis, we consider the individual Mellin transform of Dirichlet L-functions

$$\mathcal{Z}_1(s, \chi) = \int_1^{\infty} \left| L\left(\frac{1}{2} + ix, \chi\right) \right|^2 x^{-s} dx,$$

and obtain for it a meromorphic continuation to the whole complex plane. We observe that the study of $\mathcal{Z}_1(s, \chi)$ is more complicated than that of $\hat{\mathcal{Z}}_1(s, \chi)$ because the summing over all characters modulo q simplifies the problem.

Chapter I of the thesis is auxiliary one. Here formulae for the Laplace transform of Dirichlet L-functions are obtained. Relation between the Laplace and Mellin transforms for the Riemann zeta-function was observed in [14]. We recall that the Laplace transform $\mathfrak{L}(s, f)$ of the function $f(x)$ is defined by

$$\mathfrak{L}(s, f) = \int_0^{\infty} f(x) e^{-sx} dx$$

provided the integral converges for $\sigma > \sigma_0$, with some σ_0. It is well known that the Laplace transform can be applied for the investigation of the moments of zeta-functions. This is easily seen from the following simple observation [33]. Suppose that $f(x) \geq 0$ for $x \geq 0$, and, for a given $k > 0$,

$$\int_0^{\infty} f(x) e^{-\delta x} dx \sim \frac{1}{\delta} \log^k \frac{1}{\delta}$$

as $\delta \longrightarrow 0$. Then

$$\int_0^{T} f(x) dx \sim T \log^k T$$

as $T \longrightarrow \infty$.

The first formula for the Laplace transform of the function $\left|\zeta\left(\frac{1}{2}+ix\right)\right|^2$ was presented in [33]. Here it was proved that

$$\int\limits_0^\infty \left|\zeta\left(\frac{1}{2}+ix\right)\right|^2 e^{-2sx}dx = 2\pi e^{is}\sum_{m=1}^\infty d(m)e^{2\pi ime^{2is}} + \sum_{m=0}^\infty a_m s^m$$

for $|s|$ small enough and $\sigma > 0$, where, as above $d(m)$, is the divisor function. Define

$$\mathfrak{L}(s) = \int\limits_0^\infty \left|\zeta\left(\frac{1}{2}+ix\right)\right|^2 e^{-sx}dx.$$

Then a more precise formula is true [26].

Theorem A. *Let* $\{s \in \mathbb{C} : 0 < |\sigma| < \pi\}$*. Then*

$$\mathfrak{L}(s) = ie^{is/2}\left(\gamma_0 - \log 2\pi - \left(\frac{\pi}{2} - s\right)i\right) + 2\pi e^{-is/2}\sum_{m=1}^\infty d(m)e^{-2\pi ime^{-is}} + \lambda(s),$$

where the function $\lambda(s)$ *is analytic in the strip* $\{s \in \mathbb{C} : |\sigma| < \pi\}$*, and, for* $|\sigma| \le \theta$*,* $0 < \theta < \pi$*, the estimate*

$$\lambda(s) = O((1+|s|))^{-1}$$

is true.

A similar formula was also obtained for the Laplace transform

$$\mathfrak{L}_\rho(s) = \int\limits_0^\infty \left|\zeta(\rho+ix)\right|^2 e^{-sx}dx$$

with a fixed ρ, $\frac{1}{2} < \rho < 1$, in [18].

Let

$$\mathfrak{L}(s,\chi) = \int\limits_0^\infty \left|L\left(\frac{1}{2}+ix,\chi\right)\right|^2 e^{-sx}dx.$$

In Chapter 1, the formulae for $\mathfrak{L}(s,\chi)$ are obtained. To state them, we use the following notation.

Let

$$G(\chi) = \sum_{l=1}^q \chi(l)e^{2\pi il/q}$$

denote the Gauss sum,

$$d(m) = \sum_{d|m} 1,$$

the divisor function, γ_0 Euler's constant, and $\mu(m)$ Möbius function.

Moreover, let

$$b = \begin{cases} 0 & \text{if} \quad \chi(-1) = 1, \\ 1 & \text{if} \quad \chi(-1) = -1, \end{cases}$$

$$\epsilon(\chi) = \frac{G(\chi)}{\sqrt{q}}, \qquad \epsilon_1(\chi) = -\frac{G(\chi)}{\sqrt{q}},$$

and

$$E(\chi) = \begin{cases} \epsilon(\chi) & \text{if} \quad b = 0, \\ \epsilon_1(\chi) & \text{if} \quad b = 1. \end{cases}$$

Theorem 1.1. *Let* $\{s \in \mathbb{C} : 0 < \sigma < \pi\}$ *and* χ *is a primitive character mod* $q > 1$. *Then*

$$\mathfrak{L}(s, \chi) = \frac{2\pi i^b e^{-\frac{is}{2}}}{\sqrt{q} E(\chi)} \sum_{m=1}^{\infty} d(m)\chi(m) \exp\left\{ -\frac{2\pi i m}{q} e^{-is} \right\} + \lambda(s, \chi),$$

where the function $\lambda(s, \chi)$ *is analytic in the strip* $\{s \in \mathbb{C} : |\sigma| < \pi\}$, *and, for* $|\sigma| \leq \theta$, $0 < \theta < \pi$, *the estimate*

$$\lambda(s, \chi) = O((1 + |s|))^{-1}$$

is valid.

Theorem 1.2. *Let* $\{s \in \mathbb{C} : 0 < \sigma < \pi\}$ *and* χ_0 *is the principal character mod* $q > 1$. *Then*

$$\mathfrak{L}(s, \chi_0) = ie^{\frac{is}{2}} \prod_{p|q} \left(1 - \frac{1}{p}\right) \sum_{m|q} \mu(m) \left(\gamma_0 - \log 2\pi - \left(\frac{\pi}{2} - s\right) i + \sum_{p|q} \frac{\log p}{p-1} + \log m \right)$$

$$- 2\pi i e^{-\frac{is}{2}} \sum_{n|q} \sum_{m|q} \frac{\mu(m)\mu(n)}{m} \sum_{k=1}^{\infty} d(k) \exp\left\{ -\frac{2\pi i k n}{m} \right\} + \lambda(s, \chi_0),$$

where the function $\lambda(s, \chi_0)$ *has the same properties as in Theorem 1.1.*

When $q = 1$, we obtain Theorem A.

The same results are also true for the modified Laplace transform

$$\mathfrak{L}(s,\chi) = \int_1^\infty \left| L\left(\frac{1}{2}+ix,\chi\right)\right|^2 e^{-sx}dx.$$

Chapter 2 is auxiliary, too. Here we obtain transformation formulae for the functions

$$\Phi\left(z;\frac{k}{l}\right) = \sum_{m=1}^\infty d(m)e^{2\pi i m\frac{k}{l}}e^{-mz} - \frac{\gamma_0 - 2\log l - \log z}{lz},$$

where k and l are coprime positive integers, $\Im z \neq 0$, $\Re z > 0$, and

$$\Phi(z;\chi,q) = \sum_{m=1}^\infty d(m)\chi(m)e^{-2\pi i m/q}e^{-mz}$$
$$- \frac{1}{G(\overline{\chi})}\sum_{a=1}^q \overline{\chi}(a)\frac{(q,a-1)}{qz}\left(\gamma_0 - 2\log\frac{q}{(q,a-1)} - \log z\right),$$

where χ is a Dirichlet character modulo q.

Before the statements of our results, we recall a transformation formula used in [26] for the Mellin transform $\mathcal{Z}_1(s)$. Let

$$\delta = \begin{cases} 1 & \text{if} \quad Imz > 0, \\ -1 & \text{if} \quad Imz < 0, \end{cases}$$

and, for $\Im z \neq 0$ and $\Re z > 0$,

$$\Phi(z) = \sum_{m=1}^\infty d(m)e^{-mz} - \frac{\gamma_0 - \log z}{z}.$$

Theorem B. *Let $1 < b < 2$, then for the function $\Phi(z)$ transformation formula*

$$\Phi(z^{-1}) = -2\pi i\delta z\sum_{m=1}^\infty d(m)e^{-4\pi^2 mz} + \frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty}(2\pi)^{1-2w}\Gamma(w)\zeta^2(w)$$
$$\times\left(\cot\left(\frac{\pi w}{2}\right) + \delta i\right)z^{1-w}dw + \frac{1}{4}.$$

is true.

Define

$$I(z,b) = \frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty}\left(\frac{2\pi}{l}\right)^{1-2w}\Gamma(w)\Bigg((\sin(\pi w))^{-1}E\left(w;\frac{\overline{k}}{l},0\right)$$
$$+ (\cot(\pi w) + \delta i)\,E\left(w;-\frac{\overline{k}}{l},0\right)\Bigg)z^{1-w}dw,$$

and, for a Dirichlet character χ,

$$I(z;\chi,b)=\frac{1}{2\pi iG(\overline{\chi})}\sum_{a=1}^{q}\overline{\chi}(a)\int_{b-i\infty}^{b+i\infty}\left(\frac{2\pi}{q}\right)^{1-2w}\Gamma(w)$$

$$\times\left\{\sin^{-1}(\pi w)\,E\left(w;\frac{\overline{\left(\frac{a-1}{(q,a-1)}\right)}}{\frac{q}{(q,a-1)}},0\right)(q,a-1)^{1-2w}+\cot(\pi w)\right.$$

$$\left.\times E\left(w;-\frac{\overline{\left(\frac{a-1}{(q,a-1)}\right)}}{\frac{q}{(q,a-1)}},0\right)(q,a-1)^{1-2w}+\delta iE\left(w;\frac{\frac{(a-1)}{(q,a-1)}}{\frac{q}{(q,a-1)}},0\right)\right\}z^{1-w}dw,$$

$$\widetilde{I}(z;\chi,b)=\frac{1}{2\pi iG(\overline{\chi})}\sum_{a=1}^{q}\overline{\chi}(a)\int_{b-i\infty}^{b+i\infty}\left(\frac{2\pi(q,a-1)}{q}\right)^{1-2w}\Gamma(w)$$

$$\times\left\{\sin^{-1}(\pi w)\,E\left(w;\frac{\overline{\left(\frac{a-1}{(q,a-1)}\right)}}{\frac{q}{(q,a-1)}},0\right)+(\cot(\pi w)+\delta i)\,E\left(w;-\frac{\overline{\left(\frac{a-1}{(q,a-1)}\right)}}{\frac{q}{(q,a-1)}},0\right)\right\}z^{1-w}dw,$$

where $E\left(s;\frac{k}{l},\alpha\right)$, $l>1$, $(k,l)=1$, is the Estermann zeta-function defined, for $\sigma>\max(1+Re\alpha,1)$ by the series

$$E(s;\frac{k}{l},\alpha)=\sum_{m=1}^{\infty}\frac{\sigma_{\alpha}(m)}{m^{s}}\exp\{2\pi im\frac{k}{l}\},$$

k is connected to $\overline{k}$ by the congruence $k\overline{k}\equiv 1(\mod l)$, and a_0^+ and a_0^- are the constant terms in the Laurent series expansions for the functions $E\left(s;\frac{\overline{k}}{l},0\right)$ and $E\left(s;-\frac{\overline{k}}{l},0\right)$, respectively. Then our results are contained in the following theorems.

Theorem 2.1. *If $\Re z>0$ and $\Im z\neq 0$, then for the function $\Phi\left(z;\frac{k}{l}\right)$ the transformation formula*

$$\Phi\left(z^{-1};\frac{k}{l}\right)=-\frac{2\pi i\delta z}{l}\sum_{m=1}^{\infty}d(m)e^{-2\pi im\frac{\overline{k}}{l}}e^{-\frac{4\pi^2mz}{l^2}}+\frac{l}{2\pi^2}\left(a_0^+-a_0^-\right)+\frac{1}{4}+I(z,b)$$

is valid.

Theorem 2.2. *If $\Re z>0$ and $\Im z\neq 0$, then for the function $\Phi(z;\chi,q)$ the transformation formula*

$$\Phi\left(z^{-1};\chi,q\right)=-\frac{2\pi i\delta z}{q}\sum_{m=1}^{\infty}d(m)\chi(m)e^{-2\pi im/q}e^{-\frac{4\pi^2mz}{q^2}}$$

$$+\frac{q}{2\pi^2G(\overline{\chi})}\sum_{a=1}^{q}\frac{\overline{\chi}(a)}{(q,a-1)}\left(a_{0a}^+-a_{0a}^-\right)+I(z;\chi,b),$$

is valid.

Theorem 2.3. *If $\Re z > 0$ and $\Im z \neq 0$, then for the function $\Phi(z;\chi,q)$ the transformation formula*

$$\Phi\left(z^{-1};\chi,q\right) = -\frac{2\pi i\delta z}{G(\overline{\chi})q}\sum_{a=1}^{q}\overline{\chi}(a)(q,a-1)\sum_{m=1}^{\infty}d(m)e^{-2\pi i m\frac{\overline{(a-1)/(q,a-1)}}{q/(q,a-1)}}e^{-\frac{4\pi^2 m(q,a-1)^2 z}{q^2}}$$
$$+\frac{q}{2\pi^2 G(\overline{\chi})}\sum_{a=1}^{q}\frac{\overline{\chi}(a)}{(q,a-1)}(a_{0a}^{+}-a_{0a}^{-})+\widetilde{I}(z;\chi,b),$$

is valid.

In Chapter 3, using the results of Chapters 1 and 2, we obtain the meromorphic continuation for the Mellin transform $\mathcal{Z}_1(s,\chi)$. We have the following results.

Define

$$c(q) = \sum_{a=1}^{q}\overline{\chi}(a)(q,a-1),$$

and

$$a(q) = \sum_{p|q}\frac{\log p}{p-1},$$

also φ denote, as usual, Euler's totient function, and B_k Bernoulli numbers.

Theorem 3.1. *The function $\mathcal{Z}_1(s,\chi_0)$ has a meromorphic continuation to the whole complex plane. It has a double pole at the point $s=1$, and the main part of its Laurent expansion at this point is*

$$\mathcal{Z}_1(s,\chi_0) = \frac{\varphi(q)}{q}\left(\frac{1}{(s-1)^2}+\frac{2\gamma_0+2a(q)-\log 2\pi}{s-1}\right)+\ldots.$$

The other poles of $\mathcal{Z}_1(s,\chi_0)$ are the simple poles at the points $s=-(2j-1), j\in\mathbb{N}$, and

$$\operatorname*{Res}_{\substack{s=-(2j-1)\\ j\in\mathbb{N}}}\mathcal{Z}_1(s,\chi_0) = \frac{\varphi(q)i^{-2j}(1-2^{1-2j})B_{2j}}{2jq}.$$

Theorem 3.2. *The function $\mathcal{Z}_1(s,\chi)$ has a meromorphic continuation to the whole complex plane.*

1. *If* $c(q) \neq 0$ *it has a double pole at the point* $s = 1$*, and the main part of its Laurent expansion at this point is*

$$\mathcal{Z}_1(s,\chi) = \frac{i^b}{q}\sum_{a=1}^{q}\overline{\chi}(a)(q,a-1)\left(\frac{1}{(s-1)^2} + \frac{2\gamma_0 + \log{(q,a-1)^2/2\pi q}}{s-1}\right) + \ldots.$$

the other poles of $\mathcal{Z}_1(s,\chi)$ *are the simple poles at the points* $s = -(2j-1), j \in \mathbb{N}$*, and*

$$\operatorname*{Res}_{\substack{s=-(2j-1)\\ j\in\mathbb{N}}} \mathcal{Z}_1(s,\chi) = \frac{i^{b-2j}(1-2^{1-2j})B_{2j}}{2jq}\sum_{a=1}^{q}\overline{\chi}(a)(q,a-1).$$

2. *If* $c(q) = 0$*, the function* $\mathcal{Z}_1(s,\chi)$ *is entire function.*

Approbation

The results of the thesis were presented at the Conferences of Lithuanian Mathematical Society (2011-2014), at the 17th International Conference Mathematical Modelling and Analysis (Tallin, Estonia, June 6-9, 2012), 18th International Conference Mathematical Modelling and Analysis and 4th Conference Approximation Methods and Orthogonal Expansions (Tartu, Estonia, May 27-30, 2013), 19th International Conference Mathematical Modelling and Analysis (Druskininkai, Lithuania, May 26-29, 2014), XII International Conference Algebra and Number Theory: Modern Problems and Application, dedicated to 80-th anniversary of Professor V. N. Latyshev (Tula, Russia, April 21-25, 2014), at the doctorant conferences of Institute of Mathematics and Informatics and at the seminars of Number theory of the faculty of Mathematics and Informatics of Vilnius University.

Principal publications

The main results of the thesis have been published in the papers:

1. Balčiūnas A., *A transformation formula related to Dirichlet L-functions with principal character*, Lietuvos matematikos rinkinys **53**(A)(2012), 13-18.
2. Balčiūnas A., *Mellin transform of Dirichlet L-functions with principal character*, Šiauliai mathematical Seminar **8**(16) (2013), 7-26.
3. Balčiūnas A., *A transformation formula with primitive character*, Lietuvos matematikos rinkinys **54**(A)(2013), 6-11 .
4. Balčiūnas A., *Mellin transforms of Dirichlet L-functions* , Proceeding XII International Conference Algebra and Number Theory: Modern Problems and Application, Tula (2014), 13-16.
5. Balčiūnas A., Laurinčikas A.,*The Laplace transform of Dirichlet L-functions*, Nonlinear analysis : Modelling and Control **17**(2) (2012), 127-138.

Other publications

1. Balčiūnas A., *The Laplace and Mellin transforms of Dirichlet L-functions*, Mathematical modelling and analysis : 17th International Conference, June 6-9, 2012, Tallinn, Estonia: Abstracts. Tallinn: Tallinn University of Technology, 2012 p.18.
2. Balčiūnas A., *Mellin transform of Dirichlet L-functions with principal character.* 18th International Conference : Mathematical Modelling and Analysis and Fourth International Conference : Approximation Methods and Orthogonal Expansions, May 27 - 30, 2013, Tartu, Estonia: Abstracts. Tartu: University of Tartu, 2013 p.16.

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Chapter 1

The Laplace transform of Dirichlet L-functions

Let χ be a Dirichlet character modulo q, and let $L(s,\chi)$ denote the corresponding L-function which is defined, for $\sigma > 1$, by

$$L(s,\chi) \stackrel{def}{=} \sum_{m=1}^{\infty} \frac{\chi(m)}{m^s}.$$

Denote by χ_0 the principal character modulo q. Then it's well known that the function $L(s,\chi_0)$ is analytically continued to the whole complex plane, except for a simple pole at $s = 1$ with residue

$$\prod_{p|q} \left(1 - \frac{1}{p}\right),$$

where p denotes a prime number. If $\chi \neq \chi_0$, then $L\left(s,\chi\right)$ is analytically continued to an entire function.

In the theory Dirichlet L-functions, usually the moments

$$\sum_{\chi = \chi \,(\mathrm{mod}\ q)} \int_0^T |L\left(\sigma + it, \chi\right)|^{2k} dt, \qquad k \geq 0, \quad \sigma \geq \frac{1}{2},$$

are considered, see, for example, [27]. This corresponds to the Laplace transform

$$\sum_{\chi = \chi \,(\mathrm{mod}\ q)} \int_0^\infty |L\left(\sigma + it, \chi\right)|^{2k} e^{-sx} dx.$$

The aim of this chapter is an explicit formula for the individual Laplace transform $\mathfrak{L}(s,\chi)$ of $|L\left(\frac{1}{2}+ix,\chi\right)|^2$, i.e., for the function

$$\mathfrak{L}(s,\chi) \stackrel{def}{=} \int_0^\infty \left|L\left(\frac{1}{2}+ix,\chi\right)\right|^2 e^{-sx}dx. \tag{1.1}$$

The results of this chapter are published in [5]. We consider separately the cases of the principal and primitive Dirichlet characters. Later, in Chapter 3, these results will be used to get meromorphic continuation to the whole complex plane for the Mellin transform $\mathcal{Z}_1(s,\chi)$.

1.1 Statement of the main theorem

For the statements of the formulae obtained, we need some notations.
Let $G(\chi)$ denote the Gauss sum, i.e.,

$$G(\chi) = \sum_{l=1}^{q} \chi(l) e^{2\pi i l/q}.$$

Moreover, let

$$a(q) = \sum_{p|q} \frac{\log p}{p-1},$$

$$b = \begin{cases} 0 & \text{if} \quad \chi(-1) = 1, \\ 1 & \text{if} \quad \chi(-1) = -1, \end{cases}$$

$$E(\chi) = \begin{cases} \epsilon(\chi) & \text{if} \quad b = 0, \\ \epsilon_1(\chi) & \text{if} \quad b = 1, \end{cases}$$

where

$$\epsilon(\chi) = \frac{G(\chi)}{\sqrt{q}}, \qquad \epsilon_1(\chi) = -\frac{G(\chi)}{\sqrt{q}}.$$

As usual, denote by $d(m)$ the divisor function

$$d(m) = \sum_{d|m} 1,$$

γ_0 is Euler constant, and $\mu(m)$ is the Mŏbius function.

Theorem 1.1. *Let* $\{s \in \mathbb{C} : 0 < \sigma < \pi\}$*, and* χ *be a primitive character modulo* $q > 1$*. Then*

$$\mathfrak{L}(s,\chi) = \frac{2\pi i^b e^{-\frac{is}{2}}}{\sqrt{q}E(\chi)} \sum_{m=1}^{\infty} d(m)\chi(m) \exp\left\{ -\frac{2\pi i m}{q} e^{-is} \right\} + \lambda(s,\chi),$$

where the function $\lambda(s,\chi)$ *is analytic in the strip* $\{s \in \mathbb{C} : |\sigma| < \pi\}$*, and, for* $|\sigma| \leq \theta, \quad 0 < \theta < \pi$*, the estimate*

$$\lambda(s,\chi) = O((1+|s|))^{-1}$$

is valid.

Theorem 1.2. *Let* $\{s \in \mathbb{C} : 0 < \sigma < \pi\}$*, and* χ_0 *be a principal character modulo* $q > 1$*. Then*

$$\begin{aligned}\mathfrak{L}(s,\chi_0) = ie^{\frac{is}{2}} \prod_{p|q} \left(1 - \frac{1}{p}\right) \sum_{m|q} \mu(m) \left(\gamma_0 - \log 2\pi - \left(\frac{\pi}{2} - s\right) i + a(q) + \log m\right) \\ - 2\pi i e^{-\frac{is}{2}} \sum_{n|q} \sum_{m|q} \frac{\mu(m)\mu(n)}{m} \sum_{k=1}^{\infty} d(k) \exp\left\{ -\frac{2\pi i k n}{m} \right\} + \lambda(s,\chi_0),\end{aligned}$$

where the function $\lambda(s,\chi_0)$ *has the same properties as in Theorem 1.1*

The case $q = 1$ corresponds the Riemann zeta-function $\zeta(s)$. Let $\mathfrak{L}_\zeta(s)$ be the Laplace transform of $|\zeta\left(\frac{1}{2} + ix\right)|^2$.

Corollary 1.3. *Let* $\{s \in \mathbb{C} : 0 < \sigma < \pi\}$*. Then*

$$\mathfrak{L}_\zeta(s) = ie^{\frac{is}{2}} \left(\gamma_0 - \log 2\pi - \left(\frac{\pi}{2} - s\right) i\right) + 2\pi e^{-\frac{is}{2}} \sum_{m=1}^{\infty} d(m) \exp\left\{ -2\pi i m e^{-is} \right\} + \lambda(s),$$

where function $\lambda(s)$ *has the same properties as in the above theorems.*

1.2 Analytic lemmas

In the proof of Theorems 1.1 and 1.2, we will use some auxiliary results. We state these results as separate lemmas. First we remind the functional equation for $L(s,\chi)$. Let

$$l(s,\chi) = \left(\frac{\pi}{q}\right)^{-\frac{s+b}{2}} \Gamma\left(\frac{s+b}{2}\right) L(s,\chi),$$

where $\Gamma(s)$ is the gamma-function.

Lemma 1.1. *If χ is a primitive character modulo $q > 1$, then*

$$l(s, \chi) = E(\chi) l(1 - s, \bar{\chi}).$$

Proof of the lemma is given, for example, in [6].

Lemma 1.2. *For $\sigma > 1$,*

$$L^2(s, \chi) = \sum_{m=1}^{\infty} \frac{d(m)\chi(m)}{m^s}.$$

Proof. For $\sigma > 1$, using the multiplicative property of Dirichlet characters, we have that

$$L^2(s, \chi) = \sum_{m=1}^{\infty} \frac{\chi(m)}{m^s} \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \sum_{m=1}^{\infty} \frac{a(m)}{m^s},$$

where

$$a(m) = \sum_{d|m} \chi(d)\chi\left(\frac{m}{d}\right) = \chi(m) \sum_{d|m} 1 = \chi(m) d(m).$$

□

Denote by $\varphi(q)$ the Euler totient function.

Lemma 1.3. *Let σ_0 be arbitrary real number. Then, for $\sigma \geq \sigma_0$,*

$$L(s, \chi) = \frac{E_0 \varphi(q)}{q(s-1)} + O_{\sigma_0}\left(q^c \left(|t| + 2\right)^c\right),$$

where $c = c(\sigma_0) > 0$, and

$$E_0 = \begin{cases} 1 & \textit{if} \quad \chi = \chi_0, \\ 0 & \textit{if} \quad \chi \neq \chi_0. \end{cases}$$

The lemma is Theorem 7.3.2 from [31].

Lemma 1.4.

$$L'\left(\frac{1}{2} + it, \chi\right) = O\left(q^{c_1}\left(|t| + 2\right)^{c_1}\right),$$

holds with some $c_1 > 0$.

Proof. The lemma is a corollary of Lemma 1.3. and the integral Cauchy formula.

□

Lemma 1.5. *Suppose that a>0 and b>0. Then*

$$\frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \Gamma(s) b^{-s} ds = e^{-b}.$$

The lemma is the well-known Mellin formula, see, for example, [32].

1.3 Proof of theorems

Define the function $\lambda(s,\chi)$ by

$$\lambda(s,\chi) = \int_0^\infty \left| L\left(\frac{1}{2}+ix,\chi\right)\right|^2 e^{-sx}dx$$

$$-\frac{e^{-\frac{is}{2}}}{2i^{1-b}} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{L(z,\chi)L(1-z,\bar{\chi})e^{-iz(\frac{\pi}{2}-s)}}{\cos\left(\frac{\pi b}{2}-\frac{\pi z}{2}\right)}dz. \tag{1.2}$$

First suppose that $b=0$. Using the formulae

$$\cos s = \frac{e^{is}+e^{-is}}{2}$$

and

$$\overline{L(s,\chi)} = L(\bar{s},\bar{\chi}),$$

and making a substitution $z=\frac{1}{2}+ix$ in (1.2) we have

$$\begin{aligned}
&\frac{e^{-\frac{is}{2}}}{2i} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{L(z,\chi)L(1-z,\bar{\chi})e^{-iz(\frac{\pi}{2}-s)}}{\cos\frac{\pi z}{2}}dz \\
=&e^{-\frac{is}{2}} \int_{-\infty}^{\infty} \frac{L(\frac{1}{2}+ix,\chi)L(\frac{1}{2}-ix,\bar{\chi})\exp\{-\frac{\pi i}{4}+\frac{\pi x}{2}+\frac{is}{2}-xs\}}{\exp\{\frac{\pi i}{4}-\frac{\pi x}{2}\}+\exp\{-\frac{\pi i}{4}+\frac{\pi x}{2}\}}dx \\
=&e^{-\frac{is}{2}} \int_{-\infty}^{\infty} \frac{L(\frac{1}{2}+ix,\chi)\overline{L(\frac{1}{2}+ix,\chi)}\exp\{-\frac{\pi i}{4}+\frac{\pi x}{2}+\frac{is}{2}-xs\}}{\exp\{\frac{\pi i}{4}-\frac{\pi x}{2}\}+\exp\{-\frac{\pi i}{4}+\frac{\pi x}{2}\}}dx \\
=&\int_0^\infty \frac{|L(\frac{1}{2}+ix,\chi)|^2\exp\{-\frac{\pi i}{4}+\frac{\pi x}{2}-xs\}}{\exp\{\frac{\pi i}{4}-\frac{\pi x}{2}\}+\exp\{-\frac{\pi i}{4}+\frac{\pi x}{2}\}}dx \\
+&\int_0^\infty \frac{|L(\frac{1}{2}+ix,\bar{\chi})|^2\exp\{-\frac{\pi i}{4}-\frac{\pi x}{2}+xs\}}{\exp\{\frac{\pi i}{4}+\frac{\pi x}{2}\}+\exp\{-\frac{\pi i}{4}-\frac{\pi x}{2}\}}dx.
\end{aligned} \tag{1.3}$$

Since

$$\begin{aligned}
&1-\frac{\exp\{-\frac{\pi i}{4}+\frac{\pi x}{2}\}}{\exp\{\frac{\pi i}{4}-\frac{\pi x}{2}\}+\exp\{-\frac{\pi i}{4}+\frac{\pi x}{2}\}} \\
=&\frac{\exp\{\frac{\pi i}{4}-\frac{\pi x}{2}\}}{\exp\{\frac{\pi i}{4}-\frac{\pi x}{2}\}+\exp\{-\frac{\pi i}{4}+\frac{\pi x}{2}\}},
\end{aligned}$$

we find from (1.2) and (1.3) that

$$\lambda(s,\chi)=\int\limits_0^\infty\left|L\left(\frac{1}{2}+ix,\chi\right)\right|^2 e^{-sx}dx-\frac{e^{-\frac{is}{2}}}{2i}\int\limits_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\frac{L(z,\chi)L(1-z,\bar\chi)e^{-iz(\frac{\pi}{2}-s)}}{\cos\frac{\pi z}{2}}dz$$

$$=\int\limits_0^\infty\left|L\left(\frac{1}{2}+ix,\chi\right)\right|^2 e^{-sx}dx-\int\limits_0^\infty\frac{|L(\frac{1}{2}+ix,\chi)|^2\exp\{-\frac{\pi i}{4}+\frac{\pi x}{2}-xs\}}{\exp\{\frac{\pi i}{4}-\frac{\pi x}{2}\}+\exp\{-\frac{\pi i}{4}+\frac{\pi x}{2}\}}dx$$

$$-\int\limits_0^\infty\frac{|L(\frac{1}{2}+ix,\bar\chi)|^2\exp\{-\frac{\pi i}{4}-\frac{\pi x}{2}+xs\}}{\exp\{\frac{\pi i}{4}+\frac{\pi x}{2}\}+\exp\{-\frac{\pi i}{4}-\frac{\pi x}{2}\}}dx \tag{1.4}$$

$$=\int\limits_0^\infty\frac{\left|L\left(\frac{1}{2}+ix,\chi\right)\right|^2 e^{-xs}\exp\{\frac{\pi i}{4}-\frac{\pi x}{2}\}}{\exp\{\frac{\pi i}{4}-\frac{\pi x}{2}\}+\exp\{-\frac{\pi i}{4}+\frac{\pi x}{2}\}}dx-\int\limits_0^\infty\frac{\left|L\left(\frac{1}{2}+ix,\bar\chi\right)\right|^2 e^{xs}\exp\{-\frac{\pi i}{4}-\frac{\pi x}{2}\}}{\exp\{\frac{\pi i}{4}+\frac{\pi x}{2}\}+\exp\{-\frac{\pi i}{4}-\frac{\pi x}{2}\}}dx.$$

Now let $b=1$. In this case, in view of the formula

$$\sin s=\frac{e^{is}-e^{-is}}{2i},$$

we find that

$$\frac{e^{-\frac{is}{2}}}{2}\int\limits_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\frac{L(z,\chi)L(1-z,\bar\chi)e^{-iz(\frac{\pi}{2}-s)}}{\sin\frac{\pi z}{2}}dz$$

$$=-\,e^{-\frac{is}{2}}\int\limits_{-\infty}^{\infty}\frac{L(\frac{1}{2}+ix,\chi)L(\frac{1}{2}-ix,\bar\chi)\exp\{-\frac{\pi i}{4}+\frac{\pi x}{2}+\frac{is}{2}-xs\}}{\exp\{\frac{\pi i}{4}-\frac{\pi x}{2}\}-\exp\{-\frac{\pi i}{4}+\frac{\pi x}{2}\}}dx$$

$$=-\int\limits_0^\infty\frac{|L(\frac{1}{2}+ix,\chi)|^2\exp\{-\frac{\pi i}{4}+\frac{\pi x}{2}-xs\}}{\exp\{\frac{\pi i}{4}-\frac{\pi x}{2}\}-\exp\{-\frac{\pi i}{4}+\frac{\pi x}{2}\}}dx \tag{1.5}$$

$$-\int\limits_0^\infty\frac{|L(\frac{1}{2}+ix,\bar\chi)|^2\exp\{-\frac{\pi i}{4}-\frac{\pi x}{2}+xs\}}{\exp\{\frac{\pi i}{4}+\frac{\pi x}{2}\}-\exp\{-\frac{\pi i}{4}-\frac{\pi x}{2}\}}dx.$$

Clearly, we have

$$1+\frac{\exp\{-\frac{\pi i}{4}+\frac{\pi x}{2}\}}{\exp\{\frac{\pi i}{4}-\frac{\pi x}{2}\}-\exp\{-\frac{\pi i}{4}+\frac{\pi x}{2}\}}$$
$$=\frac{\exp\{\frac{\pi i}{4}-\frac{\pi x}{2}\}}{\exp\{\frac{\pi i}{4}-\frac{\pi x}{2}\}-\exp\{-\frac{\pi i}{4}+\frac{\pi x}{2}\}},$$

thus, from (1.2) and (1.5), it follows that

$$\lambda(s,\chi)=\int_0^\infty \frac{\left|L\left(\frac{1}{2}+ix,\chi\right)\right|^2 e^{-xs}\exp\{\frac{\pi i}{4}-\frac{\pi x}{2}\}}{\exp\{\frac{\pi i}{4}-\frac{\pi x}{2}\}-\exp\{-\frac{\pi i}{4}+\frac{\pi x}{2}\}}dx$$
$$+\int_0^\infty \frac{\left|L\left(\frac{1}{2}+ix,\bar{\chi}\right)\right|^2 e^{xs}\exp\{-\frac{\pi i}{4}-\frac{\pi x}{2}\}}{\exp\{\frac{\pi i}{4}+\frac{\pi x}{2}\}-\exp\{-\frac{\pi i}{4}-\frac{\pi x}{2}\}}dx. \tag{1.6}$$

By estimates for $L\left(\frac{1}{2}+ix,\chi\right)$ of Lemma 1.3, we have that the integrals in (1.4) and (1.6) converge uniformly on compact subsets of the strip$\{s\in\mathbb{C}:|\sigma|<\pi\}$, thus, the function $\lambda(s,\chi)$ is analytic in that strip.

It remains to estimate the function $\lambda(s,\chi)$. Suppose that $|\sigma|\le\theta$, where $0<\theta<\pi$. First let $|s|$ be small. Then the integrals in (1.4) and (1.6) are convergent, therefore bounded by a constant. If $|s|$ is large, then integrating by parts with respect to $e^{\pm sx}$ and using the estimate

$$\left(\left|L\left(\frac{1}{2}+ix,\chi\right)\right|^2\right)'=\left(L\left(\frac{1}{2}+ix,\chi\right)L\left(\frac{1}{2}-ix,\bar{\chi}\right)\right)'$$
$$=iL'\left(\frac{1}{2}+ix,\chi\right)L\left(\frac{1}{2}-ix,\bar{\chi}\right)-iL\left(\frac{1}{2}+ix,\chi\right)L'\left(\frac{1}{2}-ix,\bar{\chi}\right)=$$
$$=O\Big(q\big(|x|+2\big)\Big)^{c_2}$$

with some $c_2>0$, which follows from Lemmas 1.3 and 1.4, we obtain, that

$$\lambda(s,\chi)=O(|s|^{-1}).$$

So, in all cases, we have that, for $|\sigma|\le\theta$, $0<\theta<\pi$,

$$\lambda(s,\chi)=O(1+|s|)^{-1}.$$

Equality (1.2) shows that

$$\mathfrak{L}(s,\chi)=\frac{e^{-\frac{is}{2}}}{2i^{1-b}}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\frac{L(z,\chi)L(1-z,\bar{\chi})e^{-iz(\frac{\pi}{2}-s)}}{\cos\left(\frac{\pi b}{2}-\frac{\pi z}{2}\right)}dz+\lambda(s,\chi), \tag{1.7}$$

where the function $\lambda(s,\chi)$ is analytic in the strip $\{s\in\mathbb{C}:|\sigma|<\pi\}$, and for $|\sigma|\le\theta$, $0<\theta<\pi$, the estimate

$$\lambda(s,\chi)=O(1+|s|)^{-1}$$

is valid.

It remains to calculate the integral in (1.7). Using Lemma 1.1, we find

$$L(1-z,\bar{\chi}) = E^{-1}(\chi)\left(\frac{\pi}{q}\right)^{-z+\frac{1}{2}}\frac{\Gamma(\frac{z}{2}+\frac{b}{2})}{\Gamma(\frac{1}{2}+\frac{b}{2}-\frac{z}{2})}L(z,\chi). \tag{1.8}$$

Taking into account the formulae, see [25],

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin\pi z}, \qquad z\notin\mathbb{Z}, \tag{1.9}$$

and

$$\Gamma(z)\Gamma\left(z+\frac{1}{2}\right) = 2\sqrt{\pi}2^{-2z}\Gamma(2z),$$

we obtain that

$$\begin{aligned}\frac{\Gamma(\frac{z}{2})}{\Gamma(\frac{1}{2}-\frac{z}{2})} =& \frac{\Gamma(\frac{z}{2})\Gamma(\frac{1}{2}+\frac{z}{2})}{\Gamma(\frac{1}{2}-\frac{z}{2})\Gamma(\frac{1}{2}+\frac{z}{2})} = \frac{\Gamma(\frac{z}{2})\Gamma(\frac{1}{2}+\frac{z}{2})\cos\frac{\pi z}{2}}{\pi} = \\ &= \frac{2\sqrt{\pi}2^{-z}\Gamma(z)\cos\frac{\pi z}{2}}{\pi} = 2^{1-z}\pi^{-\frac{1}{2}}\Gamma(z)\cos\frac{\pi z}{2},\end{aligned} \tag{1.10}$$

and

$$\begin{aligned}\frac{\Gamma(\frac{1}{2}+\frac{z}{2})}{\Gamma(1-\frac{z}{2})} =& \frac{\Gamma(\frac{1}{2}+\frac{z}{2})\Gamma(\frac{1}{2}-\frac{z}{2})}{\Gamma(1-\frac{z}{2})\Gamma(\frac{1}{2}-\frac{z}{2})} = \frac{\pi}{\cos\frac{\pi z}{2}2\sqrt{\pi}2^{-1+z}\Gamma(1-z)} = \\ &= \frac{\Gamma(z)\sin\pi z}{2^z\sqrt{\pi}\cos\frac{\pi z}{2}} = 2^{1-z}\pi^{-\frac{1}{2}}\Gamma(z)\sin\frac{\pi z}{2}.\end{aligned}$$

Since $b=0$ or $b=1$, hence have that

$$\frac{\Gamma(\frac{z}{2}+\frac{b}{2})}{\Gamma(\frac{1}{2}+\frac{b}{2}-\frac{z}{2})} = 2^{1-z}\pi^{-\frac{1}{2}}\Gamma(z)\cos\left(\frac{\pi b}{2}-\frac{\pi z}{2}\right),$$

and, in view of (1.8), this gives

$$L(1-z,\bar{\chi}) = E^{-1}(\chi)2^{1-z}\pi^{-z}q^{z-\frac{1}{2}}\Gamma(z)\cos\left(\frac{\pi b}{2}-\frac{\pi z}{2}\right)L(z,\chi).$$

Hence, we find that

$$\begin{aligned}&\frac{e^{-\frac{is}{2}}}{2i^{1-b}}\int\limits_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\frac{L(z,\chi)L(1-z,\bar{\chi})e^{-iz(\frac{\pi}{2}-s)}}{\cos\left(\frac{\pi b}{2}-\frac{\pi z}{2}\right)}dz \\ =&\frac{e^{-\frac{is}{2}}}{i^{1-b}\sqrt{q}E(\chi)}\int\limits_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\Gamma(z)L^2(z,\chi)\left(\frac{2\pi i}{q}e^{-is}\right)^{-z}dz.\end{aligned} \tag{1.11}$$

Now we shift the line of integration to the right.

Proof of Theorem 1.1. For a non-principal character χ the integrand in (1.11) is a regular function in the strip $\{z \in \mathbb{Z} : \frac{1}{2} < \Re z < 2\}$. Therefore,

$$\begin{aligned}
&\frac{e^{-\frac{is}{2}}}{i^{1-b}\sqrt{q}E(\chi)} \int\limits_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \Gamma(z)L^2(z,\chi)\left(\frac{2\pi i}{q}e^{-is}\right)^{-z} dz \\
=&\frac{e^{-\frac{is}{2}}}{i^{1-b}\sqrt{q}E(\chi)} \int\limits_{2-i\infty}^{2+i\infty} \Gamma(z)L^2(z,\chi)\left(\frac{2\pi i}{q}e^{-is}\right)^{-z} dz. \qquad (1.12)
\end{aligned}$$

Moreover, in view of Lemmas 1.2 and 1.5,

$$\begin{aligned}
&\int\limits_{2-i\infty}^{2+i\infty} \Gamma(z)L^2(z,\chi)\left(\frac{2\pi i}{q}e^{-is}\right)^{-z} dz \\
=&\sum_{m=1}^{\infty} d(m)\chi(m) \int\limits_{2-i\infty}^{2+i\infty} \Gamma(z)\left(\frac{2\pi i m}{q}e^{-is}\right)^{-z} dz \\
=&2\pi i \sum_{m=1}^{\infty} d(m)\chi(m) \exp\left\{-\frac{2\pi i m}{q}e^{-is}\right\}.
\end{aligned}$$

This, (1.11), (1.12) and (1.7) prove Theorem 1.1.

□

Proof of Theorem 1.2. For the principal Dirichlet character χ_0, we have that

$$L(s,\chi_0) = \zeta(s)\prod_{p|q}\left(1-\frac{1}{p^s}\right).$$

Using the functional equation for $\zeta(s)$

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s),$$

and formula (1.10), we find that

$$\begin{aligned}
&L(1-z,\chi_0) = \zeta(1-z)\prod_{p|q}\left(1-\frac{1}{p^{1-z}}\right) \\
=&\frac{\pi^{-\frac{z}{2}}\Gamma(\frac{z}{2})\zeta(z)}{\pi^{-\frac{1-z}{2}}\Gamma(\frac{1-z}{2})}\prod_{p|q}\left(1-\frac{1}{p^{1-z}}\right) \\
=&\zeta(z)2^{1-z}\pi^{-z}\Gamma(z)\cos\frac{\pi z}{2}\prod_{p|q}\left(1-\frac{1}{p^{1-z}}\right).
\end{aligned}$$

Therefore,

$$
\begin{aligned}
&L(z,\chi_0)L(1-z,\chi_0)e^{-iz(\frac{\pi}{2}-s)}\\
=&\zeta^2(z)2^{1-z}\pi^{-z}\Gamma(z)\cos\frac{\pi z}{2}e^{-iz(\frac{\pi}{2}-s)}\prod_{p|q}\left(1-\frac{1}{p^z}\right)\prod_{p|q}\left(1-\frac{1}{p^{1-z}}\right).
\end{aligned}
$$

Similarly (see formula (1.2)) to the case of non-principal character, we define the function $\lambda(s,\chi_0)$

$$
\begin{aligned}
\lambda(s,\chi_0)=&\int\limits_0^\infty\left|L\left(\frac{1}{2}+ix,\chi_0\right)\right|^2e^{-sx}dx-\\
&-\frac{e^{-\frac{is}{2}}}{2i}\int\limits_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\frac{L(z,\chi_0)L(1-z,\chi_0)e^{-iz(\frac{\pi}{2}-s)}}{\cos\frac{\pi z}{2}}dz
\end{aligned}
$$

and obtain that

$$
\begin{aligned}
\mathfrak{L}(s,\chi_0)=&\frac{e^{-\frac{is}{2}}}{i}\int\limits_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\Gamma(z)\zeta^2(z)\left(2\pi ie^{-is}\right)^{-z}\prod_{p|q}\left(1-\frac{1}{p^z}\right)\prod_{p|q}\left(1-\frac{1}{p^{1-z}}\right)dz\\
&+\lambda(s,\chi_0). \qquad (1.13)
\end{aligned}
$$

We write the product

$$
\prod\nolimits_q(z)=\prod_{p|q}\left(1-\frac{1}{p^{1-z}}\right)
$$

as a sum using the Mӧbius function $\mu(m)$. Let $q=p_1^{\alpha_1}p_2^{\alpha_2}...p_k^{\alpha_k}$, then

$$
\begin{aligned}
\prod\nolimits_q(z)=&\prod_{p|q}\left(1-\frac{1}{p^{1-z}}\right)\\
&=1-p_1^{z-1}+(p_1p_2)^{z-1}+...+(-1)^k(p_1p_2...p_k)^{z-1}=\sum_{m|q}\mu(m)m^{z-1}.
\end{aligned}
$$

The function

$$
\Gamma(z)\zeta^2(z)(2\pi ie^{-is})^{-z}\prod_{p|q}\left(1-\frac{1}{p^z}\right)m^{z-1}
$$

has the pole of order 2 at $z=1$, and its Laurent expansion at this point is

$$
\zeta(z)=\frac{1}{z-1}+\gamma_0+\gamma_1(z-1)+...,
$$

therefore,

$$
\zeta^2(z)=\frac{1}{(z-1)^2}+2\gamma_0\frac{1}{z-1}+...,
$$

and

$$(z-1)^2\zeta^2(z) = 1 + 2\gamma_0(z-1) + ...,$$

moreover, since $\Gamma(1) = 1$, $\Gamma'(1) = -\gamma_0$, we find that

$$\begin{aligned}
&\operatorname*{Res}_{z=1}\Gamma(z)\zeta^2(z)\left(2\pi i e^{-is}\right)^{-z} = \lim_{z\to 1}\left((z-1)^2\Gamma(z)\zeta^2(z)\left(2\pi i e^{-is}\right)^{-z}\right)' \\
&= \lim_{z\to 1}\left(\Gamma(z)\left(2\pi i e^{-is}\right)^{-z} + 2\gamma_0(z-1)\Gamma(z)(2\pi i e^{-is})^{-z}\right)' \\
&= \lim_{z\to 1}\Big(\Gamma'(z)\left(2\pi i e^{-is}\right)^{-z} - \Gamma(z)\left(2\pi i e^{-is}\right)^{-z}\log 2\pi i e^{-is} \\
&+ 2\gamma_0(z-1)\Gamma'(z) - 2\gamma_0(z-1)\Gamma(z)(2\pi i e^{-is})^{-z}\log 2\pi i e^{-is} \\
&+ 2\gamma_0\Gamma(z)(2\pi i e^{-is})^{-z}\Big) \\
&= \frac{1}{2\pi i e^{-is}}\left(-\gamma_0 - \log 2\pi i e^{-is} + 2\gamma_0\right) \\
&= \frac{e^{is}}{2\pi i}\left(\gamma_0 - \log 2\pi - \left(\frac{\pi}{2} - s\right)i\right).
\end{aligned}$$

Finally, we get

$$\begin{aligned}
&\operatorname*{Res}_{z=1}\Gamma(z)\zeta^2(z)\left(2\pi i e^{-is}\right)^{-z}\prod_{p|q}\left(1-\frac{1}{p^z}\right)m^{z-1} \\
&= \lim_{z\to 1}\left((z-1)^2\Gamma(z)\zeta^2(z)\left(2\pi i e^{-is}\right)^{-z}\prod_{p|q}\left(1-\frac{1}{p^z}\right)m^{z-1}\right)' \\
&= \lim_{z\to 1}\left((z-1)^2\Gamma(z)\zeta^2(z)\left(2\pi i e^{-is}\right)^{-z}\right)'\prod_{p|q}\left(1-\frac{1}{p^z}\right)m^{z-1} \\
&+ \lim_{z\to 1}(z-1)^2\Gamma(z)\zeta^2(z)\left(2\pi i e^{-is}\right)^{-z}\prod_{p|q}\left(1-\frac{1}{p^z}\right)\sum_{p|q}\frac{p^{\frac{1}{z}}\log p}{1-\frac{1}{p^z}}m^{z-1} \\
&+ \lim_{z\to 1}(z-1)^2\Gamma(z)\zeta^2(z)\left(2\pi i e^{-is}\right)^{-z}\log m\prod_{p|q}\left(1-\frac{1}{p^z}\right)m^{z-1} \\
&= \frac{e^{is}}{2\pi i}\left(\gamma_0 - \log 2\pi - \left(\frac{\pi}{2} - s\right)i\right)\prod_{p|q}\left(1-\frac{1}{p}\right) \\
&+ (2\pi i e^{-is})^{-1}\prod_{p|q}\left(1-\frac{1}{p}\right)\sum_{p|q}\frac{\log p}{p-1} \\
&+ (2\pi i e^{-is})^{-1}\log m\prod_{p|q}\left(1-\frac{1}{p}\right) \\
&= \frac{e^{is}}{2\pi i}\prod_{p|q}\left(1-\frac{1}{p}\right)\left(\gamma_0 - \log 2\pi - \left(\frac{\pi}{2} - s\right)i + \sum_{p|q}\frac{\log p}{p-1} + \log m\right).
\end{aligned}$$

Therefore, in view of (1.13),

$$\mathfrak{L}(s,\chi_0)$$

$$= ie^{\frac{is}{2}}\prod_{p|q}\left(1-\frac{1}{p}\right)\sum_{m|q}\left(\gamma_0-\log 2\pi-\left(\frac{\pi}{2}-s\right)i+\sum_{p|q}\frac{\log p}{p-1}+\mu(m)\log m\right)$$

$$-\,ie^{-\frac{is}{2}}\int\limits_{2-i\infty}^{2+i\infty}\Gamma(z)\zeta^2(z)\left(2\pi ie^{-is}\right)^{-z}\prod_{p|q}\left(1-\frac{1}{p^z}\right)\left(1-\frac{1}{p^{1-z}}\right)dz+\lambda(s,\chi_0). \quad (1.14)$$

We have that

$$\prod_{p|q}\left(1-\frac{1}{p^z}\right)\left(1-\frac{1}{p^{1-z}}\right)=\sum_{n|q}\sum_{m|q}\frac{\mu(m)}{m}\mu(n)\left(\frac{m}{n}\right)^z.$$

Moreover, for $\sigma>1$, [33]

$$\zeta^2(s)=\sum_{k=1}^{\infty}\frac{d(k)}{k^s}.$$

Therefore, the application of Lemma 1.5, leads to

$$\int\limits_{2-i\infty}^{2+i\infty}\Gamma(z)\zeta^2(z)\left(2\pi ie^{-is}\right)^{-z}\prod_{p|q}\left(1-\frac{1}{p^z}\right)\left(1-\frac{1}{p^{1-z}}\right)dz$$

$$=2\pi i\sum_{n|q}\sum_{m|q}\frac{\mu(m)\mu(n)}{m}\sum_{k=1}^{\infty}d(k)\exp\left\{-\frac{2\pi ikn}{m}e^{-is}\right\}.$$

This together with (1.14) proves the theorem.

□

Further, we study the modified Laplace transform of a Dirichlet L- function, which is defined

$$\mathfrak{L}(s,\chi)\overset{def}{=}\int_1^{\infty}\left|L\left(\frac{1}{2}+ix,\chi\right)\right|^2e^{-sx}dx. \quad (1.15)$$

It differs a bit from the usual Laplace transform, where the lower limit of integration is 0. By partial integration, using Lemmas 1.3 and 1.4, similarly to the proof of Theorems 1.1 and 1.2 we find that

$$\mathfrak{L}(s,\chi)=\int_0^1\left|L\left(\frac{1}{2}+ix,\chi\right)\right|^2e^{-sx}dx=O((1+|s|))^{-1}.$$

Moreover, the last integral is an entire function. Therefore, it can be included in the function $\lambda(s)$. So, Theorems 1.1 and 1.2 also are valid for the Laplace transform defined by formula (1.15) As it was mentioned above, in Chapter 3, we will use Theorems 1.1 and 1.2 for the meromorphic continuation of the modified Mellin transform. For this, we will understand the Laplace transform as in formula (1.15).

Chapter 2

Summation formulae with exponential term

In meromorphic continuation for Mellin transform of Dirichlet L-functions, some specific integrals appear. In general case, such integrals do not converge absolutely, therefore we transform the integrand. After this, it turn out that we could decompose Mellin transform to a holomorphic part with $d(m)$ series, and parts which produce poles. For this, transformation formulae are used, and this chapter is devoted for them, see [2] and [4].

2.1 Transformation of exponential sums

Let k and l be coprime positive integers, and $\Im z \neq 0$. For $\Re z > 0$, define

$$\Phi\left(z; \frac{k}{l}\right) = \sum_{m=1}^{\infty} d(m) e^{2\pi i m \frac{k}{l}} e^{-mz} - \frac{\gamma_0 - 2\log l - \log z}{lz}, \tag{2.1}$$

and, for a Dirichlet character χ modulo q,

$$\begin{aligned} \Phi\left(z; \chi, q\right) = &\sum_{m=1}^{\infty} d(m)\chi(m) e^{-2\pi i m/q} e^{-mz} \\ &- \frac{1}{G(\overline{\chi})} \sum_{a=1}^{q} \overline{\chi}(a) \frac{(q, a-1)}{qz} \left(\gamma_0 - 2\log\frac{q}{(q, a-1)} - \log z\right). \end{aligned} \tag{2.2}$$

The aim is to obtain formulae for $\Phi\left(z^{-1}; \frac{k}{l}\right)$ and $\Phi\left(z^{-1}; \chi, q\right)$.

Let $\overline{k}$ be connected to k by the congruence $k\overline{k} \equiv 1(\mod l)$, a_0^+ and a_0^- be the constant terms in the Laurent series expansions for Estermann zeta-functions $E\left(s; \frac{\overline{k}}{l}, 0\right)$ and $E\left(s; -\frac{\overline{k}}{l}, 0\right)$, respectively, which we will define in next section. Denote

$$\delta = \begin{cases} 1 & \text{if} \quad Imz > 0, \\ -1 & \text{if} \quad Imz < 0, \end{cases}$$

and, for $1 < b < 2$, define

$$I(z,b) = \frac{1}{2\pi i} \int\limits_{b-i\infty}^{b+i\infty} \left(\frac{2\pi}{l}\right)^{1-2w} \Gamma(w) \Bigg((\sin(\pi w))^{-1} E\left(w; \frac{\overline{k}}{l}, 0\right)$$
$$+ (\cot(\pi w) + \delta i) E\left(w; -\frac{\overline{k}}{l}, 0\right)\Bigg) z^{1-w} dw,$$

and, for Dirichlet character χ,

$$I(z;\chi,b) = \frac{1}{2\pi i G(\overline{\chi})} \sum_{a=1}^{q} \overline{\chi}(a) \int\limits_{b-i\infty}^{b+i\infty} \left(\frac{2\pi}{q}\right)^{1-2w} \Gamma(w)$$
$$\times \Bigg\{ \sin^{-1}(\pi w) E\left(w; \frac{\overline{\left(\frac{a-1}{(q,a-1)}\right)}}{\frac{q}{(q,a-1)}}, 0\right) (q, a-1)^{1-2w} + \cot(\pi w)$$
$$\times E\left(w; -\frac{\overline{\left(\frac{a-1}{(q,a-1)}\right)}}{\frac{q}{(q,a-1)}}, 0\right) (q, a-1)^{1-2w} + \delta i E\left(w; \frac{\frac{(a-1)}{(q,a-1)}}{\frac{q}{(q,a-1)}}, 0\right) \Bigg\} z^{1-w} dw,$$

$$\widetilde{I}(z;\chi,b) = \frac{1}{2\pi i G(\overline{\chi})} \sum_{a=1}^{q} \overline{\chi}(a) \int\limits_{b-i\infty}^{b+i\infty} \left(\frac{2\pi(q,a-1)}{q}\right)^{1-2w} \Gamma(w)$$
$$\times \Bigg\{ \sin^{-1}(\pi w) E\left(w; \frac{\overline{\left(\frac{a-1}{(q,a-1)}\right)}}{\frac{q}{(q,a-1)}}, 0\right) + (\cot(\pi w) + \delta i)$$
$$\times E\left(w; -\frac{\overline{\left(\frac{a-1}{(q,a-1)}\right)}}{\frac{q}{(q,a-1)}}, 0\right) \Bigg\} z^{1-w} dw.$$

Theorem 2.1. *If $\Re z > 0$ and $\Im z \neq 0$, then for the function $\Phi\left(z; \frac{k}{l}\right)$ the transformation formula*

$$\Phi\left(z^{-1}; \frac{k}{l}\right) = -\frac{2\pi i \delta z}{l} \sum_{m=1}^{\infty} d(m) e^{-2\pi i m \frac{\overline{k}}{l}} e^{-\frac{4\pi^2 m z}{l^2}} + \frac{l}{2\pi^2}\left(a_0^+ - a_0^-\right) + \frac{1}{4} + I(z,b)$$

is valid.

Theorem 2.2. *If $\Re z > 0$ and $\Im z \neq 0$, then for the function $\Phi(z; \chi, q)$ the transformation formula*

$$\Phi\left(z^{-1}; \chi, q\right) = -\frac{2\pi i \delta z}{q} \sum_{m=1}^{\infty} d(m)\chi(m) e^{-2\pi i m/q} e^{-\frac{4\pi^2 m z}{q^2}}$$
$$+ \frac{q}{2\pi^2 G(\overline{\chi})} \sum_{a=1}^{q} \frac{\overline{\chi}(a)}{(q, a-1)} \left(a_{0a}^{+} - a_{0a}^{-}\right) + I(z; \chi, b)$$

is valid.

Theorem 2.3. *If $\Re z > 0$ and $\Im z \neq 0$, then for the function $\Phi(z; \chi, q)$ the transformation formula*

$$\Phi\left(z^{-1}; \chi, q\right) = -\frac{2\pi i \delta z}{G(\overline{\chi}) q} \sum_{a=1}^{q} \overline{\chi}(a)(q, a-1) \sum_{m=1}^{\infty} d(m) e^{-2\pi i m \frac{\overline{(a-1)/(q,a-1)}}{q/(q,a-1)}} e^{-\frac{4\pi^2 m (q,a-1)^2 z}{q^2}}$$
$$+ \frac{q}{2\pi^2 G(\overline{\chi})} \sum_{a=1}^{q} \frac{\overline{\chi}(a)}{(q, a-1)} (a_{0a}^{+} - a_{0a}^{-}) + \widetilde{I}(z; \chi, b)$$

is valid.

2.2 Estermann zeta-function

Let $l > 1$ and $(k, l) = 1$. The Estermann zeta-function $E\left(s; \frac{k}{l}, \alpha\right)$, for $\sigma > \max(1 + \Re\alpha, 1)$, is defined by the series

$$E\left(s; \frac{k}{l}, \alpha\right) \stackrel{def}{=} \sum_{m=1}^{\infty} \frac{\sigma_\alpha(m)}{m^s} \exp\{2\pi i m \frac{k}{l}\},$$

where, for $\alpha \in \mathbb{C}$,

$$\sigma_\alpha(m) = \sum_{d|m} d^\alpha,$$

is the generalized divisor function.

For $\lambda \in \mathbb{R}$ and $0 < \beta \leqslant 1$, denote by $L(\lambda, \beta, s)$ the Lerch zeta-function defined, for $\sigma > 1$, by

$$L(\lambda, \beta, s) \stackrel{def}{=} \sum_{m=0}^{\infty} \frac{e^{2\pi i m \lambda}}{(m+\beta)^s}$$

It is well known, see, for example, [25], $L(\lambda, \beta, s)$, for $\lambda \notin \mathbb{Z}$, is analytically continuable to an entire function, while for $\lambda \in \mathbb{Z}$, the function $L(\lambda, \beta, s)$ becomes the Hurwitz zeta-function

$$\zeta(s, \beta) \stackrel{def}{=} \sum_{m=0}^{\infty} \frac{1}{(m+\beta)^s},$$

which is meromorphically continuable to the whole complex plane where it has a simple pole at $s = 1$ with residue 1.

It is not difficult to see that, for $\sigma > max(1 + \Re\alpha, 1)$,

$$E\left(s; \frac{k}{l}, \alpha\right) = l^{\alpha - s} \sum_{v=1}^{l} \exp\{2\pi i \frac{vk}{l}\} L(1, \frac{v}{l}, s - \alpha) L(\frac{v}{l}, 1, s). \tag{2.3}$$

This equality and the mentioned properties of $L(\lambda, \beta, s)$ show that the function $E\left(s; \frac{k}{l}, \alpha\right)$ is analytic in the whole complex plane, except for two simple poles at $s = 1$ and $s = 1+\alpha$ if $\alpha \neq 0$, and a double pole $s = 1$ if $\alpha = 0$.

Equality (2.3) together with the functional equation for the Lerch zeta-function, see, for example,[25], leads to the functional equation for $E\left(s; \frac{k}{l}, \alpha\right)$

$$E\left(s; \frac{k}{l}, \alpha\right) = \frac{1}{\pi}\left(\frac{2\pi}{l}\right)^{2s-1-\alpha} \Gamma(1-s)\Gamma(1+\alpha-s) \times$$
$$\left(\cos\frac{\pi\alpha}{2} E\left(1+\alpha-s; \frac{\overline{k}}{l}, \alpha\right) - \cos\left(\pi s - \frac{\pi\alpha}{2}\right) E\left(1+\alpha-s; -\frac{\overline{k}}{l}, \alpha\right)\right). \tag{2.4}$$

The functions $L(1, \frac{v}{l}, s)$ and $L(\frac{k}{l}, 1, s)$ are expressed linearly by Hurwitz zeta-functions and the Riemann zeta-function. Therefore, it is not difficult to obtain that (see [17])

$$E\left(s; \frac{k}{l}, 0\right) = \frac{1}{l}\left(\frac{1}{(s-1)^2} + \frac{2\gamma_0 - 2\log l}{s-1}\right) + c_0\left(\frac{k}{l}\right) + \dots. \tag{2.5}$$

The function $E(s; \frac{k}{l}, \alpha)$, for $\alpha = 0$, was introduced by T. Estermann in [7] for needs of the representation of numbers as a sum of two products. In [19], the extension for $\alpha \in [-1, 0]$ was given. From formula (2.4), when $\alpha = 0$, we find that

$$E\left(1-s; \frac{k}{l}, 0\right) = \frac{1}{\pi}\left(\frac{2\pi}{l}\right)^{2-2s} \Gamma^2(s)\left(E\left(s; \frac{\overline{k}}{l}, 0\right) + \cos(\pi s) E\left(s; -\frac{\overline{k}}{l}, 0\right)\right). \tag{2.6}$$

The series of the definition for $\Phi(z; \frac{k}{l})$ contains the product $d(m)\exp\{2\pi i\frac{k}{l}\}$ which are coefficients of the Dirichlet series for $E\left(s; \frac{k}{l}, 0\right)$. Therefore, the function $E\left(s; \frac{k}{l}, 0\right)$ is involved in the formulae for $\Phi\left(z^{-1}; \frac{k}{l}\right)$ and $\Phi\left(z^{-1}; \chi, q\right)$.

2.3 Auxiliary results

In the proof of Theorem 2.2, we need to express the exponential sum $\sum_{m=1}^{\infty} d(m)\chi(m)\exp\{2\pi im/q\}m^{-s}$ by the Gaussian sum. Therefore, we will use the following lemma.

Lemma 2.4. *For $\sigma > 1$,*

$$\sum_{m=1}^{\infty} d(m)\chi(m)\exp\left\{-\frac{2\pi im}{q}\right\}m^{-s} = \frac{1}{G(\overline{\chi})}\sum_{a=1}^{q}\overline{\chi}(a)E\left(s;\frac{\frac{(a-1)}{(q,a-1)}}{\frac{q}{(q,a-1)}},0\right).$$

Proof. It is well known, see, for example, [6], that, for every $m \in \mathbb{N}$,

$$\chi(m)G(\overline{\chi}) = \sum_{a=1}^{q}\chi(a)e^{2\pi ima/q}$$

Using this, we get

$$\begin{aligned}
&G(\overline{\chi})\sum_{m=1}^{\infty} d(m)\chi(m)\exp\left\{-\frac{2\pi im}{q}\right\}m^{-s}\\
&=\sum_{m=1}^{\infty} d(m)\exp\left\{-\frac{2\pi im}{q}\right\}m^{-s}\sum_{a=1}^{q}\overline{\chi}(a)e^{2\pi ima/q}\\
&=\sum_{a=1}^{q}\overline{\chi}(a)\sum_{m=1}^{\infty} d(m)e^{2\pi im(a-1)/q}m^{-s} = \sum_{a=1}^{q}\overline{\chi}(a)E\left(s;\frac{\frac{(a-1)}{(q,a-1)}}{\frac{q}{(q,a-1)}},0\right),
\end{aligned}$$

and the claim of the lemma follows.

□

2.4 Proof of transformation formulae

Proof of Theorem 2.1. For $\Re z > 0$, the series

$$\sum_{m=1}^{\infty} d(m)\exp\left\{2\pi im\frac{k}{l}\right\}e^{-mz}$$

converges absolutely, therefore, by the Mellin formula (Lemma 1.5) and definition of the Estermann zeta-function, we have

$$\sum_{m=1}^{\infty} d(m) \exp\left\{2\pi i m \frac{k}{l}\right\} e^{-mz}$$
$$= \sum_{m=1}^{\infty} d(m) \exp\left\{2\pi i m \frac{k}{l}\right\} \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \Gamma(w)(mz)^{-w} dw \tag{2.7}$$
$$= \frac{1}{2\pi i} \int_{2-\infty}^{2+\infty} \Gamma(w) E\left(w; \frac{k}{l}, 0\right) z^{-w} dw.$$

Now we move the line of integration in (2.7) to the left. Let $0 < a < 1$. Since, as it is noted in Section 2.2, the function $E(w; \frac{k}{l}, 0)$ has a double pole at $w = 1$, we obtain that

$$\sum_{m=1}^{\infty} d(m) \exp\left\{2\pi i m \frac{k}{l}\right\} e^{-mz} = \frac{1}{2\pi i} \int_{a-\infty}^{a+\infty} \Gamma(w) E\left(w; \frac{k}{l}, 0\right) z^{-w} dw$$
$$+ \operatorname*{Res}_{w=1} \Gamma(w) E\left(w; \frac{k}{l}, 0\right) z^{-w}. \tag{2.8}$$

Clearly,

$$\Gamma(w) = 1 - \gamma_0(w-1) + \frac{\Gamma''(1)(w-1)^2}{2} + ..., \tag{2.9}$$

$$z^{-w} = z^{-1} e^{-(w-1)\log z} = z^{-1} \left(1 - (w-1)\log z + \frac{(w-1)^2 \log^2 z}{2} + ...\right). \tag{2.10}$$

Therefore, in view of (2.5),

$$\operatorname*{Res}_{w=1} \Gamma(w) E\left(w; \frac{k}{l}, 0\right) z^{-w} = \frac{\gamma_0 - 2\log l - \log z}{lz}.$$

This, (2.8) and the definition of $\Phi\left(z; \frac{k}{l}\right)$ show that

$$\Phi\left(z; \frac{k}{l}\right) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \Gamma(w) E\left(w; \frac{k}{l}, 0\right) z^{-w} dw. \tag{2.11}$$

Hence,

$$\Phi\left(z^{-1}; \frac{k}{l}\right) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \Gamma(w) E\left(w; \frac{k}{l}, 0\right) z^{w} dw$$
$$= \frac{1}{2\pi i} \int_{1-a-i\infty}^{1-a-+i\infty} \Gamma(1-w) E\left(1-w; \frac{k}{l}, 0\right) z^{1-w} dw. \tag{2.12}$$

Substituting formulae (2.6) and (1.11) into (2.12), we have

$$
\begin{aligned}
\Phi\left(z^{-1};\frac{k}{l}\right) =& \frac{1}{2\pi i}\int\limits_{1-a-i\infty}^{1-a+i\infty}\left(\frac{2\pi}{l}\right)^{1-2w}\Gamma(w)(\sin\pi w)^{-1}E\left(w;\frac{\overline{k}}{l},0\right)z^{1-w}d \\
&+\frac{1}{2\pi i}\int\limits_{1-a-i\infty}^{1-a+i\infty}\left(\frac{2\pi}{l}\right)^{1-2w}\Gamma(w)\cot(\pi w)E\left(w;-\frac{\overline{k}}{l},0\right)z^{1-w}dw \\
&=-\frac{\delta i 2\pi z}{l}\left(\frac{1}{2\pi i}\int\limits_{1-a-i\infty}^{1-a+i\infty}\Gamma(w)E\left(w;-\frac{\overline{k}}{l},0\right)\left(\frac{4\pi^2 z}{l^2}\right)^{-w}dw\right) \qquad (2.13)\\
&+\frac{1}{2\pi i}\int\limits_{1-a-i\infty}^{1-a+i\infty}\left(\frac{2\pi}{l}\right)^{1-2w}\Gamma(w)\left((\sin(\pi w))^{-1}E\left(w;\frac{\overline{k}}{l},0\right)\right.\\
&\left.+(\cot(\pi w)+\delta i)E\left(w;-\frac{\overline{k}}{l},0\right)\right)z^{1-w}dw \\
&=-\frac{2\pi i\delta z}{l}\Phi\left(\frac{4\pi^2 z}{l^2};-\frac{\overline{k}}{l}\right)+\frac{1}{2\pi i}\int\limits_{1-a-i\infty}^{1-a+i\infty}\left(\frac{2\pi}{l}\right)^{1-2w}\Gamma(w)\\
&\times\left((\sin(\pi w))^{-1}E\left(w;\frac{\overline{k}}{l},0\right)+(\cot(\pi w)+\delta i)E\left(w;-\frac{\overline{k}}{l},0\right)\right)z^{1-w}dw\\
&\overset{def}{=}-\frac{2\pi i\delta z}{l}\Phi\left(\frac{4\pi^2 z}{l^2};-\frac{\overline{k}}{l}\right)+I,
\end{aligned}
$$

in virtue of (2.11), since $0<1-a<1$.

It remains to transform the integral I in (2.13). For this, we move the line of integration to the right. Let $1<b<2$. Then we have that

$$
\begin{aligned}
I=&\frac{1}{2\pi i}\int\limits_{b-i\infty}^{b+i\infty}\left(\frac{2\pi}{l}\right)^{1-2w}\Gamma(w)\left((\sin(\pi w))^{-1}E\left(w;\frac{\overline{k}}{l},0\right)\right.\\
&\left.+(\cot(\pi w)+\delta i)E\left(w;-\frac{\overline{k}}{l},0\right)\right)z^{1-w}dw-\operatorname*{Res}_{w=1}(...). \qquad (2.14)
\end{aligned}
$$

Let us consider the residue term in formula (2.14). We will apply (2.9), and the expansions

$$\frac{1}{\sin(\pi w)}=-\frac{1}{\pi(w-1)}-\frac{\pi}{6}(w-1)+..., \qquad (2.15)$$

$$\cot(\pi w) = \frac{1}{\pi(w-1)} - \frac{\pi}{3}(w-1) + ..., \tag{2.16}$$

$$z^{1-w} = 1 - (w-1)\log z + ..., \tag{2.17}$$

$$\left(\frac{2\pi}{l}\right)^{1-2w} = \frac{l}{2\pi} - \frac{l}{\pi}\log\frac{2\pi}{l}(w-1) + ..., \tag{2.18}$$

as $w \to 1$. We see that the first terms of $\frac{1}{\sin(\pi w)}$ and $\cot(\pi w)$ differ only by sign. Having in mind that the Laurent series of $E\left(w; \frac{\overline{k}}{l}, 0\right)$and $E\left(w; -\frac{\overline{k}}{l}, 0\right)$ (see formula (2.5)) have the same main parts, we see that all terms, which arise from the fractions $\pm\frac{1}{\pi(w-1)}$, vanish, except for the term $-\frac{l}{2\pi^2}(a_0^+ - a_0^-)$. The second terms of $\frac{1}{\sin(\pi w)}$ and $\cot(\pi w)$ give us the term $\frac{1}{l}(-\frac{\pi}{6} - \frac{\pi}{3})\frac{l}{2\pi} = -\frac{1}{4}$. There are no other terms arising from $\frac{1}{\sin(\pi w)}$ and $\cot(\pi w)$. The other terms of the residue term arise only from $\delta i \left(\frac{2\pi}{l}\right)^{1-2w}\Gamma(w)E\left(w; -\frac{\overline{k}}{l}, 0\right)z^{1-w}$, and give

$$\frac{\delta i}{2\pi}\left(2\gamma_0 - 2\log l - \gamma_0 - \log z - 2\log\frac{2\pi}{l}\right) = \frac{\delta i}{2\pi}\left(\gamma_0 - \log 4\pi^2 z\right).$$

Finally, we get

$$\operatorname*{Res}_{w=1}\left(\frac{2\pi}{l}\right)^{1-2w}\Gamma(w)\left((\sin(\pi w))^{-1}E\left(w; \frac{\overline{k}}{l}, 0\right) + (\cot(\pi w) + \delta i)E\left(w; -\frac{\overline{k}}{l}, 0\right)\right)z^{1-w}$$

$$= -\frac{l}{2\pi^2}(a_0^+ - a_0^-) + \frac{\delta i}{2\pi}\left(\gamma_0 - \log 4\pi^2 z\right) - \frac{1}{4}. \tag{2.19}$$

Thus, by (2.14) and (2.19),

$$I = \frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty}\left(\frac{2\pi}{l}\right)^{1-2w}\Gamma(w)\left((\sin(\pi w))^{-1}E\left(w; \frac{\overline{k}}{l}, 0\right)\right.$$

$$\left. + (\cot(\pi w) + \delta i)E\left(w; -\frac{\overline{k}}{l}, 0\right)\right)z^{1-w}dw + \frac{l}{2\pi^2}(a_0^+ - a_0^-) - \frac{\delta i}{2\pi}\left(\gamma_0 - \log 4\pi^2 z\right) + \frac{1}{4}.$$

Now, from (2.13), we have

$$\Phi\left(z^{-1}; \frac{k}{l}\right) = -\frac{2\pi i\delta z}{l}\Phi\left(\frac{4\pi^2 z}{l^2}; -\frac{\overline{k}}{l}\right) + I(z;b) + \frac{l}{2\pi^2}(a_0^+ - a_0^-) - \frac{\delta i}{2\pi}\left(\gamma_0 - \log 4\pi^2 z\right) + \frac{1}{4}$$

$$= -\frac{2\pi i\delta z}{l}\sum_{m=1}^{\infty}d(m)e^{-2\pi i m\frac{\overline{k}}{l}}e^{-\frac{4\pi^2 mz}{l^2}} + \frac{2\pi i\delta z}{l}\left(\frac{\gamma_0 - 2\log l - \log\frac{4\pi^2 z}{l^2}}{l\frac{4\pi^2 z}{l^2}}\right) + I(z;b)$$

$$+\frac{l}{2\pi^2}(a_0^+ - a_0^-) - \frac{\delta i}{2\pi}\left(\gamma_0 - \log 4\pi^2 z\right) + \frac{1}{4},$$

and this finishes the proof.

□

Proof of Theorem 2.2. By Lemmas 1.5 and 2.4, we have the formula similar to (2.7), namely,

$$\sum_{m=1}^{\infty} d(m)\chi(m)\exp\{-2\pi i m/q\}e^{-mz}$$
$$= \frac{1}{2\pi i G(\overline{\chi})}\sum_{a=1}^{q}\overline{\chi}(a)\int_{2-i\infty}^{2+i\infty}\Gamma(w)E\left(w;\frac{\frac{(a-1)}{(q,a-1)}}{\frac{q}{(q,a-1)}},0\right)z^{-w}dw. \tag{2.20}$$

Since

$$\operatorname*{Res}_{w=1}\Gamma(w)E\left(w;\frac{\frac{(a-1)}{(q,a-1)}}{\frac{q}{(q,a-1)}},0\right)z^{-w} = \frac{(q,a-1)}{qz}\left(\gamma_0 - 2\log\frac{q}{(q,a-1)} - \log z\right),$$

moving the line of integration in (2.20) to the left, for $0 < c < 1$, we obtain that

$$\Phi\left(z;\chi,q\right) = \frac{1}{2\pi i G(\overline{\chi})}\sum_{a=1}^{q}\overline{\chi}(a)\int_{c-i\infty}^{c+i\infty}\Gamma(w)E\left(w;\frac{\frac{(a-1)}{(q,a-1)}}{\frac{q}{(q,a-1)}},0\right)z^{-w}dw. \tag{2.21}$$

We change z^{-1} to z, and the integration variable $w \mapsto 1-w$ in (2.21). This gives

$$\Phi\left(z^{-1};\chi,q\right) = \frac{1}{2\pi i G(\overline{\chi})}\sum_{a=1}^{q}\overline{\chi}(a)\int_{c-i\infty}^{c+i\infty}\Gamma(w)E\left(w;\frac{\frac{(a-1)}{(q,a-1)}}{\frac{q}{(q,a-1)}},0\right)z^{w}dw$$
$$= \frac{1}{2\pi i G(\overline{\chi})}\sum_{a=1}^{q}\overline{\chi}(a)\int_{1-c-i\infty}^{1-c+i\infty}\Gamma(1-w)E\left(1-w;\frac{\frac{(a-1)}{(q,a-1)}}{\frac{q}{(q,a-1)}},0\right)z^{1-w}dw. \tag{2.22}$$

Substituting formulae (1.9) and (2.6) into (2.22), we get

$$\Phi\left(z^{-1};\chi,q\right) = \frac{1}{2\pi i G(\overline{\chi})}\sum_{a=1}^{q}\overline{\chi}(a)\int_{1-c-i\infty}^{1-c+i\infty}\left(\frac{2\pi(q,a-1)}{q}\right)^{1-2w}\Gamma\left(w\right)\Bigg\{\sin^{-1}(\pi w)$$
$$\times E\left(w;\frac{\overline{\left(\frac{a-1}{(q,a-1)}\right)}}{\frac{q}{(q,a-1)}},0\right) + \cot\left(\pi w\right)E\left(w;-\frac{\overline{\left(\frac{a-1}{(q,a-1)}\right)}}{\frac{q}{(q,a-1)}},0\right)\Bigg\}z^{1-w}dw$$
$$= -\frac{2\pi i\delta z}{q}\Phi\left(\frac{4\pi^2 z}{q^2};\chi,q\right) + \frac{1}{2\pi i G(\overline{\chi})}\sum_{a=1}^{q}\overline{\chi}(a)\int_{1-c-i\infty}^{1-c+i\infty}\left(\frac{2\pi}{q}\right)^{1-2w}$$

$$\times \Gamma(w) \left\{ \sin^{-1}(\pi w) E\left(w; \frac{\overline{\left(\frac{a-1}{(q,a-1)}\right)}}{\frac{q}{(q,a-1)}}, 0 \right) (q, a-1)^{1-2w} + \cot(\pi w) \right.$$

$$\left. \times E\left(w; -\frac{\overline{\left(\frac{a-1}{(q,a-1)}\right)}}{\frac{q}{(q,a-1)}}, 0 \right) (q, a-1)^{1-2w} + \delta i E\left(w; \frac{\frac{(a-1)}{(q,a-1)}}{\frac{q}{(q,a-1)}}, 0 \right) \right\} z^{1-w} dw \tag{2.23}$$

$$\stackrel{def}{=} -\frac{2\pi i \delta z}{q} \Phi\left(\frac{4\pi^2 z}{q^2}; \chi, q \right) + I(\chi).$$

In the same way as in Theorem 2.1, using formulae (2.5), (2.9) and (2.15)-(2.18), we find that

$$\operatorname*{Res}_{w=1} \left(\frac{2\pi}{q}\right)^{1-2w} \Gamma(w) \left\{ \sin^{-1}(\pi w) E\left(w; \frac{\overline{\left(\frac{a-1}{(q,a-1)}\right)}}{\frac{q}{(q,a-1)}}, 0 \right) (q, a-1)^{1-2w} \right.$$

$$\left. + \cot(\pi w) E\left(w; -\frac{\overline{\left(\frac{a-1}{(q,a-1)}\right)}}{\frac{q}{(q,a-1)}}, 0 \right) (q, a-1)^{1-2w} + \delta i E\left(w; \frac{\frac{(a-1)}{(q,a-1)}}{\frac{q}{(q,a-1)}}, 0 \right) \right\} z^{1-w}$$

$$= -\frac{1}{4} + \frac{\delta i (q, a-1)}{2\pi} \left(\gamma_0 - \log \frac{4\pi^2 z}{(q, a-1)^2} \right) - \frac{q}{2\pi^2 (q, a-1)} (a_{0a}^+ - a_{0a}^-),$$

where a_{0a}^+ and a_{0a}^- are the constant terms in Laurent series expansions of $E\left(w; \frac{\overline{\left(\frac{a-1}{(q,a-1)}\right)}}{\frac{q}{(q,a-1)}}, 0 \right)$ and $E\left(w; -\frac{\overline{\left(\frac{a-1}{(q,a-1)}\right)}}{\frac{q}{(q,a-1)}}, 0 \right)$, respectively. Therefore,

$$I(\chi) = \frac{1}{2\pi i G(\overline{\chi})} \sum_{a=1}^{q} \overline{\chi}(a) \int_{1-c-i\infty}^{1-c+i\infty} \left(\frac{2\pi}{q}\right)^{1-2w} \Gamma(w) \left\{ \sin^{-1}(\pi w) E\left(w; \frac{\overline{\left(\frac{a-1}{(q,a-1)}\right)}}{\frac{q}{(q,a-1)}}, 0 \right) \right.$$

$$\times (q, a-1)^{1-2w} + \cot(\pi w) E\left(w; -\frac{\overline{\left(\frac{a-1}{(q,a-1)}\right)}}{\frac{q}{(q,a-1)}}, 0 \right) (q, a-1)^{1-2w}$$

$$\left. + \delta i E\left(w; \frac{\frac{(a-1)}{(q,a-1)}}{\frac{q}{(q,a-1)}}, 0 \right) \right\} z^{1-w} dw = \frac{1}{2\pi i G(\overline{\chi})} \sum_{a=1}^{q} \overline{\chi}(a) \int_{b-i\infty}^{b+i\infty} \left(\frac{2\pi}{q}\right)^{1-2w} \Gamma(w)$$

$$\times \left\{ \sin^{-1}(\pi w) E\left(w; \frac{\overline{\left(\frac{a-1}{(q,a-1)}\right)}}{\frac{q}{(q,a-1)}}, 0 \right) (q, a-1)^{1-2w} + \cot(\pi w) E\left(w; -\frac{\overline{\left(\frac{a-1}{(q,a-1)}\right)}}{\frac{q}{(q,a-1)}}, 0 \right) \right.$$

$$\left. \times (q, a-1)^{1-2w} + \delta i E\left(w; \frac{\frac{(a-1)}{(q,a-1)}}{\frac{q}{(q,a-1)}}, 0 \right) \right\} z^{1-w} dw$$

$$+\frac{1}{4G(\overline{\chi})}\sum_{a=1}^{q}\overline{\chi}(a)-\frac{\delta i}{2\pi G(\overline{\chi})}\sum_{a=1}^{q}\overline{\chi}(a)(q,a-1)\left(\gamma_0-\log\frac{4\pi^2 z}{(q,a-1)^2}\right)$$
$$+\frac{q}{2\pi^2 G(\overline{\chi})}\sum_{a=1}^{q}\frac{\overline{\chi}(a)}{(q,a-1)}(a_{0a}^{+}-a_{0a}^{-})$$
$$=I(z;\chi,b)-\frac{\delta i}{2\pi G(\overline{\chi})}\sum_{a=1}^{q}\overline{\chi}(a)(q,a-1)\left(\gamma_0-\log\frac{4\pi^2 z}{(q,a-1)^2}\right)$$
$$+\frac{q}{2\pi^2 G(\overline{\chi})}\sum_{a=1}^{q}\frac{\overline{\chi}(a)}{(q,a-1)}(a_{0a}^{+}-a_{0a}^{-}),$$

where $1<b<2$. Now, from (2.23), we find

$$\Phi\left(z^{-1};\chi,q\right)=-\frac{2\pi i\delta z}{q}\Phi\left(\frac{4\pi^2 z}{q^2};\chi,q\right)+I(z;\chi,b)$$
$$-\frac{\delta i}{2\pi G(\overline{\chi})}\sum_{a=1}^{q}\overline{\chi}(a)(q,a-1)\left(\gamma_0-\log\frac{4\pi^2 z}{(q,a-1)^2}\right)$$
$$+\frac{q}{2\pi^2 G(\overline{\chi})}\sum_{a=1}^{q}\frac{\overline{\chi}(a)}{(q,a-1)}(a_{0a}^{+}-a_{0a}^{-})$$
$$=-\frac{2\pi i\delta z}{q}\sum_{m=1}^{\infty}d(m)\chi(m)e^{-2\pi i m/q}e^{-\frac{4\pi^2 m z}{q^2}}$$
$$+\frac{2\pi i\delta z}{qG(\overline{\chi})}\sum_{a=1}^{q}\frac{\overline{\chi}(a)(q,a-1)}{q\cdot\frac{4\pi^2 z}{q^2}}\left(\gamma_0-2\log\frac{q}{(q,a-1)}-\log\frac{4\pi^2 z}{q^2}\right)+I\left(z;\chi,b\right)$$
$$-\frac{\delta i}{2\pi G(\overline{\chi})}\sum_{a=1}^{q}\overline{\chi}(a)(q,a-1)\left(\gamma_0-\log\frac{4\pi^2 z}{(q,a-1)^2}\right)$$
$$+\frac{q}{2\pi^2 G(\overline{\chi})}\sum_{a=1}^{q}\frac{\overline{\chi}(a)}{(q,a-1)}(a_{0a}^{+}-a_{0a}^{-}),$$

and the claim of Theorem 2.2 follows.

□

Proof of Theorem 2.3. We express the series

$$\sum_{m=1}^{\infty}d(m)\chi(m)e^{-2\pi i m/q}e^{-mz}$$

as a finite sum of functions $\Phi(z;\frac{k}{l})$. For this, write formula (2.23) in the form

$$\Phi\left(z^{-1};\chi,q\right)=\frac{1}{2\pi iG(\overline{\chi})}\sum_{a=1}^{q}\overline{\chi}(a)\int\limits_{1-c-i\infty}^{1-c+i\infty}\left(\frac{2\pi(q,a-1)}{q}\right)^{1-2w}\Gamma\left(w\right)\Bigg\{\sin^{-1}(\pi w)$$

$$\times E\left(w;\overline{\frac{\left(\frac{a-1}{(q,a-1)}\right)}{\frac{q}{(q,a-1)}}},0\right)+\cot\left(\pi w\right)E\left(w;-\overline{\frac{\left(\frac{a-1}{(q,a-1)}\right)}{\frac{q}{(q,a-1)}}},0\right)\Bigg\}z^{1-w}dw$$

$$=-\frac{\delta i}{2\pi iG(\overline{\chi})}\sum_{a=1}^{q}\overline{\chi}(a)\int\limits_{1-c-i\infty}^{1-c+i\infty}\left(\frac{2\pi(q,a-1)}{q}\right)^{1-2w}\Gamma\left(w\right)E\left(w;-\overline{\frac{\left(\frac{a-1}{(q,a-1)}\right)}{\frac{q}{(q,a-1)}}},0\right)z^{1-w}dw$$

$$+\frac{1}{2\pi iG(\overline{\chi})}\sum_{a=1}^{q}\overline{\chi}(a)\int\limits_{1-c-i\infty}^{1-c+i\infty}\left(\frac{2\pi(q,a-1)}{q}\right)^{1-2w}\Gamma\left(w\right)\Bigg\{\sin^{-1}(\pi w)$$

$$\times E\left(w;\overline{\frac{\left(\frac{a-1}{(q,a-1)}\right)}{\frac{q}{(q,a-1)}}},0\right)+\left(\cot\left(\pi w\right)+\delta i\right)E\left(w;-\overline{\frac{\left(\frac{a-1}{(q,a-1)}\right)}{\frac{q}{(q,a-1)}}},0\right)\Bigg\}z^{1-w}dw \tag{2.24}$$

$$\overset{def}{=}-\frac{2\pi i\delta z}{G(\overline{\chi})q}\sum_{a=1}^{q}\overline{\chi}(a)(q,a-1)\Phi\left(\frac{4\pi^{2}(q,a-1)^{2}z}{q^{2}};-\overline{\frac{\left(\frac{a-1}{(q,a-1)}\right)}{\frac{q}{(q,a-1)}}}\right)+\widetilde{I}(\chi).$$

Since, in this case,

$$\operatorname*{Res}_{w=1}\left(\frac{2\pi(q,a-1)}{q}\right)^{1-2w}\Gamma\left(w\right)\Bigg\{\sin^{-1}\left(\pi w\right)E\left(w;\overline{\frac{\left(\frac{a-1}{(q,a-1)}\right)}{\frac{q}{(q,a-1)}}},0\right)$$

$$+\left(\cot\left(\pi w\right)+\delta i\right)E\left(w;-\overline{\frac{\left(\frac{a-1}{(q,a-1)}\right)}{\frac{q}{(q,a-1)}}},0\right)\Bigg\}z^{1-w}$$

$$=-\frac{1}{4}+\frac{\delta i}{2\pi}\left(\gamma_{0}-\log 4\pi^{2}z\right)-\frac{q}{2\pi^{2}(q,a-1)}(a_{0a}^{+}-a_{0a}^{-}),$$

we have that

$$\widetilde{I}(\chi)=\widetilde{I}(z;\chi,b)+\frac{q}{2\pi^{2}G(\overline{\chi})}\sum_{a=1}^{q}\frac{\overline{\chi}(a)}{(q,a-1)}(a_{0a}^{+}-a_{0a}^{-}),$$

where $\widetilde{I}(z;\chi,b)$ is defined by the same formula as $\widetilde{I}(\chi)$, but we integrate from $b-i\infty$ to $b+i\infty$. Therefore, by (2.24),

$$\Phi\left(z^{-1};\chi,q\right)=-\frac{2\pi i\delta z}{G(\overline{\chi})q}\sum_{a=1}^{q}\overline{\chi}(a)(q,a-1)\sum_{m=1}^{\infty}d(m)e^{-2\pi im\overline{\frac{(a-1)/(q,a-1)}{q/(q,a-1)}}}e^{-\frac{4\pi^{2}m(q,a-1)^{2}z}{q^{2}}}$$

$$+\frac{2\pi i\delta z}{G(\overline{\chi})q}\sum_{a=1}^{q}\overline{\chi}(a)(q,a-1)\frac{\gamma_0-2\log\frac{q}{(q,a-1)}-\log\frac{4\pi^2(q,a-1)^2z}{q^2}}{\frac{q}{(q,a-1)}\cdot\frac{4\pi^2(q,a-1)^2z}{q^2}}+\widetilde{I}(z;\chi,b)$$
$$+\frac{q}{2\pi^2G(\overline{\chi})}\sum_{a=1}^{q}\frac{\overline{\chi}(a)}{(q,a-1)}(a_{0a}^{+}-a_{0a}^{-}).$$

The second term here vanishes, and we get the claim of the theorem.

□

Chapter 3

Mellin transforms of Dirichlet L-functions

3.1 Statements of the theorems

In this chapter, applying transformation formulae obtained in Chapter 2, we indicate singularities, and give the meromorphic continuation to the whole complex plane for the modified Mellin transform defined by the integral, for $\sigma > 1$,

$$\mathcal{Z}_1(s,\chi) \stackrel{\text{def}}{=} \int_1^\infty \left| L\left(\frac{1}{2}+ix,\chi\right)\right|^2 x^{-s}dx. \tag{3.1}$$

As it was mentioned above, the Estermann zeta-function $E\left(s;\frac{k}{l},0\right)$, is analytically continuable to the whole complex plane, except for a double pole at the point $s = 1$. This makes the influence that $\mathcal{Z}_1(s,\chi)$ also has a double pole at the same point, and simple poles at the points $s = -(2k-1)$. Also we find the main part of the Laurent series expansions of $\mathcal{Z}_1(s,\chi)$ at the point $s = 1$, and the residues at other singularities. We consider the transform (3.1) separately for the principal χ_0 and primitive χ characters modulo q, and use the previous notation. In the statements of the theorems, we use the Bernoulli numbers B_k, which can be defined by

$$\frac{x}{e^x-1} = \sum_{n=0}^\infty \frac{B_n x^n}{n!},$$

and also define

$$c(q) = \sum_{a=1}^q \overline{\chi}(a)(q,a-1).$$

Then the following result is presented in [3].

Theorem 3.1. *The function $\mathcal{Z}_1(s,\chi_0)$ has a meromorphic continuation to the whole complex plane. It has a double pole at the point $s=1$, and the main part of its Laurent expansion at this point is*

$$\mathcal{Z}_1(s,\chi_0) = \frac{\varphi(q)}{q}\left(\frac{1}{(s-1)^2} + \frac{2\gamma_0 + 2a(q) - \log 2\pi}{s-1}\right) + \dots.$$

The other poles of $\mathcal{Z}_1(s,\chi_0)$ are the simple poles at the points $s=-(2j-1), j\in\mathbb{N}$, and

$$\operatorname*{Res}_{\substack{s=-(2j-1)\\ j\in\mathbb{N}}} \mathcal{Z}_1(s,\chi_0) = \frac{\varphi(q) i^{-2j}(1-2^{1-2j})B_{2j}}{2jq}.$$

Theorem 3.2. *The function $\mathcal{Z}_1(s,\chi)$ has a meromorphic continuation to the whole complex plane.*

1. *If $c(q)\neq 0$, it has a double pole at the point $s=1$, and the main part of its Laurent expansion at this point is*

$$\mathcal{Z}_1(s,\chi) = \frac{i^b}{q}\sum_{a=1}^{q}\overline{\chi}(a)(q,a-1)\left(\frac{1}{(s-1)^2} + \frac{2\gamma_0 + \log(q,a-1)^2/2\pi q}{s-1}\right) + \dots;$$

   the other poles of $\mathcal{Z}_1(s,\chi)$ are the simple poles at the points $s=-(2j-1), j\in\mathbb{N}$, and

$$\operatorname*{Res}_{\substack{s=-(2j-1)\\ j\in\mathbb{N}}} \mathcal{Z}_1(s,\chi) = \frac{i^{b-2j}(1-2^{1-2j})B_{2j}}{2jq}\sum_{a=1}^{q}\overline{\chi}(a)(q,a-1).$$

2. *If $c(q)=0$, the function$\mathcal{Z}_1(s,\chi)$ is an entire function.*

3.2 Connection between Laplace and Mellin transforms

Now we study analytic properties of the Mellin transform $\mathcal{Z}_1(s,\chi)$ using the modified Laplace transform $\mathfrak{L}(s,\chi)$ (see formula (1.15)). For $\sigma>1$, the definitions of $\mathcal{Z}_1(s,\chi)$,

$\mathfrak{L}(s,\chi)$, and of $\Gamma(s)$ imply

$$\int_0^\infty \mathfrak{L}(w,\chi)w^{s-1}dw = \int_0^\infty \left(\int_1^\infty \left| L\left(\frac{1}{2}+ix,\chi\right)\right|^2 e^{-wx}dx \right) w^{s-1}dw$$
$$= \int_1^\infty \left(\left| L\left(\frac{1}{2}+ix,\chi\right)\right|^2 \int_0^\infty e^{-wx}w^{s-1}dw \right) dx = \Gamma(s)\int_1^\infty \left| L\left(\frac{1}{2}+ix,\chi\right)\right|^2 x^{-s}dx$$
$$= \mathcal{Z}_1(s,\chi)\Gamma(s).$$

Hence, for $\sigma > 1$,

$$\mathcal{Z}_1(s,\chi) = \frac{1}{\Gamma(s)}\int_0^\infty \mathfrak{L}(w,\chi)w^{s-1}dw. \tag{3.2}$$

Since Theorems 1.1, and 1.2 are valid in the vertical strip $0 < \Re w < \pi$, we change the integration over the positive real axis in the above integral using Cauchy's integral theorem.

Lemma 3.3. *Let $0 \le \alpha < \frac{\pi}{2}$ and $L = \{w \in \mathbb{C} : w = Re^{i\varphi}, 0 \le \varphi \le \alpha\}$. Then, for $\sigma > 1$*

$$\lim_{R\to\infty} \int\limits_L \mathfrak{L}(w,\chi)w^{s-1}dw = 0.$$

Proof. By Lemma 1.7, we find $C > 0$ such that, for $\Re w \ge C$,

$$\mathfrak{L}(w,\chi) = \int_1^\infty \left| L\left(\frac{1}{2}+ix,\chi\right)\right|^2 e^{-wx}dx = -\frac{1}{w}\int_1^\infty \left| L\left(\frac{1}{2}+ix,\chi\right)\right|^2 de^{-wx}$$
$$= -\frac{1}{w}\left| L\left(\frac{1}{2}+ix,\chi\right)\right|^2 e^{-wx}\Bigg|_1^\infty + \frac{1}{w}\int_1^\infty e^{-wx}\left(\left| L\left(\frac{1}{2}+ix,\chi\right)\right|^2\right)' dx, \tag{3.3}$$

where the second term in above integral vanishes, therefore, $\mathfrak{L}(w,\chi) = O(|w|^{-1})$.

Let $m > \sigma$ arbitrary positive integer. Integrating m times in formula (3.3), we find that

$$\mathfrak{L}(w,\chi) = O(|w|^{-m}).$$

Thus

$$\int\limits_L \mathfrak{L}(w,\chi)w^{s-1}dw = O\left(R^{-m+\sigma-1}\int\limits_L |dw|\right) = O\left(R^{-m+\sigma}\right),$$

and the lemma follows. □

Lemma 3.4. *Let* $0 \le \alpha < \frac{\pi}{2}$ *and* $L_1 = \{w \in \mathbb{C} : w = re^{i\varphi}, 0 \le \varphi \le \alpha\}$. *Then, for* $\sigma > 1$,

$$\lim_{r \to 0} \int_{L_1} \mathfrak{L}(w, \chi) w^{s-1} dw = 0.$$

Proof. Integrating by parts in the definition of $\mathfrak{L}(w, \chi)$, we obtain

$$\mathfrak{L}(w,\chi) = \int_1^\infty e^{-wx} d\left(\int_1^x \left|L\left(\frac{1}{2}+it,\chi\right)\right|^2 dt\right) = \left(e^{-wx}\int_1^x \left|L\left(\frac{1}{2}+it,\chi\right)\right|^2 dt\right)\Bigg|_1^\infty$$
$$+w\int_1^\infty e^{-wx}\left(\int_1^x \left|L\left(\frac{1}{2}+it,\chi\right)\right|^2 dt\right)dx, \tag{3.4}$$

where the right-hand side of (3.4) vanishes. It is well-known that

$$\int_1^x \left|L\left(\frac{1}{2}+it,\chi\right)\right|^2 dt = O\left(x \log x\right).$$

Now, integrating by parts in the remaining integral leads to

$$\mathfrak{L}(w,\chi) = |w| \int_1^\infty e^{-x\Re w} x \log x dx = O\left(\frac{e^{-\Re w}\left(2 + \Re w - \log \Re w\right)}{|w| \cos^2 \alpha}\right).$$

Thus,

$$\int_{L_1} \mathfrak{L}(w,\chi) w^{s-1} dw = O\left(r^{\sigma-1}\log r\right),$$

and the claim of the lemma follows, because $\sigma > 1$.

□

Now we move the line of integration in (3.2) to the ray $w = Re^{i\alpha}$, where $0 \le \alpha < \frac{\pi}{2}$, and obtain, that for $\sigma > 1$,

$$\mathcal{Z}_1(s,\chi) = \frac{1}{\Gamma(s)} \int_0^{\infty e^{i\alpha}} \mathfrak{L}(w,\chi) w^{s-1} dw. \tag{3.5}$$

Next we split the integral (3.5) into two parts by a point $w_0 = |w_0|e^{i\alpha}$ with $0 < \Re w_0 < \pi$. The first part of the path of integration is contained in the vertical strip $0 < \Re w < \pi$, and we use there Theorems 1.1, and 1.2. In the second part, we write the Laplace transform according to its definition.

3.3 Entire parts of Mellin transforms

As mentioned above, we fix a point $w_0 = |w_0|e^{i\alpha}$ with $0 < \Re w_0 < \pi$, and define the functions

$$\mathcal{Z}_{11}(s,\chi_0) = \frac{i\varphi(q)}{\Gamma(s)} \sum_{m|q} \mu(m) \int_0^{w_0} \left(\gamma - \log 2\pi - \left(\frac{\pi}{2} - w\right) i + a(q) + \log m\right) e^{\frac{iw}{2}} w^{s-1} dw,$$

$$\mathcal{Z}_{12}(s,\chi_0) = \frac{1}{\Gamma(s)} \int_0^{w_0} \lambda(w,\chi_0) w^{s-1} dw,$$

$$\mathcal{Z}_{13}(s,\chi_0) = \frac{1}{\Gamma(s)} \int_{w_0}^{\infty e^{i\alpha}} \left(\int_1^{\infty} \left| L\left(\frac{1}{2} + ix, \chi_0\right) \right|^2 e^{-wx} dx \right) w^{s-1} dw,$$

$$\mathcal{Z}_{14}(s,\chi_0) = \frac{2\pi}{\Gamma(s)} \sum_{n|q} \sum_{m|q} \frac{\mu(m)\mu(n)}{m} \sum_{k=1}^{\infty} d(k) \int_0^{w_0} e^{-\frac{iw}{2}} \exp\left\{ -\frac{2\pi i k n}{m} e^{-iw} \right\} w^{s-1} dw,$$

and the functions $\mathcal{Z}_{11}(s,\chi) - \mathcal{Z}_{13}(s,\chi)$

$$\mathcal{Z}_{11}(s,\chi) = \frac{1}{\Gamma(s)} \int_0^{w_0} \lambda(w,\chi) w^{s-1} dw,$$

$$\mathcal{Z}_{12}(s,\chi) = \frac{1}{\Gamma(s)} \int_{w_0}^{\infty e^{i\alpha}} \left(\int_1^{\infty} \left| L\left(\frac{1}{2} + ix, \chi\right) \right|^2 e^{-wx} dx \right) w^{s-1} dw,$$

$$\mathcal{Z}_{13}(s,\chi) = \frac{2\pi i^b}{\Gamma(s)\sqrt{q}E(\chi)} \sum_{k=1}^{\infty} d(k)\chi(k) \int_0^{w_0} e^{-\frac{iw}{2}} \exp\left\{ -\frac{2\pi i k}{q} e^{-iw} \right\} w^{s-1} dw,$$

where χ_0, and χ are the principal and the primitive characters modulo q, respectively. Then, taking into account (3.5), we find, by Theorem 1.1, that, for $\sigma > 1$,

$$\mathcal{Z}_1(s,\chi_0) = \sum_{j=1}^{4} \mathcal{Z}_{1j}(s,\chi_0), \tag{3.6}$$

and, by Theorem 1.2, that, for $\sigma > 1$,

$$\mathcal{Z}_1(s,\chi) = \sum_{j=1}^{3} \mathcal{Z}_{1j}(s,\chi). \tag{3.7}$$

Lemma 3.5. *Let the function $f(w)$ be holomorphic in half-plane $\Re w \geq 0$, and have bounded derivatives. For $\sigma > 1$, define*

$$F(s) \stackrel{def}{=} \frac{1}{\Gamma(s)} \int_0^{w_0} f(w) w^{s-1} dw.$$

Then the function $F(s)$ is entire.

Proof. Let $k \in \mathbb{N}$ be arbitrary. Integrating by parts k times and using the functional equation for the gamma-function, we find that

$$\frac{1}{\Gamma(s)}\int_0^{w_0} f(w)w^{s-1}dw =$$
$$\frac{1}{s\Gamma(s)}\int_0^{w_0} f(w)dw^s = \frac{1}{\Gamma(s+1)}f(w_0)w_0^s - \frac{1}{\Gamma(s+1)}\int_0^{w_0} f'(w)w^s dw =$$
$$\sum_{j=1}^{k}\frac{(-1)^{j-1}f^{(j-1)}(w_0)w_0^{s+j-1}}{\Gamma(s+j)} + \frac{(-1)^k}{\Gamma(s+k)}\int_0^{w_0} f^{(k)}(w)w^{s+k-1}dw. \tag{3.8}$$

Since $\frac{1}{\Gamma(s)}$ is entire, we see that the right-hand side of (3.8) is a holomorphic function for $\sigma > -k+1$. Since k is arbitrary, hence $F(s)$ is an entire function.

□

From Lemma 3.5 follows that the functions $\mathcal{Z}_{11}(s,\chi_0), \mathcal{Z}_{12}(s,\chi_0)$, and $\mathcal{Z}_{11}(s,\chi)$ are entire functions. Functions $\mathcal{Z}_{13}(s,\chi_0)$, and $\mathcal{Z}_{12}(s,\chi)$ are entire by its definitions. It remains to study the functions $\mathcal{Z}_{14}(s,\chi_0)$, and $\mathcal{Z}_{13}(s,\chi)$.

Functions $\mathcal{Z}_{14}(s,\chi_0)$ and $\mathcal{Z}_{13}(s,\chi)$ produce poles of the corresponding Mellin transforms. Using theorems of Chapter 2, we transform these parts to a holomorphic parts, and an integral parts producing poles. Firstly, in the definition of the functions $\mathcal{Z}_{14}(s,\chi_0)$ and $\mathcal{Z}_{13}(s,\chi)$, we make a substitution $e^{-iw} = 1+\frac{1}{z}$, and let $z_0 = (e^{-iw_0}-1)^{-1}$. This leads to the formulae

$$\mathcal{Z}_{14}(s,\chi_0) = \frac{2\pi i^s}{\Gamma(s)}\sum_{n|q}\sum_{m|q}\frac{\mu(m)\mu(n)}{m}$$
$$\times\sum_{k=1}^{\infty} d(k)e^{-\frac{2\pi ikn}{m}}\int_{z_0}^{\infty} z^{-2}\left(1+\frac{1}{z}\right)^{-\frac{1}{2}}\left(\log\left(1+\frac{1}{z}\right)\right)^{s-1} e^{-\frac{2\pi ikn}{mz}}dz \tag{3.9}$$

and

$$\mathcal{Z}_{13}(s,\chi) = \frac{2\pi i^{b+s}}{\Gamma(s)\sqrt{q}E(\chi)}$$
$$\times\sum_{k=1}^{\infty} d(k)\chi(k)e^{-\frac{2\pi ik}{q}}\int_{z_0}^{\infty} z^{-2}\left(1+\frac{1}{z}\right)^{-\frac{1}{2}}\left(\log\left(1+\frac{1}{z}\right)\right)^{s-1} e^{-\frac{2\pi ik}{qz}}dz, \tag{3.10}$$

where the integrals are taken over the curve $z = (e^{-ire^{i\alpha}}-1)^{-1}$.

3.4 Entire parts of $\mathcal{Z}_{14}(s,\chi_0)$

Let consider the integral (3.9) separately for small and large values of $|z|$, suppose here that $|z_0|<1$, and apply Theorem 2.1 for the function $\mathcal{Z}_{14}(s,\chi_0)$ in formula (3.9). In view of the definition of $\Phi(z;\frac{n}{m})$, we find that

$$\sum_{k=1}^{\infty} d(k) e^{-\frac{2\pi ikn}{m}} e^{-\frac{2\pi ikn}{zm}} = \Phi\left(\frac{2\pi in}{zm}; -\frac{\frac{n}{(m,n)}}{\frac{m}{(m,n)}}\right) + \frac{\gamma_0 - 2\log\frac{m}{(m,n)} - \log\frac{2\pi in}{mz}}{\frac{2\pi in}{z(m,n)}}, \tag{3.11}$$

because in the definition of $\Phi(z;\frac{n}{m})$ the numbers n and m are coprime.

First let $|z|>1$. We apply Theorem 2.1 with $\frac{mz}{2\pi in}$ in place of z. In this case, $\Im(\frac{mz}{2\pi in})<0$, therefore $\delta=-1$, and Theorem 2.1 yields

$$\Phi\left(\frac{2\pi in}{zm}; -\frac{\frac{n}{(m,n)}}{\frac{m}{(m,n)}}\right) = \frac{z(m,n)}{n}\sum_{k=1}^{\infty} d(k) e^{\frac{2\pi ik\overline{(n/(m,n))}}{m/(m,n)}} e^{\frac{2\pi ikz(m,n)^2}{mn}} + \frac{m}{2\pi^2(m,n)}(a_0^+ - a_0^-) + \frac{1}{4} + I\left(\frac{mz}{2\pi in}, b\right).$$

This, together with (3.11), shows that

$$\sum_{k=1}^{\infty} d(k) e^{-\frac{2\pi ikn}{m}} e^{-\frac{2\pi ikn}{zm}} = \frac{z(m,n)}{n}\sum_{k=1}^{\infty} d(k) e^{\frac{2\pi ik\overline{(n/(m,n))}}{m/(m,n)}} e^{\frac{2\pi ikz(m,n)^2}{mn}} + \frac{m}{2\pi^2(m,n)}(a_0^+ - a_0^-)$$
$$+\frac{1}{4} + \frac{z(m,n)}{2\pi in}\left(\gamma_0 - 2\log\frac{m}{(m,n)} - \log\frac{2\pi in}{mz}\right) + I\left(\frac{mz}{2\pi in}, b\right). \tag{3.12}$$

Now let $|z|\le 1$. We aplly Theorem 2.1 with $\frac{imn}{2\pi z(m,n)^2}$ in place of z. In this case, $\Im(\frac{imn}{2\pi z(m,n)^2})>0$, therefore, $\delta=1$, and, by Theorem 2.1, we find that

$$\Phi\left(\frac{2\pi z(m,n)^2}{imn}; \frac{\overline{(n/(m,n))}}{m/(m,n)}\right) = \frac{n}{z(m,n)}\sum_{k=1}^{\infty} d(k) e^{-\frac{2\pi ikn}{m}} e^{-\frac{2\pi ikn}{zm}} + \frac{m}{2\pi^2(m,n)}(a_0^+ - a_0^-) + \frac{1}{4} + I\left(\frac{imn}{2\pi z(m,n)^2}, b\right).$$

Hence, by the definition of $\Phi\left(z;\frac{n}{m}\right)$,

$$\sum_{k=1}^{\infty} d(k) e^{-\frac{2\pi ikn}{m}} e^{-\frac{2\pi ikn}{zm}}$$
$$= \frac{z(m,n)}{n}\Phi\left(\frac{2\pi z(m,n)^2}{imn}; \frac{\overline{(n/(m,n))}}{m/(m,n)}\right) - \frac{mz}{2\pi^2 n}(a_0^+ - a_0^-) - \frac{z(m,n)}{4n} - \frac{z(m,n)}{n} I\left(\frac{imn}{2\pi z(m,n)^2}, b\right)$$
$$= \frac{z(m,n)}{n}\sum_{k=1}^{\infty} d(k) e^{\frac{2\pi ik\overline{(n/(m,n))}}{m/(m,n)}} e^{\frac{2\pi ikz(m,n)^2}{mn}} - \left(\gamma_0 - 2\log\frac{m}{(m,n)} - \log\frac{2\pi z(m,n)^2}{imn}\right)\frac{i}{2\pi}$$
$$-\frac{mz}{2\pi^2 n}(a_0^+ - a_0^-) - \frac{z(m,n)}{4n} - \frac{z(m,n)}{n} I\left(\frac{imn}{2\pi z(m,n)^2}, b\right). \tag{3.13}$$

Now let $\hat{z}$ be a point on the path of integration in formula (3.9) such that $|\hat{z}| = 1$. Then, in view of (3.12) and (3.13), we have that

$$\mathcal{Z}_{14}(s,\chi_0) = \sum_{j=1}^{9} I_{j4}(s,\chi_0),$$

where

$$I_{14}(s,\chi_0) = \frac{2\pi i^s}{\Gamma(s)} \sum_{n|q} \sum_{m|q} \frac{\mu(m)\mu(n)(m,n)}{mn} \sum_{k=1}^{\infty} d(k) e^{\frac{2\pi i k \overline{(n/(m,n))}}{m/(m,n)}}$$
$$\times \int_{z_0}^{\infty} z^{-1}\left(1+\frac{1}{z}\right)^{-\frac{1}{2}} \left(\log\left(1+\frac{1}{z}\right)\right)^{s-1} e^{\frac{2\pi i k z (m,n)^2}{mn}} dz,$$

$$I_{24}(s,\chi_0) = -\frac{i^s}{\pi\Gamma(s)} \sum_{n|q} \sum_{m|q} \frac{\mu(m)\mu(n)(a_0^+ - a_0^-)}{n} \int_{z_0}^{\hat{z}} z^{-1}\left(1+\frac{1}{z}\right)^{-\frac{1}{2}} \left(\log\left(1+\frac{1}{z}\right)\right)^{s-1} dz,$$

$$I_{34}(s,\chi_0) = -\frac{2\pi i^s}{4\Gamma(s)} \sum_{n|q} \sum_{m|q} \frac{\mu(m)\mu(n)(m,n)}{mn} \int_{z_0}^{\hat{z}} z^{-1}\left(1+\frac{1}{z}\right)^{-\frac{1}{2}} \left(\log\left(1+\frac{1}{z}\right)\right)^{s-1} dz,$$

$$I_{44}(s,\chi_0) = -\frac{i^{s+1}}{\Gamma(s)} \sum_{n|q} \sum_{m|q} \frac{\mu(m)\mu(n)}{m} \int_{z_0}^{\hat{z}} z^{-2}\left(1+\frac{1}{z}\right)^{-\frac{1}{2}}$$
$$\times \left(\gamma_0 - 2\log\frac{m}{(m,n)} - \log\frac{2\pi z (m,n)^2}{imn}\right) \left(\log\left(1+\frac{1}{z}\right)\right)^{s-1} dz,$$

$$I_{54}(s,\chi_0) = -\frac{2\pi i^s}{\Gamma(s)} \sum_{n|q} \sum_{m|q} \frac{\mu(m)\mu(n)(m,n)}{mn} \int_{z_0}^{\hat{z}} z^{-1}\left(1+\frac{1}{z}\right)^{-\frac{1}{2}}$$
$$\times I\left(\frac{imn}{2\pi z (m,n)^2}, b\right) \left(\log\left(1+\frac{1}{z}\right)\right)^{s-1} dz,$$

$$I_{64}(s,\chi_0) = \frac{i^s}{\pi\Gamma(s)} \sum_{n|q} \sum_{m|q} \frac{\mu(m)\mu(n)(a_0^+ - a_0^-)}{(m,n)} \int_{\hat{z}}^{\infty} z^{-2}\left(1+\frac{1}{z}\right)^{-\frac{1}{2}} \left(\log\left(1+\frac{1}{z}\right)\right)^{s-1} dz,$$

$$I_{74}(s,\chi_0) = \frac{2\pi i^s}{4\Gamma(s)} \sum_{n|q} \sum_{m|q} \frac{\mu(m)\mu(n)}{m} \int_{\hat{z}}^{\infty} z^{-2}\left(1+\frac{1}{z}\right)^{-\frac{1}{2}} \left(\log\left(1+\frac{1}{z}\right)\right)^{s-1} dz,$$

$$I_{84}(s,\chi_0) = \frac{i^{s-1}}{\Gamma(s)} \sum_{n|q} \sum_{m|q} \frac{\mu(m)\mu(n)(m,n)}{mn} \int_{\hat{z}}^{\infty} z^{-1} \left(1+\frac{1}{z}\right)^{-\frac{1}{2}}$$

$$\times \left(\gamma_0 - 2\log\frac{m}{(m,n)} - \log\frac{2\pi in}{zm}\right) \left(\log\left(1+\frac{1}{z}\right)\right)^{s-1} dz,$$

$$I_{94}(s,\chi_0) = \frac{2\pi i^s}{\Gamma(s)} \sum_{n|q} \sum_{m|q} \frac{\mu(m)\mu(n)}{m} \int_{\hat{z}}^{\infty} z^{-2} \left(1+\frac{1}{z}\right)^{-\frac{1}{2}}$$

$$\times I\left(\frac{mz}{2\pi in}, b\right) \left(\log\left(1+\frac{1}{z}\right)\right)^{s-1} dz.$$

Since $0 < r\cos\alpha < \pi$, we have that

$$\Im z = \Im \frac{1}{e^{-ire^{i\alpha}} - 1} = \Im \frac{1}{e^{r\sin\alpha}(\cos(r\cos\alpha) - i\sin(r\cos\alpha)) - 1}$$

$$= \frac{e^{r\sin\alpha}\sin(r\cos\alpha)}{(e^{r\sin\alpha}\cos(r\cos\alpha)-1)^2 + (e^{r\sin\alpha}\sin(r\cos\alpha))^2} > 0.$$

Therefore, $\Re(2\pi iz) < 0$ in the integral $I_{14}(s,\chi_0)$. Hence, the series and integral both converge absolutely and uniformly in s on compact subsets of $\mathbb{C}$. Thus, the function $I_{14}(s,\chi_0)$ is entire. The functions $I_{24}(s,\chi_0) - I_{44}(s,\chi_0)$ are entire by their definitions.

The formula for the Laplace transform in Theorem 1.1 is valid only for $\Re w > 0$. The integrand in the definition of $I(z,b)$ has simple poles at $w = j, j \in \mathbb{N}$. If we move the integration line in the integral for $I(z,b)$ to the right , we get residues at these points. The contribution of a double pole at $w = 1$ is incorporated in the formula for $\Phi(z^{-1}; \frac{m}{n})$. If we move the integration in the function $I(z,b)$ more to the right, we get residues at the points $w = j+1, j \in \mathbb{N}$. Since

$$\sin\pi w = (-1)^{j+1}\sin\pi(w-j-1),$$

and

$$\cos\pi w = (-1)^{j+1}\cos\pi(w-j-1),$$

we find that

$$\operatorname*{Res}_{w=j+1} = \left(\frac{2\pi}{l}\right)^{1-2w} \Gamma(w)\sin^{-1}(\pi w) E\left(w; \frac{\overline{k}}{l}, 0\right) z^{1-w}$$

$$= \lim_{w\to j+1} \left(\frac{2\pi}{l}\right)^{1-2w} \Gamma(w) \frac{\pi(w-j-1)}{(-1)^{j+1}\pi\sin\pi(w-j-1)} E\left(w; \frac{\overline{k}}{l}, 0\right) z^{1-w}$$

$$= \left(\frac{2\pi}{l}\right)^{-2j-1} (-1)^{j+1}\frac{j!}{\pi} E\left(1+j; \frac{\overline{k}}{l}, 0\right) z^{-j},$$

and

$$
\begin{aligned}
&\operatorname*{Res}_{w=j+1} = \left(\frac{2\pi}{l}\right)^{1-2w} \Gamma(w)\cot(\pi w) E\left(w; -\frac{\overline{k}}{l}, 0\right) z^{1-w} \\
&= \lim_{w\to j+1} \left(\frac{2\pi}{l}\right)^{1-2w} \Gamma(w) \frac{\pi(w-j-1)\cos\pi(w-j-1)}{\pi\sin\pi(w-j-1)} E\left(w; -\frac{\overline{k}}{l}, 0\right) z^{1-w} \\
&= \left(\frac{2\pi}{l}\right)^{-2j-1} \frac{j!}{\pi} E\left(1+j; -\frac{\overline{k}}{l}, 0\right) z^{-j}.
\end{aligned}
$$

Thus, by the residue theorem,

$$
I(z,b) = a_1 z^{-1} + a_2 z^{-2} + ... + a_j z^{-j} + I(z, b_1), \tag{3.14}
$$

where

$$
a_j = \left(\frac{2\pi}{l}\right)^{-2j-1} (-1)^{j+1} \frac{j!}{\pi} \left(E\left(j+1; \frac{\overline{k}}{l}, 0\right) + E\left(j+1; -\frac{\overline{k}}{l}, 0\right)\right).
$$

Here the function $I(z, b_1)$ is defined by the same formula as $I(z,b)$ with $j+1 < b_1 < j+2$. By the latter formula, we have that

$$
I\left(\frac{imn}{2\pi z(m,n)^2}, b\right) = \hat{a}_1 z + \hat{a}_2 z^2 + ... + \hat{a}_j z^j + I\left(\frac{imn}{2\pi z(m,n)^2}, b_1\right)
$$

with

$$
\hat{a}_j = \left(\frac{2\pi(m,n)^2}{imn}\right)^j a_j.
$$

Therefore, $I_{54}(s, \chi_0)$ is also an entire function of s.

3.5 Poles of $\mathcal{Z}_{14}(s, \chi_0)$

The reminder functions $I_{64}(s, \chi_0) - I_{94}(s, \chi_0)$ can produce poles.

Proof of Theorem 3.1. Using the Taylor series expansion, we find that

$$
\begin{aligned}
&\left(1+\frac{1}{z}\right)^{-\frac{1}{2}} \left(\log\left(1+\frac{1}{z}\right)\right)^{s-1} \\
&= \left(1 - \frac{1}{2z} + \frac{3}{4\cdot 2! z^2} - ...\right) \cdot \left(\frac{1}{z} - \frac{1}{2z^2} + \frac{1}{3z^3} - ...\right)^{s-1} \\
&= z^{-s+1}\left(1 - \frac{s}{2z} + b_2(s)\frac{1}{z^2} + ...\right).
\end{aligned} \tag{3.15}
$$

This and the definition of $I_{64}(s,\chi_0)$, for $\sigma > 0$, yield

$$I_{64}(s,\chi_0) = \frac{i^s}{\pi\Gamma(s)} \sum_{n|q} \sum_{m|q} \frac{\mu(m)\mu(n)(a_0^+ - a_0^-)}{(m,n)} \sum_{k=0}^{\infty} b_k(s) \int_{\hat{z}}^{\infty} z^{-s-1-k} dz.$$

Obviously,

$$\int_{\hat{z}}^{\infty} z^{-s-1-k} dz = \frac{\hat{z}^{-s-k}}{s+k}. \tag{3.16}$$

Thus, the poles at $s = -k, k \in \mathbb{N}_0$, are canceled by the zeros of the function $\Gamma^{-1}(s)$, and the function $I_{64}(s,\chi_0)$ is entire. The same arguments show that the function $I_{74}(s,\chi_0)$ is entire, too.

By formula (3.14), we obtain

$$I\left(\frac{mz}{2\pi in}, b\right) = \hat{\hat{a}}_1 z^{-1} + \hat{\hat{a}}_2 z^{-2} + ... + \hat{\hat{a}}_j z^{-j} + I\left(\frac{mz}{2\pi in}, b_1\right)$$

with

$$\hat{\hat{a}}_j = \left(\frac{m}{2\pi in}\right)^{-j} a_j.$$

Hence, by formula (3.16), possible poles are canceled by the zeros of the function $\Gamma^{-1}(s)$, and the function $I_{94}(s,\chi_0)$ is entire in s. Thus, it remains to consider the function $I_{84}(s,\chi_0)$, and to obtain the poles of the function $\mathcal{Z}_1(s,\chi_0)$. Using expansion (3.15), we can write $I_{84}(s,\chi_0)$ in the form

$$I_{84}(s,\chi_0) = \frac{i^{s-1}}{\Gamma(s)} \sum_{n|q} \sum_{m|q} \frac{\mu(m)\mu(n)(m,n)}{mn} \sum_{k=0}^{\infty} b_k(s)$$
$$\times \int_{\hat{z}}^{\infty} z^{-k-s} \left(\gamma_0 - \log \frac{2\pi imn}{(m,n)^2} + \log z\right) dz. \tag{3.17}$$

Suppose that $\sigma > 1$. Then, clearly,

$$\int_{\hat{z}}^{\infty} z^{-s} \log z dz = \frac{1}{s-1} \hat{z}^{-s+1} \log \hat{z} + \frac{\hat{z}^{-s+1}}{(s-1)^2}. \tag{3.18}$$

Therefore,

$$\int_{\hat{z}}^{\infty} \left(\gamma_0 - \log \frac{2\pi imn}{(m,n)^2} + \log z\right) z^{-s} dz = \frac{\hat{z}^{-s+1}}{(s-1)^2} + \frac{\left(\gamma_0 - \log \frac{2\pi imn}{(m,n)^2 \hat{z}}\right) \hat{z}^{-s+1}}{s-1}.$$

Since $b_0(s) = 1$, hence, the first term in (3.15) is

$$\frac{i^{s-1}}{\Gamma(s)} \sum_{n|q} \sum_{m|q} \frac{\mu(m)\mu(n)(m,n)}{mn} \left(\frac{\hat{z}^{-s+1}}{(s-1)^2} + \frac{\left(\gamma_0 - \log \frac{2\pi imn}{(m,n)^2 \hat{z}}\right) \hat{z}^{-s+1}}{s-1}\right).$$

This shows that the function $I_{84}(s,\chi_0)$ has a double pole at the point $s=1$. The properties of the gamma-function imply

$$\frac{i^{s-1}\hat{z}^{1-s}}{\Gamma(s)} = e^{(s-1)\log i}e^{-(s-1)\log\hat{z}}\Gamma^{-1}(s) = (1+(s-1)\log i+...)(1-(s-1)\log\hat{z}+...)$$
$$\times(1+\gamma_0(s-1)+...) = 1+(\gamma_0+\log i-\log\hat{z})(s-1)+.... \qquad (3.19)$$

Therefore, the Laurent series expansion of the function $I_{84}(s,\chi_0)$ and thus, of the function $\mathcal{Z}_1(s,\chi_0)$, is

$$\mathcal{Z}_1(s,\chi_0) = \sum_{n|q}\sum_{m|q}\frac{\mu(m)\mu(n)(m,n)}{mn}\frac{1}{(s-1)^2}$$
$$+\sum_{n|q}\sum_{m|q}\frac{\mu(m)\mu(n)(m,n)}{mn}\frac{\left(2\gamma_0-\log\frac{mn}{(m,n)^2}-\log 2\pi\right)}{s-1}+.... \qquad (3.20)$$

From this, the first part of the theorem follows. Now we find simple poles of $\mathcal{Z}_{14}(s,\chi_0)$, and residues at these points.

Suppose that $\sigma>1-k$ with $k\in\mathbb{N}$. Then we have that

$$\int_{\hat{z}}^{\infty} z^{-k-s}dz = \frac{\hat{z}^{-s-k+1}}{s+k-1}.$$

The poles at the points $s=-k, k\in\mathbb{N}_0$, are canceled by the zeros of $\Gamma^{-1}(s)$. For the same σ and k as above, we have that

$$\int_{\hat{z}}^{\infty} z^{-s-k}\log z dz = \frac{\hat{z}^{-s-k+1}}{s+k-1}\log\hat{z}+\frac{\hat{z}^{-s-k+1}}{(s+k-1)^2}.$$

In virtue of $\Gamma^{-1}(s)$, this shows that the points $s=-k, k\in\mathbb{N}_0$, are the possible poles of $I_{84}(s,\chi_0)$. However, equality (3.15) involves the coefficients $b_k(s)$, and some of them can cancel the above poles. Indeed, in view of (3.13), $b_1(s)=-\frac{s}{2}$. Thus, the pole at $s=0$ is cancelled by $b_1(s)$. To show this in general case, we return to the initial definition of the function $\mathcal{Z}_{14}(s,\chi_0)$

$$\mathcal{Z}_{14}(s,\chi_0) = \frac{2\pi}{\Gamma(s)}\sum_{n|q}\sum_{m|q}\frac{\mu(m)\mu(n)}{m}\sum_{k=1}^{\infty}d(k)\int_0^{w_0} e^{-\frac{iw}{2}}\exp\{-\frac{2\pi ikn}{m}e^{-iw}\}w^{s-1}dw,$$

where $w_0=|w_0|e^{i\alpha}, 0<|\Re w_0|<\pi$, is a fixed point. We put $e^{-iw}=1-iz$ in the integral of the above formula, and let z_0 correspond the point w_0. Then we obtain that, for $\sigma > 1$,

$$\mathcal{Z}_{14}(s,\chi_0) = \frac{2\pi}{\Gamma(s)} \sum_{n|q} \sum_{m|q} \frac{\mu(m)\mu(n)}{m} \sum_{k=1}^{\infty} d(k)$$
$$\times \int_0^{z_0} (1-iz)^{-\frac{1}{2}} e^{-\frac{2\pi i k n}{m}} e^{-\frac{2\pi k n z}{m}} (i\log(1-iz))^{s-1} dz. \tag{3.21}$$

By the residue theorem, we write

$$\int_0^{z_0} (1-iz)^{-\frac{1}{2}} e^{-\frac{2\pi k n z}{m}} (i\log(1-iz))^{s-1} dz = \int_0^{|z_0|} (1-iz)^{-\frac{1}{2}} e^{-\frac{2\pi k n z}{m}} (i\log(1-iz))^{s-1} dz$$
$$+ \int_{|z_0|}^{z_0} (1-iz)^{-\frac{1}{2}} e^{-\frac{2\pi k n z}{m}} (i\log(1-iz))^{s-1} dz, \tag{3.22}$$

where the second integral in the right-hand side of (3.22) is taken along the path connecting the points $|z_0|$ and z_0, and is an entire function. Thus, it remains to consider the integral over $(0, |z_0|)$ in (3.22). Suppose that $|z|$ is small enough. Then we have, for the part of the integrand of this integral,

$$(1-iz)^{-\frac{1}{2}} (i\log(1-iz))^{s-1} = \left(1 - \frac{1}{2}(-iz) + \frac{\frac{1}{2}\cdot\frac{3}{2}}{2!}(-iz)^2 + ...\right)$$
$$\times \left(i\left(-iz - \frac{(-iz)^2}{2} + \frac{(-iz)^3}{3} + ...\right)\right)^{s-1} = \left(1 - \frac{1}{2}(-iz) + \frac{3}{4\cdot 2!}(-iz)^2 + ...\right)$$
$$\times i^{s-1}(-iz)^{s-1}\left(1 - (s-1)\frac{(-iz)}{2} + (s-1)\frac{(-iz)^3}{3} + ...\right) \tag{3.23}$$
$$= z^{s-1}\left(1 - \frac{s}{2}(-iz) + b_2(s)(-iz)^2 + ...\right).$$

Here the polynomials $b_k(s)$ are the same as above. For $\sigma > -l$, the integral

$$\int_0^{|z_0|} z^{s-1} \sum_{j=l+1}^{\infty} e^{-\frac{2\pi k n z}{m}} b_j(s)(-iz)^j dz$$

is an analytic function. Therefore, for $\sigma > -l$, we consider only the integral

$$\int_0^{|z_0|} e^{-\frac{2\pi k n z}{m}} z^{s-1} \sum_{j=0}^{l} b_j(s)(-iz)^j dz$$
$$= \int_0^{\infty} e^{-\frac{2\pi k n z}{m}} z^{s-1} \sum_{j=0}^{l} b_j(s)(-iz)^j dz - \int_{|z_0|}^{\infty} e^{-\frac{2\pi k n z}{m}} z^{s-1} \sum_{j=0}^{l} b_j(s)(-iz)^j dz. \tag{3.24}$$

The second integral in the above formula reduces to the incomplete gamma-function. The incomplete gamma-function differs from ordinary gamma-function, where lower limit of integration is 0, and defines an analytic function. Thus, in view of (3.21)-(3.24), we obtain that, for $\sigma > -l$, the function

$$J(s) \overset{def}{=} \frac{2\pi}{\Gamma(s)} \sum_{n|q} \sum_{m|q} \frac{\mu(m)\mu(n)}{m} \sum_{k=1}^{\infty} d(k) e^{-\frac{2\pi ikn}{m}} \times \int_0^{\infty} e^{-\frac{2\pi knz}{m}} \left(\sum_{j=0}^{l} b_j(s)(-iz)^j \right) z^{s-1} dz \tag{3.25}$$

gives the main part of the function $\mathcal{Z}_{14}(s, \chi_0)$. We have that, for $\sigma > 1-j$,

$$\int_0^{\infty} e^{-\frac{2\pi knz}{m}} z^{s+j-1} dz = (2\pi)^{-s-j} \left(\frac{m}{n}\right)^{s+j} \Gamma(s+j), \tag{3.26}$$

$$\sum_{k=1}^{\infty} \frac{d(k)}{k^{s+j}} e^{-\frac{2\pi ikn}{m}} = E\left(s+j; \frac{\frac{n}{(m,n)}}{\frac{m}{(m,n)}}, 0\right), \tag{3.27}$$

and

$$\frac{\Gamma(s+j)}{\Gamma(s)} = (s+j-1)...(s+1)s. \tag{3.28}$$

Therefore, for $\sigma > 1$, by (3.24),

$$J(s) = \sum_{j=0}^{l} \sum_{n|q} \sum_{m|q} \frac{\mu(m)\mu(n)}{m} (-i)^j b_j(s) (2\pi)^{1-s-j} s(s+1)...(s+j-1) \times \left(\frac{m}{n}\right)^{s+j} E\left(s+j; \frac{\frac{n}{(m,n)}}{\frac{m}{(m,n)}}, 0\right). \tag{3.29}$$

Since the function $E\left(s+j; \frac{m}{n}, 0\right)$ is meromorphic with a double pole at $s = 1-j$, and

$$E\left(s+j; \frac{\frac{n}{(m,n)}}{\frac{m}{(m,n)}}, 0\right) = \frac{(m,n)}{m} \left(\frac{1}{(s-(1-j))^2} + \frac{2\gamma - 2\log\frac{m}{(m,n)}}{s-(1-j)} \right) + ...,$$

equality (3.29) gives a meromorphic continuation for $J(s)$, and thus, for $\mathcal{Z}_1(s, \chi_0)$, to the whole complex plane. Since,

$$J(s) = \sum_{j=1}^{l} \sum_{n|q} \sum_{m|q} \frac{\mu(m)\mu(n)}{m} (-i)^j b_j(s) (2\pi)^{1-s-j} s(s+1)...(s+j-1) \left(\frac{m}{n}\right)^{s+j} \times E\left(s+j; \frac{\frac{n}{(m,n)}}{\frac{m}{(m,n)}}, 0\right) + \sum_{n|q} \sum_{m|q} \frac{\mu(m)\mu(n)}{mn} \left(\frac{1}{(s-1)^2} + \frac{2\gamma - 2\log\frac{mn}{(m,n)^2} - \log 2\pi}{s-1} \right) + ..., \tag{3.30}$$

from (3.30), having in mind the factor $(s+j-1)$, we conclude that $J(s)$ has a simple pole at $s=1-j, j \in \mathbb{N}$, if $b_j(1-j) \neq 0$, and exactly no poles if $b_j(1-j)=0$. We take $s=1-j$ in formula (3.23). This gives

$$z^j(1-iz)^{-\frac{1}{2}}(i\log(1-iz))^{-j} = \sum_{k=0}^{\infty} b_k(1-j)(-iz)^k.$$

Let $w=-iz$. Then, by the formula of Taylor's coefficients,

$$\begin{aligned} b_j(1-j) &= \frac{1}{2\pi i} \int\limits_{|w|=r} \frac{(iw)^j(1+w)^{-\frac{1}{2}}(i\log(1+w))^{-j}}{w^{j+1}} \\ &= \frac{1}{2\pi i} \int\limits_{|w|=r} \frac{dw}{w(1+w)^{\frac{1}{2}}(\log(1+w))^j}, \end{aligned} \tag{3.31}$$

where $r>0$ is sufficiently small. We make the substitution $\log(1+w)=-z$ in (3.31) and get

$$b_j(1-j) = \frac{(-1)^j}{2\pi i} \int\limits_{|z|=r} \frac{dz}{z^j(e^{z/2}-e^{-z/2})} = \frac{(-1)^j}{2\pi i} \int\limits_{|z|=r} \frac{dz}{z^j 2\sinh\frac{z}{2}}, \tag{3.32}$$

where

$$\sinh\frac{z}{2} = \frac{e^{\frac{z}{2}}-e^{-\frac{z}{2}}}{2}.$$

Therefore, $(-1)^j b_j(1-j)$ is the $(j-1)$ th coefficient of the Laurent series expansion for the function $\left(2\sinh\frac{z}{2}\right)^{-1}$ at the point $z=0$. It is well-known that Laurent series expansion of this function we can write using the Bernoulli numbers B_{2k}, namely,

$$\frac{1}{2\sinh\frac{z}{2}} = \frac{1}{z} - \sum_{k=1}^{\infty} \frac{(2^{2k-1}-1)B_{2k}}{2^{2k-1}(2k)!} z^{2k-1}. \tag{3.33}$$

In formula (3.33), we have only odd powers of z. From one hand, this means that the coefficients standing in front of the term $z^{2k}, k \in \mathbb{N}$, in formula (3.33) are equals to zero. Therefore $b_{2j+1}(-2j)=0, j \in \mathbb{N}_0$, and $J(s)$ has not simple poles at these points. From the other hand, the coefficient standing in front of the term $z^{2k-1}, k \in \mathbb{N}$, in formula (3.33) corresponds the coefficient $2j, j \in \mathbb{N}$, in formula (3.32), and we have simple poles at points $1-2j, j \in \mathbb{N}$, because $b_{2j}(1-2j) \neq 0, j \in \mathbb{N}$.

Now we get residues at the points $1-2j, j \in \mathbb{N}$. From formulae (3.33) and (3.32), it follows that

$$(-1)^{2j} b_{2j}(1-2j) = -\frac{(1-2^{-(2j-1)})B_{2j}}{(2j)!}. \tag{3.34}$$

Then (3.30) and (3.34) imply

$$\operatorname*{Res}_{\substack{s=1-2j\\ j\in\mathbb{N}}} \mathcal{Z}_1(s,\chi_0) = \sum_{n|q}\sum_{m|q} \frac{\mu(m)\mu(n)}{m}(-i)^{2j} b_{2j}(1-2j)(-1)^{2j-1}(2j-1)!\frac{m}{n}\frac{(m,n)}{m}$$
$$= \sum_{n|q}\sum_{m|q} \frac{\mu(m)\mu(n)}{mn}(m,n)\frac{i^{-2j}(1-2^{-(2j-1)})B_{2j}}{2j}. \qquad (3.35)$$

Properties of the Möbus function show that

$$\sum_{n|q}\sum_{m|q} \frac{\mu(m)\mu(n)}{mn}(m,n) = \frac{\varphi(q)}{q},$$

$$\sum_{n|q}\sum_{m|q} \frac{\mu(m)\mu(n)(m,n)}{mn}\log\left(\frac{mn}{(m,n)^2}\right) = -\frac{2\varphi(q)a(q)}{q},$$

and this complets the proof of Theorem 3.1.

□

3.6 Entire parts of $\mathcal{Z}_{13}(s,\chi)$

For the Mellin transform with primitive character we have the decomposition defined by formula (3.7). As it was shown in Section 3.3, the functions $\mathcal{Z}_{11}(s,\chi)$ and $\mathcal{Z}_{12}(s,\chi)$ are holomorphic, and possible poles of the Mellin transform here arise only from the part $\mathcal{Z}_{13}(s,\chi)$ in formula (3.10).

For $|z|>1$, from formula (2.1) and Theorem 2.1, taking $\frac{qz}{2\pi i}$ in place of z, having in mind that in this case $\Im(\frac{qz}{2\pi i})<0$, and therefore, $\delta=-1$, we have

$$\sum_{k=1}^{\infty} d(k)\chi(k)e^{-\frac{2\pi ik}{q}}e^{-\frac{2\pi ik}{qz}} = \frac{1}{G(\overline{\chi})}\sum_{a=1}^{q}\overline{\chi}(a)\Phi\left(\frac{2\pi i}{qz};\frac{\frac{a-1}{(q,a-1)}}{\frac{q}{(q,a-1)}}\right)$$
$$+\frac{\left(\gamma_0-\log\frac{2\pi iq}{(q,a-1)^2z}\right)z(q,a-1)}{2\pi i} = \frac{1}{G(\overline{\chi})}\sum_{a=1}^{q}\overline{\chi}(a)\Phi\left(\left(\frac{qz}{2\pi i}\right)^{-1};\frac{\frac{a-1}{(q,a-1)}}{\frac{q}{(q,a-1)}}\right)$$
$$+\frac{\left(\gamma_0-\log\frac{2\pi iq}{(q,a-1)^2z}\right)z(q,a-1)}{2\pi i}$$

$$= \frac{1}{G(\overline{\chi})} \sum_{a=1}^{q} \overline{\chi}(a) \left(\frac{2\pi i(q,a-1)}{q} \cdot \frac{qz}{2\pi i} \right) \sum_{k=1}^{\infty} d(k) e^{-2\pi i k \frac{\overline{(a-1)/(q,a-1)}}{q/(q,a-1)}} e^{-\frac{4\pi^2 k (q,a-1)^2 \cdot \frac{qz}{2\pi i}}{q^2}}$$

$$+ \frac{q}{2\pi^2(q,a-1)}(a_{0a}^{+} - a_{0a}^{-}) + \frac{1}{4} + I\left(\frac{qz}{2\pi i}, b\right) + \frac{\left(\gamma_0 - \log \frac{2\pi i q}{(q,a-1)^2 z}\right) z(q,a-1)}{2\pi i}$$

$$= \frac{z}{G(\overline{\chi})} \sum_{a=1}^{q} \overline{\chi}(a)(q,a-1) \sum_{k=1}^{\infty} d(k) e^{-2\pi i k \frac{\overline{(a-1)/(q,a-1)}}{q/(q,a-1)}} e^{\frac{2\pi i k (q,a-1)^2 z}{q}} \tag{3.36}$$

$$+ \frac{q}{2\pi^2(q,a-1)}(a_{0a}^{+} - a_{0a}^{-}) + \frac{\left(\gamma_0 - \log \frac{2\pi i q}{(q,a-1)^2 z}\right) z(q,a-1)}{2\pi i}.$$

Applying formula (2.1), we obtain

$$\Phi\left(\frac{4\pi^2 z}{m^2}; -\frac{\overline{n}}{m}\right) = \sum_{k=1}^{\infty} d(k) e^{-2\pi i k \frac{\overline{n}}{m}} e^{-\frac{4\pi^2 k z}{m^2}} - \frac{(\gamma_0 - \log 4\pi^2 z)m}{4\pi^2 z}. \tag{3.37}$$

On the other hand, Theorem 2.1 yields

$$\sum_{k=1}^{\infty} d(k) e^{-2\pi i k \frac{\overline{n}}{m}} e^{-\frac{4\pi^2 k z}{m^2}} = -\frac{m}{2\pi i \delta z} \Phi\left(z^{-1}; \frac{n}{m}\right) + \frac{m^2}{4\pi^3 i \delta z}(a_{0a}^{+} - a_{0a}^{-})$$
$$+ \frac{m}{8\pi i \delta z} + \frac{m}{2\pi i \delta z} I(z,b). \tag{3.38}$$

Therefore, combining (3.37) and (3.38) gives

$$\Phi\left(\frac{4\pi^2 z}{m^2}; -\frac{\overline{n}}{m}\right) = -\frac{m}{2\pi i \delta z} \Phi\left(z^{-1}; \frac{n}{m}\right) + \frac{m^2}{4\pi^3 i \delta z}(a_{0a}^{+} - a_{0a}^{-})$$
$$+ \frac{m}{8\pi i \delta z} + \frac{m}{2\pi i \delta z} I(z,b) - \frac{(\gamma_0 - \log 4\pi^2 z)m}{4\pi^2 z}. \tag{3.39}$$

Now if $|z| < 1$, we take $\frac{qi}{2\pi z(q,a-1)^2}$ in place of z. Since $\Im\left(\frac{qi}{2\pi z(q,a-1)^2}\right) > 0$, we have that $\delta = 1$, therefore, from (2.1) and(3.39), it follows

$$\sum_{k=1}^{\infty} d(k)\chi(k) e^{-\frac{2\pi i k}{q}} e^{-\frac{2\pi i k}{qz}} = \frac{1}{G(\overline{\chi})} \sum_{a=1}^{q} \overline{\chi}(a) \Phi\left(\frac{2\pi i}{qz}; \frac{\frac{a-1}{(q,a-1)}}{\frac{q}{(q,a-1)}}\right) + \frac{\left(\gamma_0 - \log \frac{2\pi i q}{(q,a-1)^2 z}\right) z(q,a-1)}{2\pi i}$$

$$= \frac{1}{G(\overline{\chi})} \sum_{a=1}^{q} \overline{\chi}(a) \Phi\left(\left(\frac{4\pi^2 (q,a-1)^2}{q^2} \cdot \frac{qi}{2\pi z(q,a-1)^2}\right); \frac{\frac{a-1}{(q,a-1)}}{\frac{q}{(q,a-1)}}\right)$$

$$+ \frac{\left(\gamma_0 - \log \frac{2\pi i q}{(q,a-1)^2 z}\right) z(q,a-1)}{2\pi i}$$

$$
\begin{aligned}
&= \frac{1}{G(\overline{\chi})}\sum_{a=1}^{q}\overline{\chi}(a)\Bigg\{-\frac{q}{2\pi i(q,a-1)\cdot\frac{qi}{2\pi z(q,a-1)^2}}\Phi\left(\frac{2\pi z(q,a-1)^2}{qi};-\frac{\overline{\left(\frac{(a-1)}{(q,a-1)}\right)}}{\frac{q}{(q,a-1)}}\right)\\
&+\frac{q^2}{4\pi^3 i(q,a-1)^2\cdot\frac{qi}{2\pi z(q,a-1)^2}}(a_{0a}^{+}-a_{0a}^{-})+\frac{q}{8\pi i(q,a-1)\cdot\frac{qi}{2\pi z(q,a-1)^2}}\\
&+\frac{q}{2\pi i(q,a-1)\cdot\frac{qi}{2\pi z(q,a-1)^2}}I\left(\frac{qi}{2\pi z(q,a-1)^2},b\right)-\frac{\left(\gamma_0-\log\frac{4\pi^2 qi}{2\pi z(q,a-1)^2}\right)q}{4\pi^2(q,a-1)\cdot\frac{qi}{2\pi z(q,a-1)^2}}\\
&+\frac{\left(\gamma_0-\log\frac{2\pi iq}{(q,a-1)^2 z}\right)z(q,a-1)}{2\pi i}\Bigg\} \qquad (3.40)\\
&=\frac{z}{G(\overline{\chi})}\sum_{a=1}^{q}\overline{\chi}(a)\Bigg((q,a-1)\sum_{k=1}^{\infty}d(k)e^{-2\pi ik\frac{\overline{(a-1)/(q,a-1)}}{q/(q,a-1)}}e^{\frac{2\pi ik(q,a-1)^2 z}{q}}\\
&-z(q,a-1)\frac{\left(\gamma_0-\log\left(\frac{q^2}{(q,a-1)^2}\cdot\frac{2\pi z(q,a-1)^2}{qi}\right)\right)}{\frac{q}{(q,a-1)}\cdot\frac{2\pi z(q,a-1)^2}{qi}}-\frac{qz}{2\pi^2}(a_{0a}^{+}-a_{0a}^{-})\\
&-\frac{z(q,a-1)}{4}-z(q,a-1)I\left(\frac{qi}{2\pi z(q,a-1)^2},b\right)\Bigg)\\
&=\frac{z}{G(\overline{\chi})}\sum_{a=1}^{q}\overline{\chi}(a)\Bigg((q,a-1)\sum_{k=1}^{\infty}d(k)e^{-2\pi ik\frac{\overline{(a-1)/(q,a-1)}}{q/(q,a-1)}}e^{\frac{2\pi ik(q,a-1)^2 z}{q}}\\
&-\frac{qz}{2\pi^2}(a_{0a}^{+}-a_{0a}^{-})-\frac{z(q,a-1)}{4}-z(q,a-1)I\left(\frac{qi}{2\pi z(q,a-1)^2},b\right)\Bigg).
\end{aligned}
$$

We denote, as in the above case, by $\hat{z}$ the point on the path of integration in formula (3.10) such that $|\hat{z}|=1$. Then, by (3.36), and (3.40), we express $\mathscr{Z}_{13}(s,\chi)$ as a sum of the functions I_{j3},

$$\mathscr{Z}_{13}(s,\chi)=\sum_{j=1}^{6}I_{j3}(s,\chi),$$

where

$$
\begin{aligned}
I_{13}(s,\chi)&=\frac{2\pi i^{b+s}}{\Gamma(s)\sqrt{q}E(\chi)G(\overline{\chi})}\sum_{a=1}^{q}\overline{\chi}(a)(q,a-1)\sum_{k=1}^{\infty}d(k)e^{-2\pi ik\frac{\overline{(a-1)/(q,a-1)}}{q/(q,a-1)}}\\
&\times\int_{z_0}^{\infty}z^{-1}\left(1+\frac{1}{z}\right)^{-\frac{1}{2}}\left(\log\left(1+\frac{1}{z}\right)\right)^{s-1}e^{\frac{2\pi ik(q,a-1)^2 z}{q}}dz,
\end{aligned}
$$

$$I_{23}(s,\chi)=-\frac{\sqrt{q}i^{b+s}}{\pi\Gamma(s)E(\chi)G(\overline{\chi})}\sum_{a=1}^{q}\overline{\chi}(a)(a_{0a}^{+}-a_{0a}^{-})\int_{z_0}^{\hat{z}}z^{-1}\left(1+\frac{1}{z}\right)^{-\frac{1}{2}}\left(\log\left(1+\frac{1}{z}\right)\right)^{s-1}dz,$$

$$I_{33}(s,\chi) = -\frac{2\pi i^{b+s}}{4\Gamma(s)\sqrt{q}E(\chi)G(\overline{\chi})} \sum_{a=1}^{q} \overline{\chi}(a)(q,a-1) \int_{z_0}^{\hat{z}} z^{-1} \left(1+\frac{1}{z}\right)^{-\frac{1}{2}} \left(\log\left(1+\frac{1}{z}\right)\right)^{s-1} dz,$$

$$I_{43}(s,\chi) = -\frac{2\pi i^{b+s}}{\Gamma(s)\sqrt{q}E(\chi)G(\overline{\chi})} \sum_{a=1}^{q} \overline{\chi}(a)(q,a-1) \int_{z_0}^{\hat{z}} z^{-1} \left(1+\frac{1}{z}\right)^{-\frac{1}{2}}$$
$$\times I\left(\frac{qi}{2\pi z(q,a-1)^2}, b\right) \left(\log\left(1+\frac{1}{z}\right)\right)^{s-1} dz,$$

$$I_{53}(s,\chi) = \frac{\sqrt{q} i^{b+s}}{\pi\Gamma(s)E(\chi)G(\overline{\chi})} \sum_{a=1}^{q} \overline{\chi}(a) \frac{(a_{0a}^{+} - a_{0a}^{-})}{(q,a-1)} \int_{\hat{z}}^{\infty} z^{-2} \left(1+\frac{1}{z}\right)^{-\frac{1}{2}} \left(\log\left(1+\frac{1}{z}\right)\right)^{s-1} dz,$$

$$I_{63}(s,\chi) = \frac{i^{b+s-1}}{\Gamma(s)\sqrt{q}E(\chi)G(\overline{\chi})} \sum_{a=1}^{q} \overline{\chi}(a)(q,a-1) \int_{\hat{z}}^{\infty} z^{-1} \left(1+\frac{1}{z}\right)^{-\frac{1}{2}}$$
$$\times \left(\gamma_0 - \log\frac{2\pi iq}{(q,a-1)^2 z}\right) \left(\log\left(1+\frac{1}{z}\right)\right)^{s-1} dz.$$

All these parts, are holomorphic, except for $I_{63}(s,\chi)$, which produces the poles of $\mathcal{Z}_{13}(s,\chi)$.

3.7 Poles of $\mathcal{Z}_{13}(s,\chi)$

The poles of $\mathcal{Z}_1(s,\chi)$ we deduce considering the function $I_{63}(s,\chi)$.

Proof of Theorem 3.1. Using Taylor series expansion (3.15), we have a new form of $I_{63}(s,\chi)$

$$I_{63}(s,\chi) = \frac{i^{b+s-1}}{\Gamma(s)\sqrt{q}E(\chi)G(\overline{\chi})} \sum_{a=1}^{q} \overline{\chi}(a)(q,a-1) \sum_{k=0}^{\infty} b_k(s) \int_{\hat{z}}^{\infty} z^{-s-k} \left(\gamma_0 - \log\frac{2\pi iq}{(q,a-1)^2 z}\right) dz.$$

Hence, integrating the first term with $k=0$ and using formula (3.18), we get a double pole at point $s=1$

$$\frac{i^{b+s-1}}{\Gamma(s)\sqrt{q}E(\chi)G(\overline{\chi})} \sum_{a=1}^{q} \overline{\chi}(a)(q,a-1) \int_{\hat{z}}^{\infty} z^{-s} \left(\gamma_0 - \log\frac{2\pi iq}{(q,a-1)^2 z}\right) dz$$
$$= \frac{i^{b+s-1}}{\Gamma(s)\sqrt{q}E(\chi)G(\overline{\chi})} \sum_{a=1}^{q} \overline{\chi}(a)(q,a-1) \left(\frac{\hat{z}^{-s+1}}{(s-1)^2} + \frac{\left(\gamma_0 - \log\frac{2\pi iq}{(q,a-1)^2\hat{z}}\right)\hat{z}^{-s+1}}{s-1}\right).$$

Now, from (3.19), we find that the main part of the Laurent series expansion for $\mathcal{Z}_{13}(s,\chi)$ at this point is

$$\mathcal{Z}_{13}(s,\chi)=\frac{i^b}{\sqrt{q}E(\chi)G(\overline{\chi})}\sum_{a=1}^{q}\overline{\chi}(a)(q,a-1)\left(\frac{1}{(s-1)^2}+\frac{2\gamma_0+\log\frac{(q,a-1)^2}{2\pi q}}{s-1}\right)+.... \tag{3.41}$$

Clearly, the polynomials $b_k(s)$ which appear in $\mathcal{Z}_{13}(s,\chi)$ from the Taylor series expansion (3.15) are the same as in $\mathcal{Z}_{14}(s,\chi_0)$, therefore, we conclude that other singularities of $\mathcal{Z}_1(s,\chi)$ are the simple poles at points $1-2j, j\in\mathbb{N}$. With purpose to get residues at these points, we consider the function

$$J(s)\overset{def}{=}\frac{2\pi i^b}{\Gamma(s)\sqrt{q}E(\chi)}\sum_{k=1}^{\infty}d(k)\chi(k)e^{-\frac{2\pi ik}{q}}$$
$$\times\int_0^{\infty}e^{-\frac{2\pi kz}{q}}\left(\sum_{j=0}^{l}b_j(s)(-iz)^j\right)z^{s-1}dz,$$

which get a non-holomorphic part of $\mathcal{Z}_1(s,\chi)$. In the same way as in proof of Theorem 3.1, taking into account Lemma 2.4, we write $J(s)$ in the form of finite sum of Estermann zeta-functions

$$J(s)=\frac{2\pi i^b}{\sqrt{q}E(\chi)}\sum_{j=0}^{l}(-i)^jb_j(s)\left(\frac{q}{2\pi}\right)^{s+j}s(s+1)...(s+j-1)$$
$$\times\frac{1}{G(\overline{\chi})}\sum_{a=1}^{q}\overline{\chi}(a)E\left(s+j;\frac{\frac{(a-1)}{(q,a-1)}}{\frac{q}{(q,a-1)}},0\right).$$

Properties of the Estermann zeta-function and polynomials $b_k(s)$ imply

$$\operatorname*{Res}_{\substack{s=1-2j\\ j\in\mathbb{N}}}\mathcal{Z}_1(s,\chi)=\frac{i^{b-2j}(1-2^{-(2j-1)})B_{2j}}{\sqrt{q}E(\chi)G(\overline{\chi})2j}\sum_{a=1}^{q}\overline{\chi}(a)(q,a-1). \tag{3.42}$$

Using properties of Dirichlet characters, write the Gauss sum in the form

$$G(\chi)=\sum_{l=1}^{q}\chi(l)e^{2\pi il/q}=\sum_{l=0}^{q-1}\chi(l)e^{2\pi il/q}.$$

Then we have

$$\chi(-1)\overline{G(\chi)}=\chi(-1)\sum_{l=1}^{q}\overline{\chi}(l)e^{-2\pi il/q}=\chi(-1)\sum_{l=1}^{q}\overline{\chi}(-1)\overline{\chi}(-l)e^{-2\pi il/q}$$
$$=|\chi(-1)|^2\sum_{l=1}^{q}\overline{\chi}(-l)e^{-2\pi il/q}=\sum_{l=1}^{q}\overline{\chi}(q-l)e^{2\pi i(q-l)/q}=\sum_{m=0}^{q-1}\overline{\chi}(m)e^{2\pi im/q}=G(\overline{\chi}).$$

Also, it is well-known, see, for example [1], that

$$|G(\chi)|^2 = q.$$

Thus,

$$\frac{i^b}{\sqrt{q}E(\chi)G(\overline{\chi})} = \frac{i^b}{\sqrt{q}(-1)^b\frac{G(\chi)}{\sqrt{q}}\chi(-1)\overline{G(\chi)}} = \frac{(-i)^b}{\chi(-1)|G(\chi)|^2} = \frac{(-i)^b}{(-1)^b q} = \frac{i^b}{q}.$$

Now this, (3.42), and (3.41) give the assertion of the theorem.

□

Conclusions

In the thesis, the following statements were established:

1. The modified Mellin transform $\mathcal{Z}_1(s,\chi_0) = \int\limits_1^\infty \left| L\left(\frac{1}{2}+ix,\chi_0\right)\right|^2 x^{-s}dx$ has a meromorphic continuation to the whole complex plane. It has a double pole at $s = 1$, and other poles of $\mathcal{Z}_1(s,\chi_0)$ are the simple poles at the points $s = -(2j-1), j \in \mathbb{N}$.

2. The modified Mellin transform $\mathcal{Z}_1(s,\chi)$ with primitive character $\chi \mod q$ also has a meromorphic continuation to the whole complex plane.

3. Possible poles of $\mathcal{Z}_1(s,\chi)$ depends on the sum $c(q) = \sum_{a=1}^{q} \overline{\chi}(a)(q, a-1)$:

   if $c(q) \neq 0$, then $\mathcal{Z}_1(s,\chi)$ has the same poles as $\mathcal{Z}_1(s,\chi_0)$;

   if $c(q) = 0$, then $\mathcal{Z}_1(s,\chi)$ is an entire function.

Bibliography

[1] T.M. Apostol, Introduction to analytic Number Theory, Springer-Verlag (1976),166.

[2] A. Balčiūnas, A transformation formula related to Dirichlet L-fubctions with principal character, Lietuvos matematikos rinkinys 53(A)(2012), 13-18.

[3] A. Balčiūnas, Mellin transform of Dirichlet L-functions with principal character, Šiauliai Math. Semin. 8(16), (2013), 7-26.

[4] A. Balčiūnas, A transformation formula with primitive character, Lietuvos matematikos rinkinys 54(A), (2013), 6-11.

[5] A. Balčiūnas, A. Laurinčikas, The Laplace transform of Dirichlet L-functions, Nonlinear Anal., Model. Control 17, (2012), No. 2, 127-138.

[6] K. Chandrasekharan, Arithmetical Functions, Nayka, Moskva, 1945

[7] T. Estermann, On the representation of a number as the sum of two products, Proc. Lond. Math. Soc. 31 (1930), 123-133.

[8] A. Ivič, Mean Values of the Riemann zeta-function,Tata Inst.Fund.Res. Lectures 82, Bombay, 1991.

[9] A. Ivič, The mean square of the error term for the fourth moment of the zeta-function, Proc. Lond. Math.Soc.(3) 66 (1994), 309-329.

[10] A. Ivič, The fourth moment of the Riemann zeta-function, J. Number Theory 51 (1995), 16-45.

[11] A. Ivič, The Mellin transform and the Riemann zeta-function, in: Proc. Conf. on Elementary and Analytic Number Theory, W.G. Nowak and J. Shoiβengeier (Eds),Univ. Wienn & Univ. f*ü*r Bodenkultur, Vienna, (1996), 112-127.

[12] A. Ivič, On some conjectures and results for the Riemann zeta-function and Hecke series, Acta Arith. 99, (2001), No. 2, 115-145.

[13] A. Ivič, The Melin transform of the square of Riemann's zeta-function, Int. J. Number Theory 1(1) (2005), 65-73.

[14] A. Ivič, The Laplace and Mellin transforms of powers of the Riemann zeta-function, Int. J.Math. Anal.1, (2006), No. 2, 113-140.

[15] A. Ivič, M. Jutila and Y. Motohashi, The Mellin transform of powers of the zeta-function, Acta Arith. 95 (2000), No. 4, 305-342.

[16] M. Jutila, The Mellin transforms of the square of Riemann's zeta-function, Period. Math. Hung. 42, (2001), 179-190.

[17] M. Jutila, The Mellin transforms of the fourth power of Riemann's zeta-function, in: Number Theory, S.D. Adhikari, R. Balasubramanian and R. Srinivas (Eds), Ramanujan Math. Soc., Lect. Ser.1, Ramanujan Math. Soc., Mysore, (2005), 15-29.

[18] R. Kačinskaitė, A. Laurinčikas, The Laplace transform of the Riemann zeta-function in the critical strip, Integral Transforms Spec. Funct. 20, (2009), No.9, 643-648.

[19] I. Kiuchi, On an exponential sum involving the arithmetic function $\sigma_a(m)$, Math. J. Okayama Univ. 29 (1987), 193-205.

[20] A. Laurinčikas, A growth estimate for the Mellin transform of the Riemann zeta-function, Mat. Zametki, 89, (2011), No.1, 70-81.

[21] A. Laurinčikas, Mean square of the Mellin transform of the Riemann zeta-function, Integral Transforms Spec. Funct. 22, (2011), No.9, 617-629.

[22] A. Laurinčikas, One transformation formula related to the Riemann zeta-function, Integral Transforms. Spec. Funct. 19, (2008), No.8, 577-583.

[23] A. Laurinčikas, The Mellin transform of the square of the Riemann zeta-function in the critical strip, in: Voronoi's Impact on Modern Science, Book 4, vol.1, Proc. 4 th. Intern. Conf. Analytic Number Theory and Spatial Tessellations, A. Laurinčikas and J. Studing (Eds), Institute of Math. NAS of Ukraine, Kyiv, (2008), 89-96.

[24] A. Laurinčikas, The Mellin transform of the square of the Riemann zeta-function in the critical strip, Integral Transforms Spec. Funct. 22, (2011), No.7, 467-476.

[25] A. Laurinčikas, R. Garunkštis, The Lerch Zeta-function, Kluwer, Dordrecht, 2002.

[26] M. Lukkarinen, The Mellin transform of the square of Riemann's zeta-function and Atkinson's formula, Ann. Acad. Scie. Fenn. Math. Diss. 140, Suomalainen Tiedeakatemia, Helsinki 2005.

[27] H.L. Montgomery, Topics in Multiplicative Number Theory, Lecture Notes 227, Springer, Berlin, Heidelberg, New York, 1971.

[28] Y. Motohashi, A note on the mean value of the zeta and L-functions, Proc. Japan Acad. Math. Sci. No. 61(A),(1985), 313-316.

[29] Y. Motohashi, A relation between the Riemann zeta-function and the hyperbolic Laplacian, Ann. Sc. Norm. Super., Pisa,Cl. Sci. 22(4) (1995), 299-313.

[30] Y. Motohashi, Spectral Theory of the Riemann Zeta-Function, Cambridge University Press, Cambridge, 1997.

[31] K. Prachar, Primzahlverteilung, Springer-Verlag, Berlin, 1957.

[32] E.C. Titchmarsh, The theory of Functions, Oxford University Press, London, 1939.

[33] E.C. Titchmarsh, The Theory of Riemann Zeta- Function,2nd ed., revised by D. R. Heath-Brown, Clarendor Press, Oxford, 1986.