VILNIUS GEDIMINAS TECHNINICAL UNIVERSITETY
INSTITUTE OF MATHEMATICS AND INFORMATICS

Vaidas KEBLIKAS

TIME PERIODIC PROBLEMS FOR NAVIER-STOKES EQUATIONS IN DOMAINS WITH CYLINDRICAL OUTLETS TO INFINITY

Doctoral Dissertation

Physical Sciences, Mathematics (01P)

Vilnius VGTU LEIDYKLA TECHNIKA 2008

VILNIAUS GEDIMINO TECHNIKOS UNIVERSITETAS
MATEMATIKOS IR INFORMATIKOS INSTITUTAS

Vaidas KEBLIKAS

NAVJĖ-STOKSO LYGČIŲ PERIODINIAI LAIKO ATŽVILGIU UŽDAVINIAI SRITYSE SU CILINDRINIAIS IŠĖJIMAIS Į BEGALYBĘ

Daktaro disertacija

Fiziniai mokslai, matematika (01P)

Vilnius LEIDYKLA TECHNIKA 2008

The Dissertation was prepared at Institute of Mathematics and Informatics in 2003–2008.

Scientific supervisor

Prof Dr Habil Konstantinas PILECKAS (Institute of mathematics and informatics, physical sciences, mathematics – 01P).

http://leidykla.vgtu.lt
VGTU leidyklos TECHNIKA 1490-M mokslo literatūros knyga

ISBN 978-9955-28-213-6

Abstract

The research area of current PhD thesis is the analysis of time periodic Navier–Stokes equations in domains with cylindrical outlets to infinity (system of pipes). The objects of investigation is so called non-stationary Poiseuille solution in the straight cylinder and Navier-Stokes equations in domains with cylindrical outlets to infinity. First of all, in this thesis is proved the existence and uniqueness of the non-stationary Poiseuille solution in Hölder spaces. Then the existence and uniqueness of the time periodic Stokes problem obtained in weighted Sobolev spaces. Finally, the existence of time periodic solutions to Navier-Stokes problem in weighted Sobolev spaces is proved. The weight-function describes asymptotical behavior of solutions, then $|x| \to \infty$. The obtained results are theoretical. However, they could be used to solve practical problems of fluid dynamics.

Reziumė

Disertacijoje nagrinėjami Navjė-Stokso lygčių periodiniai laiko atžvilgiu uždaviniai srityse su cilindriniais išėjimais į begalybę. Pagrindiniai tyrimo objektai yra taip vadinami Puazelio sprendiniai tiesiame cilindre ir Stokso, bei Navjė-Stokso lygčių sistemos cilindrų sistemoje. Pirmiausia darbe įrodomas Puazelio sprendinio egzistavimas ir vienatis Hiolderio erdvėse. Tada įrodomas periodinis Stokso uždavinio sprendinio egzistavimas svorinėse Sobolevo erdvėse. Ir galiausiai, įrodytas periodinio sprendinio Navjė-Stokso uždaviniui egzistavimas svorinėse Sobolevo erdvėse. Svorinė funkcija apibūdina sprendinių nykimo greitį, kai $|x| \to \infty$. Gauti rezultatai yra teoriniai, tačiau gali būti pritaikyti skysčių dinamikos praktiniams uždaviniams spręsti.

Contents

Introduction

Topicality of the problem

The research area of this work is the analysis of time periodic Navier–Stokes equations in domains with cylindrical outlets to infinity. In this thesis the existence and uniqueness of the non-stationary Poiseuille solution to the Navier–Stokes equations is proved. Time periodic problems for Stokes and Navier–Stokes equations are studied in domains with cylindrical outlets to infinity. The existence of the solutions to these problems is proved in weighted function spaces.

Actuality

Mathematical models of fluid dynamics are systems of linear and nonlinear partial differential equations, known as Navier–Stokes equations. The rigorous mathematical analysis of Navier-Stokes equations started at the beginning of the XX century from works of the famous French mathematician J. Leray. This analysis consists of studies concerning the correct formulations of initial boundary value problems for Navier-Stokes equations, proofs of the existence and uniqueness of solutions in different functional spaces, investigation of solutions regularity, construction of asymptotics, etc. Such questions have been studied in many papers and monographs [16], [27], [85].

The existence theory developed in literature mainly deals with the flows of viscous incompressible fluids in domains with compact boundaries (i. e. in bounded and exterior domains). Although some of these results do not depend on the shape of the boundary, many problems of scientific and practical interest (e. g. water or petroleum flow in pipes system or blood flow in veins) related to flows of viscous incompressible fluid in domains with noncompact boundaries were unsolved. Therefore, it is not surprising that during the last 30 years the special attention was given to problems in unbounded domains. However, during this time only stationary problems were exhaustively investigated, while not much is known about the non-stationary ones. Even the existence of a non-stationary analog of the Poiseuille flow in a straight cylinder was not proved (the stationary Poiseuille solution was constructed as far back as in XIX century). In this thesis we prove the existence of the non-stationary Poiseuille solution in a straight cylinder and study the time–periodic solutions of Stokes and Navier–Stokes equations in a domains with cylindrical outlets to infinity (i. e. in a system of pipes).

Aims and problems

The main aim of the dissertation is the analysis of the time periodic Navier-Stokes equations in domains with cylindrical outlets to infinity. To achieve this goal, we have to solve following problems:

1. To investigate the existence and uniqueness questions for the Poiseuille flow of an incompressible viscous fluid in an infinite cylinder.

2. To investigate the existence and uniqueness questions of a time periodic Stokes problem with cylindrical outlets to infinity.

3. To investigate the existence questions of a time periodic Navier-Stokes problem for two and three dimensional domains with cylindrical outlets to infinity.

Novelty

All results obtained in the dissertation are new. The existence of the non-stationary Poiseuille solution in Hölder spaces was not know before. The time periodic solutions to Stokes and Navier–Stokes equations in domains with cylindrical outlets to infinity are investigated for the first time.

Defended propositions

1. The existence and uniqueness of solutions for the Poiseuille flow of a incompressible viscous fluid in an infinite cylinder is proved.

2. The existence and uniqueness of solutions for time periodic Stokes problem with cylindrical outlets to infinity is proved.

3. The existence of solutions for time periodic Navier-Stokes problem in domains with cylindrical outlets to infinity is proved.

History of the problem

The solvability of boundary and initial boundary value problems for Navier-Stokes equations is one of the most important questions in mathematical hydrodynamics. It has been studied in many papers and monographs (e. g. [16], [27], [85]). The existence theory developed there deals mainly with flows of viscous fluids in domains with compact boundaries (i. e., bounded and and exterior domains). Although some of these results do not depend on the shape of the boundary, many problems of scientific and practical interest related to flows of viscous incompressible fluid in domains with noncompact boundaries have remained unsolved. Therefore, it is not surprising that during the last 30 years special attention has been given to problems in such domains. J. Heywood has drawn attention to the question of correct formulation of boundary value problems for Navier-Stokes equations in domains with noncompact boundaries. In 1976 he demonstrated [20] that in domains with noncompact boundaries the motion of a viscous fluid is not always uniquely determined by the applied external forces and usual initial and boundary conditions. Moreover, certain physically important quantities (such as fluxes of the velocity field or limiting values of the pressure at infinity) have to be prescribed additionally.

After the publication of J. Heywood's paper the theory of Navier-Stokes equations in domains with noncompact boundaries received a great impulse in a row of papers. Such problems were investigated in a wide class of domains having "outlets" to infinity. First, it was proved in [24], [29], [30], [76], [82] that looking for solutions with a finite Dirichlet integral it is necessary to prescribe additional conditions (fluxes over the sections of outlets to infinity or pressure drops) in all "wide" outlets which grow at infinity "sufficiently rapidly". The weak solvability of the steady Navier-Stokes problem was proved in the above mentioned papers for arbitrary data. For the three-dimensional time-dependent Navier-Stokes problem unique solvability is obtained only for small data.

The next step in the evolution of the mathematical theory of viscous incompressible fluids in domains with outlets to infinity was to consider natural physical problems with prescribed fluxes in "narrow" outlets to infinity also (for example in pipes). However, since any divergence-free vector-field with a finite Dirichlet integral necessarily has zero fluxes over the sections of "narrow" outlets to infinity, the usual energy estimate method becomes insufficient in this case. The Navier-Stokes problem with additionally prescribed fluxes in "narrow" outlets to infinity has to be studied in a class of functions having infinite Dirichlet integrals. The basic results concerning such problems were obtained by O. A. Ladyzhenskaya and V. A. Solonnikov [28]. In this paper a special technique of integral estimates (the so called "techniques of Saint-Venant's principle") is developed and the existence of solutions having an infinite Dirichlet integral was proved. This result is obtained without any restrictions on data, assuming as the only necessary compatibility condition that the total flux is equal to zero. In particular, the solvability of the steady Navier-Stokes problem is proved for arbitrary data in domains with cylindrical outlets to infinity and it is shown that for sufficiently small fluxes this solution is unique and tends exponentially (as $|x| \to \infty$) in each cylinder to the corresponding Poiseuille flow. Notice that for domains with two cylindrical outlets to infinity the steady Navier-Stokes problem with prescribed flux F has also been studied by C. J. Amick [3] where a solution approaching the Poiseuille flow as $|x| \to \infty$ is constructed for small $|F|$. In [49], similar results were obtained in domains with "layer-like" outlets to infinity.

The time-dependent Navier-Stokes system in domains with outlets to infinity is much less studied. In [31], [77], [80], [81] the existence of solutions with prescribed fluxes is proved. These solutions have a finite or infinite energy integral, dependent on the geometry of the outlets to infinity. In particular, if the outlets are cylindrical and the energy integral is infinite, then the time-dependent Navier-Stokes problem is proved in [31], [77], [80], [81] either for small data or for small time intervals.

In [77], [80], [81], V. A. Solonnikov studied both steady and time-dependent Navier-Stokes problems in a very general class of domains with outlets to infinity. He developed an axiomatic approach for such problems without making assumptions abaut the shape of the outlets and imposing only certain general restrictions. The function spaces used there are also very general. Many papers were devoted to the investigation of related questions, such as regularity, asymptotic behavior and uniqueness of solutions to the steady Stokes and Navier-Stokes problems in noncompact domains. It is evident that the behavior of solutions to the Navier-Stokes problem as $|x| \to \infty$ strongly depends on the geometry of outlets to infinity. Therefore, studying the properties of solutions to the Navier-Stokes problem in such domains it is convenient to study the problem in weighted function spaces which reflect the decay properties of solutions as $|x| \to \infty$. We mention papers [37], [54]–[56], where steady Stokes and Navier-Stokes problems are studied in weighted function spaces assuming the "parabolic-like" structure of the outlets to infinity. Likewise in papers [39], [40], [42], [58], [72] where asymptotics properties of solutions to steady Stokes and Navier-Stokes problems are studied in domains with outlets to infinity coinciding for large $|x|$ with an infinite layer. In papers [9]–[11], [17], [35], [43], [73], [12]–[15], analogous questions were studied for the aperture domain and the domain having sector-like outlets to infinity. Finally, we mention papers [1], [23], [36], [38], [50]–[52], [68], [69], [79], where certain existence theorems for regular solutions are proved in domains with strip-like outlets to infinity and in domains with outlets having periodically varying sections. For two-dimensional domains with outlets to infinity, more general results concerning the pointwise decay and asymptotic properties of the solutions are obtained in [4]–[5].

Navier-Stokes problems in cylindrical outlets to infinity with the usual initial condition were studied in [66]. However, the time periodic problems of these equations, which are very important to practical applications, have not yet been studied in domains with cylindrical outlets to infinity.

Approbation

The results of the dissertation were published in the followings periodical scientific papers:

- KEBLIKAS, V.; PILECKAS, K. On the existence of non-stationary Poiseuille solution, *Siberian Math. J.*, 2005, 46(3), p. 514–526.

- KEBLIKAS, V. On the time-periodic problem for the Stokes system in domains with cylindrical outlets to infinity, *Lithuanian Math. J.*, 2007, 47(2), p. 147–163.

The results of the dissertation were presented at:

- 9^{th} International conference on mathematical modelling and analysis, Latvia, Jurmala, 2004.

- 10^{th} International conference on mathematical modelling and analysis, Lithuania, Trakai, 2005.

Contributing talks were given at the seminars at Institute of Mathematics and Informatics and Vilnius Gediminas Technical University.

Structure of the dissertation

The dissertation consist of an introduction, five chapters, conclusions and the bibliography.

• In the first chapter, the necessary function spaces are defined and certain known auxiliary results are formulated.

• In the second chapter, we study a non-stationary Poiseuille solution. Poiseuille flow is an exact solution of the steady Navier-Stokes system in an infinite straight pipe of constant cross-section σ and has the prescribed flux F over σ. Moreover, it is natural to study time dependent analog of Leray's problem. However, in this case there already appears to be a problem with the definition of time dependent Poiseuille flow. In prescribing the flux $F(t)$, we have to solve for $U(x', t)$ and $q(t)$ more complicated nonstandard inverse parabolic problem. The theory of inverse problems for the parabolic equations was studied by many authors [44]–[46]. Nevertheless, we failed to find in the literature any result concerning the solvability of our problem. The analogical inverse periodic Poiseuille flow problem was solved in [8], [19].

• In the third chapter, we study the time periodic problem for the Stokes system in domains with cylindrical outlets to infinity.

• In the forth chapter, we study the two dimensional time periodic Navier–Stokes problem. Here, it was proved that at least one solution in weighted Sobolev spaces exists. However, it's existence was proved only if some of the data is small enough. The analogical problem with the usual initial conditions was proved without any small data in [64], [66]. Time periodic Navier-Stokes problems previously was studied in many papers (e. g. [21], [26], [74], [84], [86]). But we could not find there any method for solving our problem without smallness of the data.

• In the fifth chapter, we study the three-dimensional time periodic Navier-Stokes problem. The solvability in weighted Sobolev norms was obtained, of course, for all small data (including external force).

1

Function spaces and auxiliary results

In this chapter we introduce function spaces which are used in the thesis and formulate certain embedding theorems and multiplicative inequalities that are important for further considerations. In particular, we define weighted function spaces in domains with cylindrical outlets to infinity, where the weight-function regulates the behavior of elements of these spaces as $|x| \to \infty$.

1.1 Function spaces and main notations

By $c, c_j, j = 1, 2, ...$, etc., we denote different constants whose possible dependence of parameters $a_1, ..., a_n$ will be specified whenever it is necessary. In such a case, we shall write $c = c(a_1, ..., a_n)$.

Let V be a Banach space. The norm of an element u in the function space V is denoted by $\|u; V\|$. Vector-valued functions are denoted by bold letters, however the spaces of scalar and vector-valued functions are not distinguished in notations. The vector-valued function $\mathbf{u} = (u_1, ..., u_n)$ belongs to the space V, if $u_i \in V, i = 1, ..., n$, and $\|\mathbf{u}; V\| = \sum\limits_{i=1}^{n} \|u_i; V\|$.

Let G be an arbitrary domain in $\mathbb{R}^n, n > 1$, with the boundary ∂G. We shall use the following notations:

- $C^\infty(G)$ is the set of all infinitely many times differentiable in G functions;

- $C_0^\infty(G)$ is the subset of functions from $C^\infty(G)$ with compact supports in G;

- $W_q^l(G), l \geq 0, q \in [1, \infty)$, is the Sobolev space of functions with the finite norm:

$$\|u; W_q^l(G)\| = \sum_{|\alpha| \leq l} \|D^\alpha u; L_q(G)\|,$$

where $D_x^\alpha = \dfrac{\partial^{|\alpha|}}{\partial^{\alpha_1} x_1 \partial^{\alpha_2} x_2 \partial^{\alpha_3} x_3}$, $\ |\alpha| = \alpha_1 + \alpha_2 + \alpha_3$.

- $\overset{\circ}{W}{}_q^l(G)$ is the closure of $C_0^\infty(G)$ in the norm $\|\cdot\, ; W_q^l(G)\|$.

- $L_q(G) = W_q^0(G)$.

- $L_\infty(G)$ is a space of all real Lebesgue measurable functions defined in G with the finite norm:

$$\|u; L_\infty(G)\| = \operatorname*{ess\,sup}_{x \in G} |u(x)| < \infty.$$

- $C^l(\overline{G})$ (l is an integer) is a Banach space of functions $u(x)$ for which $D^\alpha u(x)$ is bounded and uniformly continuous in $\overline{G}$ for all $0 \leq |\alpha| \leq l$. The norm in $C^l(\overline{G})$ is defined by the formula:

$$\|u; C^l(\overline{G})\| = \sum_{|\alpha|=0}^{l} \sup_{x \in \overline{G}} |D^\alpha u(x)|.$$

- $C^{l+\delta}(\overline{G})$ ($\delta \in (0,1)$) is the subspace of $C^l(\overline{G})$ consisting of functions whose derivatives up to the order l are Hölder continuous in $\overline{G}$. $C^{l+\delta}(\overline{G})$ is a Banach space with the finite norm:

$$\|u; C^{l+\delta}(\overline{G})\| = \|u; C^l(\overline{G})\| + \sum_{|\alpha|=l} \sup_{x,y \in \overline{G}} \frac{|D^\alpha u(x) - D^\alpha u(y)|}{|x-y|^\delta}.$$

Consider now functions dependent on the space variable $x \in G$ and the time $t \in (0, T)$.

- $W_2^{2l,l}(G \times (0,T)), l \geq 0$ is an integer, is a Hilbert space of functions that have generalized derivatives $D_t^r D_x^\alpha$ with every r and α such that $2r + |\alpha| \leq 2l$. The norm in $W_2^{2l,l}(G \times (0,T))$ is defined by the formula:

$$\|u; W_2^{2l,l}(G \times (0,T))\| = \left(\sum_{j=0}^{2l} \sum_{2r+|\alpha|} \int_0^T \int_G |D_t^r D_x^\alpha u(x,t)|^2 dxdt \right)^{1/2}.$$

- $W_2^{1,1}(G \times (0,T))$ and $W_2^{1,0}(G \times (0,T))$ are spaces of functions with finite norms:

$$\|u; W_2^{1,1}(G \times (0,T))\| = \left(\int_0^T \int_G (|u_t(x,t)|^2 + |u(x,t)|^2 + |\nabla u(x,t)|^2) dxdt \right)^{1/2}$$

and

$$\|u; W_2^{1,0}(G \times (0,T))\| = \left(\int_0^T \int_G (|u(x,t)|^2 + |\nabla u(x,t)|^2) dxdt \right)^{1/2}.$$

- $\mathring{W}_2^{1,1}(G \times (0,T))$ and $\mathring{W}_2^{1,0}(G \times (0,T))$ are subspaces of $W_2^{1,1}(G \times (0,T))$ and $W_2^{1,0}(G \times (0,T))$ consisting of functions satisfying the condition $u(x,t)|_{\partial G} = 0$.

- $C^{2l+2\delta, l+\delta}(\overline{G} \times [0,T])$, $l \geq 0$ is an integer, $\delta \in (0, 1/2)$, is the Hölder space of continuous functions having in $\overline{G} \times [0,T]$ continuous derivatives D_x^α with respect to x up to order $2l$ and continuous derivatives D_t^r with respect to t up to the order l. The norm in $C^{2l+2\delta, l+\delta}(\overline{G} \times [0,T])$ is defined by

$$\|u; C^{2l+2\delta, l+\delta}(\overline{G} \times (0,T))\| = \sum_{|\alpha|=0}^{2l} \sup_{x \in \overline{G}, t \in [0,T]} |D_x^\alpha u(x,t)|$$

$$+ \sum_{|r|=0}^{l} \sup_{x \in \overline{G}, t \in [0,T]} |D_t^r u(x,t)| + \sum_{|\alpha|=2l} \sup_{(x,y) \in \overline{G}, t \in [0,T]} \frac{|D_x^\alpha u(x,t) - D_y^\alpha u(y,t)|}{|x-y|^{2\delta}}$$

$$+ \sum_{r=l} \sup_{x \in \overline{G}, (t,s) \in [0,T]} \frac{|D_t^r u(x,t) - D_s^r u(x,s)|}{|t-s|^{\delta}}.$$

Let $\Omega \subset \mathbb{R}^n$, $(n = 2, 3)$ be a domain with J cylindrical outlets to infinity:

$$\Omega = \Omega_{(0)} \bigcup \big(\bigcup_{j=1}^{J} \Omega_j \big),$$

i. e., outside the sphere $|x| = r_0$ the domain Ω splits into J disjoint components Ω_j (outlets to infinity) which in some coordinate systems $x^{(j)}$ are given by the relations

$$\Omega_j = \{x^{(j)} \in \mathbb{R}^n : x^{(j)'} \in \sigma_j, \;\; x_n^{(j)} > 0\}, \;\; j = 1, ..., J, \tag{1.1}$$

where $x^{(j)'} = (x_1^{(j)}, x_2^{(j)})$ for $n = 3$, $x_1^{(j)'} = x_1^{(j)}$ for $n = 2$ and $\sigma_j \subset \mathbb{R}^{n-1}$ is a bounded domain, i. e. for $x_n^{(j)} > 0$ outlets to infinity coincide with infinite pipes $\Pi_j = \{x^{(j)} \in \mathbb{R} : x^{(j)'} \in \sigma_j, \;\; -\infty < x_n^{(j)} < \infty\}$ (if $n = 2$, the outlets Ω_j coincide with infinite strips and cross-sections $\sigma_j = (0, h_j)$ are bounded intervals).

We introduce the following notations:

$$\Omega_{jk} = \{x \in \Omega_j : x_n^{(j)} < k\}, \;\; \omega_{jk} = \Omega_{jk+1} \setminus \Omega_{jk}, \;\; j = 1, ..., J,$$
$$\widehat{\omega}_{jk} = \omega_{jk-1} \cup \omega_{jk} \cup \omega_{jk+1}, \;\; j = 1, ..., J, \tag{1.2}$$
$$\Omega_{(k)} = \Omega_{(0)} \cup \Big(\bigcup_{j=1}^{J} \Omega_{jk} \Big), \;\; \Omega_{(0)} = \Omega \setminus \Big(\bigcup_{j=1}^{J} \Omega_j \Big),$$

where $k \geq 0$ is an integer.

Denote $\boldsymbol{\beta} = (\beta_1, ..., \beta_J)$ and let $E_{\beta_j}(x) = E_{\beta_j}(x_n^{(j)})$ be smooth monotone weight-functions in Ω_j such that

$$E_{\beta_j}(x) > 0, \;\; a_1 \leq E_{-\beta_j}(x) E_{\beta_j}(x) \leq a_2 \;\; \forall x \in \Omega_j, \;\; E_{\beta_j}(0) = 1, \tag{1.3$_1$}$$

$$b_1 E_{\beta_j}(k) \leq E_{\beta_j}(x) \leq b_2 E_{\beta_j}(k), \;\; \forall x \in \omega_{jk}, \tag{1.3$_2$}$$

$$|\nabla E_{\beta_j}(x)| \leq b_3 \gamma_* E_{\beta_j}(x), \;\; \forall x \in \Omega_j, \tag{1.3$_3$}$$

$$\lim_{x_n^{(j)} \to \infty} E_{\beta_j}(x) = \infty, \;\; \text{if} \;\; \beta > 0, \tag{1.3$_4$}$$

where the constants a_1, a_2, b_1, b_2, b_3 are independent of k and b_3 is independent of β_j. Simple examples of such weight-functions are

$$E_{\beta_j}(x) = (1 + \delta |x_n^{(j)}|^2)^{\beta_j} \;\; \text{and} \;\; E_{\beta_j}(x) = \exp(2\beta_j x_n^{(j)}).$$

Note that the inequality (1.3_3) holds for the first weight-function with $\gamma_* = |\beta_j| \delta$, and for the second one with $\gamma_* = |\beta_j|$. Below, in proofs of solvability for Navier-Stokes system, we need γ_* to be "sufficiently small". For the exponential weight-function $E_{\beta_j}(x) = \exp(2\beta_j x_n^{(j)})$ this is the case, if we assume that $|\beta_j|$ is sufficiently small. In the case of the power weight-function $E_{\beta_j}(x) = (1 + \delta|x_n^{(j)}|^2)^{\beta_j}$ this assumption could be satisfied taking sufficiently small δ and there are no restrictions on β_j.

Set

$$E_{\boldsymbol{\beta}}(x) = \begin{cases} 1, \ \ x \in \Omega_{(0)}, \\ \\ E_{\beta_j}(x_n^{(j)}), \ \ x \in \Omega_j, \ \ j = 1, ..., J. \end{cases} \tag{1.4}$$

Let us introduce weighted function spaces:

• $C_0^\infty(\overline{\Omega})$ is the set of all functions from $C^\infty(\Omega)$ that are equal to zero for large $|x|$ (not necessary on $\partial\Omega$).

• $\mathcal{W}^l_{2,\boldsymbol{\beta}}(\Omega)$, $l \geq 0$, is the space of functions obtained as the closure of $C_0^\infty(\overline{\Omega})$ in the norm:

$$\|u; \mathcal{W}^l_{2,\boldsymbol{\beta}}(\Omega)\| = \left(\sum_{|\alpha|=0}^{l} \int_\Omega E_{\boldsymbol{\beta}}(x) |D^\alpha u(x)|^2 dx \right)^{1/2}.$$

• $\mathcal{W}^0_{2,\boldsymbol{\beta}}(\Omega) = \mathcal{L}_{2,\boldsymbol{\beta}}(\Omega)$.

• $\mathcal{W}^{2l,l}_{2,\boldsymbol{\beta}}(\Omega \times (0,T))$ ($l \geq 0$ is an integer), $\mathcal{W}^{1,1}_{2,\boldsymbol{\beta}}(\Omega \times (0,T))$ and $\mathcal{W}^{1,0}_{2,\boldsymbol{\beta}}(\Omega \times (0,T))$ are the spaces of functions obtained as closures of the set of all infinitely many times differentiable with respect to x and t functions, that are equal to zero for large $|x|$, in the norms

$$\|u; \mathcal{W}^{2l,l}_{2,\boldsymbol{\beta}}(\Omega \times (0,T))\| = \left(\sum_{j=0}^{2l} \sum_{2r+|\alpha|=j} \int_0^T \int_\Omega E_{\boldsymbol{\beta}}(x) |D_t^r D_x^\alpha u(x,t)|^2 dx dt \right)^{1/2},$$

$$\|u; \mathcal{W}^{1,1}_{2,\boldsymbol{\beta}}(\Omega \times (0,T))\| = \left(\int_0^T \int_\Omega E_{\boldsymbol{\beta}}(x) (|u_t(x,t)|^2 + |u(x,t)|^2 + |\nabla u(x,t)|^2) dx dt \right)^{1/2}$$

and

$$\|u;\mathcal{W}^{1,0}_{2,\boldsymbol{\beta}}(\Omega^T)\| = \Big(\int\limits_0^T\int\limits_\Omega E_{\boldsymbol{\beta}}(x)(|u(x,t)|^2 + |\nabla u(x,t)|^2)dxdt\Big)^{1/2},$$

respectively.

Note that, if $\beta_j > 0$, the weight-indices β_j shows a decay rate of elements $u \in \mathcal{W}^l_{2,\boldsymbol{\beta}}(\Omega)$ and their derivatives as $|x| \to \infty$, $x \in \Omega_j$.

We will need also a "step" weight-function

$$E^{(k)}_{\boldsymbol{\beta}}(x) = \begin{cases} 1, \ \ x \in \Omega_{(0)}, \\ E_{\beta_j}(x^{(j)}_n), \ \ x \in \Omega_{jk}, \ \ j = 1, ..., J, \\ E_{\beta_j}(k), \ \ x \in \Omega\backslash\Omega_{jk}, \ \ j = 1, ..., J. \end{cases} \tag{1.5}$$

It is easy to see that $E^{(k)}_{\boldsymbol{\beta}}(x) = E_{\boldsymbol{\beta}}(x)$ for $x \in \Omega_{(k)}$,

$$|\nabla E^{(k)}_{\boldsymbol{\beta}}(x)| \leq b_3\gamma_* E^{(k)}_\beta(x), \tag{1.6}$$

and, if $\beta_j \geq 0$, then

$$E^{(k)}_{\beta_j}(x) \leq E_{\beta_j}(x), \quad \forall x \in \Omega.$$

Moreover, by definition

$$\frac{\partial}{\partial x^{(j)}_l} E^{(k)}_{\boldsymbol{\beta}}(x) = \begin{cases} 0, \ \ l = 1, ..., n, \ \ x \in \Omega_{(0)}, \\ 0, \ \ l = 1, ..., n, \ \ x \in \Omega\backslash\Omega_k, \\ 0, \ \ l = 1, ..., n-1, \ \ x \in \Omega_{jk}, j = 1, ..., J, \\ \dfrac{\partial}{\partial x^{(j)}_l} E_{\boldsymbol{\beta}}(x), \ \ l = n, \ \ x \in \Omega_{jk}, j = 1, ..., J. \end{cases}$$

Thus,

$$\nabla E^{(k)}_{\boldsymbol{\beta}}(x) \subset \bigcup_{j=1}^{J} \overline{\Omega}_{jk}. \tag{1.7}$$

Remark. The general weight functions $E_{\boldsymbol{\beta}}(x)$ and the corresponding weighted spaces first were introduced by K. Pileckas in [65]–[67].

1.2 Auxiliary results

In this section we collect known results which will be used below.

Lemma 1.1. *If $u \in W_2^{2l,l}(G \times (0,T))$ then $D_t^r D_x^\alpha u(x,t)$ with $2r + |\alpha| < 2l - 1$ belongs to the space $W_2^{2l-2r-|\alpha|-1}(G)$ and there holds the inequality*

$$\|D_t^r D_x^\alpha u(\cdot,t); W_2^{2l-2r-|\alpha|-1}(G)\| \le c\|u; W_2^{2l,l}(G \times (0,T))\|, \tag{1.8}$$

where the constant c is independent of $t \in [0,T]$.

Lemma 1.2. *Let $G \subset \mathbb{R}^2$ be a bounded domain. For any $u \in W_2^1(G)$ holds the multiplicative inequality*

$$\|u; L_4(G)\| \le c\|u; W_2^1(G)\|^{1/2}\|u; L_2(G)\|^{1/2}. \tag{1.9}$$

Lemma 1.3. *Let $G \subset \mathbb{R}^3$ be a bounded domain. For any $u \in W_2^1(G)$ holds the multiplicative inequality*

$$\|u; L_3(G)\| \le c\|u; W_2^1(G)\|^{1/2}\|u; L_2(G)\|^{1/2}. \tag{1.10}$$

If $u \in W_2^2(G)$, then the following multiplicative inequality

$$\begin{aligned}\|u; L_\infty(G)\| &\le c\|\nabla u; L_6(G)\|^{1/2}\|u; L_6(G)\|^{1/2}\\ &\le c\|\nabla u; W_2^1(G)\|^{1/2}\|u; W_2^1(G)\|^{1/2}\end{aligned} \tag{1.11}$$

is valid. The constants in (1.10), (1.11) depend only on G.

Lemma 1.4. *Let $G \subset \mathbb{R}^2$ be a bounded domain, $T < \infty$. Then the embedding operator $I : W_2^1(G \times (0,T)) \hookrightarrow L_4(G \times (0,T))$ is compact.*

Proofs of Lemmas 2.1–2.4 could be found in [2] and [27] books.

Lemma 1.5. *Let $G \subset \mathbb{R}^2$ be a bounded domain, $T < \infty$ and let $\{u_n(x,t)\}$ be a weakly convergent in the space $W_2^{2,1}(G \times (0,T))$ sequence of functions. Then*

$$\int\limits_0^T \|\nabla u_n(\cdot,t) - \nabla u_m(\cdot,t); L_4(G)\|^2 dt \to 0, \ \ as \ \ n,m \to 0.$$

Let Ω be a domain with cylindrical outlets to infinity (see section 1).

Lemma 1.6. *For any function $\mathbf{u} \in \mathcal{W}^1_{2,\boldsymbol{\beta}}(\Omega)$ which is equal to zero on $\partial\Omega$ holds the following weighted Poincaré inequality*

$$\int\limits_\Omega E_{\boldsymbol{\beta}}(x)|u(x)|^2 dx \leq c \int\limits_\Omega E_{\boldsymbol{\beta}}(x)|\nabla u(x)|^2 dx. \tag{1.12}$$

Note, that if we take in (1.12) $\boldsymbol{\beta} = 0$, we get usual Poincaré inequality

$$\int\limits_\Omega |u(x)|^2 dx \leq c_0 \int\limits_\Omega |\nabla u(x)|^2 dx.$$

Lemma 1.7. *Let $\mathbf{u}(\cdot,t) \in \mathring{W}_2^1(\Omega)$, $\mathbf{u}_t(\cdot,t) \in L_2(\Omega)$, $\mathrm{div}\mathbf{u}(x,t) = 0$, $\forall t \in [0,T]$,*

$$\int\limits_{\sigma_j} \mathbf{u}(x,t) \cdot \mathbf{n}(x) ds = 0, \ \ j = 1,...,J.$$

Then there exists a vector-field $\mathbf{W}^{(k)}(\cdot,t) \in \mathring{W}_2^1(\Omega)$ with $\mathbf{W}_t^{(k)}(\cdot,t) \in \mathring{W}_2^1(\Omega)$ such that $\mathrm{supp}_x \mathbf{W}^{(k)}(x,t) \subset \bigcup\limits_{j=1}^J \overline{\Omega}_{jk}$ and

$$\mathrm{div}\mathbf{W}^{(k)}(x,t) = -\mathrm{div}(E_{\boldsymbol{\beta}}^{(k)}(x)\mathbf{u}(x,t)).$$

There holds the estimates

$$\int_\Omega E^{(k)}_{-\boldsymbol{\beta}}(x)|\nabla \mathbf{W}^{(k)}(x,t)|^2dx \le c\gamma_*^2 \int_\Omega E^{(k)}_{\boldsymbol{\beta}}(x)|\mathbf{u}(x,t)|^2dx$$

$$\le c\gamma_*^2 \int_\Omega E^{(k)}_{\boldsymbol{\beta}}(x)|\nabla \mathbf{u}^{(k)}(x,t)|^2dx, \tag{1.13}$$

$$\int_\Omega E^{(k)}_{-\boldsymbol{\beta}}(x)|\nabla \mathbf{W}^{(k)}_t(x,t)|^2dx \le c\gamma_* \int_\Omega E^{(k)}_{\boldsymbol{\beta}}(x)|\mathbf{u}^{(k)}_t(x,t)|^2dx. \tag{1.14}$$

Moreover, if $\mathbf{u}(x,t)$ *is time periodic with a period* T*, i. e.* $\mathbf{u}(x,0) = \mathbf{u}(x,T)$*, then also* $\mathbf{W}^{(k)}(x,t)$ *is time periodic with the same period. The constants in* (1.13) *and* (1.14) *are independent of k and* $t \in [0,T]$.

Let G be a bounded domain in $\mathbb{R}^n, n = 2,3$. Consider in G the following problem:

$$\begin{cases} \mathrm{div}\mathbf{W}(x,t) = g(x,t), \\ \\ \qquad \mathbf{W}(x,t)|_{\partial G} = 0 \end{cases} \tag{1.15}$$

assuming that

$$\int_G g(x,t)dx = 0, \quad \forall t \in [0,T]. \tag{1.16}$$

Lemma 1.8. *Let* $G \subset \mathbb{R}^n$ *be a bounded domain with the Lipschitz boundary* ∂G*. If* $g(\cdot,t) \in \mathring{W}^1_2(G)$*,* $g_t(\cdot,t) \in L_2(G)$ *and (1.16) holds* $\forall t \in [0,T]$*, then problem (1.15) admits at least one solution* $\mathbf{W}(\cdot,t) \in \mathring{W}^2_2(G)$ *with* $\mathbf{W}_t(\cdot,t) \in \mathring{W}^1_2(G)$*.There holds the estimates*

$$\|\mathbf{W}(\cdot,t); W^2_2(G)\| \le \|g(\cdot,t); W^1_2(G)\|, \tag{1.17}$$

$$\|\nabla \mathbf{W}_t(\cdot,t); L_2(G)\| \le c\|g_t(\cdot,t); L_2(G)\|. \tag{1.18}$$

with a constant c independent of $g(x,t)$ *and* $t \in [0,T]$.

Lemmas 2.5–2.8 are proved in [66].

2

Non-stationary Poiseuille solution

2.1 Formulation of the problem

In this chapter we study the initial-boundary value problem for the non-stationary Navier-Stokes system in an infinite cylinder $\Pi = \{x \in \mathbb{R}^3 : x' = (x_1, x_2) \in \sigma, x_3 \in \mathbb{R}\}$:

$$\begin{cases} \mathbf{u}_t(x,t) - \nu\Delta\mathbf{u}(x,t) + (\mathbf{u}(x,t)\cdot\nabla\mathbf{u}(x,t)) + \nabla p(x,t) = 0, \\ \mathrm{div}\mathbf{u}(x,t) = 0, \\ \mathbf{u}(x,t) = 0, \\ \mathbf{u}(x,0)|_{\partial\Pi} = \mathbf{a}(x). \end{cases} \tag{2.1}$$

We look for a solution of problem (2.1) having a prescribed time-dependent flux $F(t)$ through the cross-section σ:

$$\int\limits_{\sigma} u_3(x',t)\,dx' = F(t). \tag{2.2}$$

We assume that the initial data has the form $\mathbf{a} = (0,\,0,\,a_3)$, where $a_3 = a_3(x')$ does not depend on x_3. Finally, we suppose that there holds the necessary compatibility condition

$$\int\limits_{\sigma} a_3(x')\,dx' = F(0). \tag{2.3}$$

We look for the solution $(\mathbf{u}(x,t), p(x,t))$ of problem (2.1), (2.2) in the form

$$\mathbf{u}(x,t) = (0,\,0,\,v(x',t)), \qquad p(x,t) = -q(t)x_3 + p_0(t), \tag{2.4}$$

where $p_0(t)$ is an arbitrary function of t.

Definition. *The solution of the problem (2.1), (2.2) having the form (2.4) is called a Poiseuille solution.*

Substituting (2.4) into (2.1), we get for $v(x',t)$ and $q(t)$ the following initial-boundary value problem for the heat equation

$$\begin{cases} v_t(x',t) - \nu\Delta' v(x',t) = q(t), \\ v(x',t) = 0, \\ v(x',0) = a_3(x'), \end{cases} \tag{2.5}$$

where Δ' denotes the Laplace operator with respect to variables $x' = (x_1, x_2)$.

The right-hand side $q(t)$ of equation (2.5) is not given; it has to be determined so that the solution $v(x',t)$ of (2.5) satisfies flux condition (2.2), i. e.,

$$\int\limits_{\sigma} v(x',t)\,dx' = F(t). \tag{2.6}$$

Thus, we arrive at the following inverse problem:

Given $a_3(x')$ *and* $F(t)$, *find a pair of functions* $(v(x',t), q(t))$, *satisfying the initial-boundary value problem* (2.5) *and the integral condition* (2.6).

Note that the the existence of the stationary Poiseuille solution is well known

[33]. It solves the stationary Navier-Stokes system

$$\begin{cases} -\nu\Delta\mathbf{u}(x) + (\mathbf{u}(x)\cdot\nabla\mathbf{u}(x)) + \nabla p = 0, \\ \operatorname{div}\mathbf{u}(x) = 0, \\ \mathbf{u}(x)|_{\partial\Omega} = 0, \\ \int\limits_{\sigma} u_3(x)dx = F, \end{cases} \tag{2.7}$$

in the cylinder Π and has the form:

$$\mathbf{u}_F(x) = (0, 0, q_F v(x')), \;\; p_F(x) = -\nu q_F x_3 + p_0, \tag{2.8}$$

where p_0 – is an arbitrary constant, and $v(x')$ is the solution of the Dirichlet boundary value problem for the Poisson equation

$$\begin{cases} -\nu\Delta' v(x') = 1, \\ v(x')|_{\partial\sigma} = 0. \end{cases} \tag{2.9}$$

Since

$$\int\limits_{\sigma} v(x')dx' = \nu \int\limits_{\sigma} |\nabla' v(x')|^2 dx' := \kappa_0 > 0,$$

the constant q_F could be chosen so that the Poiseuille solution satisfies the flux condition $\int\limits_{\sigma} v(x')dx' = F$, i. e. $q_F = F\kappa_0^{-1}$.

In the stationary case the constant q_F defining the pressure drop is proportional to the flux F. So the stationary problem with the prescribed pressure drop is equivalent to the one with the prescribed flux. In contrast, for the non-stationary case the function $q(t)$ defining the pressure drop is determined as a solution to inverse problem (2.5), (2.6). Thus, the non-stationary case the problem with a given pressure drop $q(t)$ is not equivalent to one with a given flux $F(t)$. Analogously, could be considered the Poiseuille solution in a two-dimensional strip $\Pi = \{x = (x_1, x_2), \;\; x_1 \in (0, h), \;\; x_2 \in \mathbb{R}\}$. In this case we suppose that the initial data has the form $\mathbf{a} = (0, a_2(x_1))$ and the flux condition is:

$$\int\limits_0^h u_2(x_1, t)dx_1 = F(t),$$

where $a(x')$ and $F(t)$ satisfy the compatibility condition:

$$\int_0^h a_2(x_1)dx_1 = F(0).$$

The corresponding non-stationary Poiseuille solution $(\mathbf{u}, p)$ has the form

$$\mathbf{u}(x,t) = (0, v(x_1,t)), \;\; p(x,t) = -q(t)x_2 + p_0(t),$$

where $v(x_1,t)$ and $q(t)$ satisfy the inverse problem

$$\begin{cases} v_t(x_1,t) - \nu \dfrac{\partial^2 v(x_1,t)}{\partial x_1^2} = q(t), \\ v(0,t) = v(h,t) = 0, \\ v(x_1,0) = a_2(x_1), \\ \int_0^h v(x_1,t)dx_1 = F(t). \end{cases} \qquad (2.10)$$

2.2 Construction of an approximate solution

Let us consider in $\sigma \times (0,T)$ the inverse problem

$$\begin{cases} v_t(x',t) - \nu\Delta v(x',t) = q(t), \\ v(x',t)|_{\partial\sigma} = 0, \\ v(x',0) = a(x'), \\ \int_\sigma v(x',t)\,dx' = F(t), \\ \int_\sigma a(x')\,dx' = F(0). \end{cases} \qquad (2.11)$$

We shall study problem (2.11) in Hölder spaces. First, we consider problem (2.11) with $a(x',t) = 0$.

Denote by $u_k(x')$ and λ_k the eigenfunctions and eigenvalues of the Laplace operator in the Sobolev space $\overset{\circ}{W}{}_2^1(\sigma)$:

$$\begin{cases} -\nu\Delta' u_k(x') = \lambda_k u_k(x'), \\ u_k(x')|_{\partial\sigma} = 0. \end{cases} \tag{2.12}$$

It is well known (see, for example [6]) that $u_k(x')$ are orthonormal in $L_2(\sigma)$, i. e.,

$$\int_\sigma u_k(x')u_l(x')dx' = \delta_{kl},$$

where δ_{kl} is Kronecker symbol. Moreover, $\int_\sigma \nabla u_k(x)\nabla u_l(x)dx' = \lambda_k\delta_{kl}$. Note that $\lambda_k > 0$ and $\{\lambda_k\} \to \infty$ as $k \to \infty$.

The constant function 1 can be decomposed into the Fourier series:

$$1 = \sum_{k=1}^{\infty} \beta_k u_k(x'),$$

where $\beta_k = \int_\sigma u_k(x')dx', k = 1, 2, \ldots,$ and $\sum_{k=1}^{\infty} \beta_k^2 = |\sigma|$. We look for an approximate solution $(v^{(N)}, q^{(N)})$ of the problem (2.11) in the form

$$v^{(N)}(x', t) = \sum_{k=1}^{N} w_k^{(N)}(t)u_k(x'). \tag{2.13}$$

Inserting (2.13) into (2.11), for functions $w_k^{(N)}(t)$ we get ordinary differential equations:

$$\begin{cases} w_k^{(N)\prime}(t) + \lambda_k w_k^{(N)}(t) = \beta_k q^{(N)}(t), \\ w_k^{(N)}(0) = 0, \quad k = 1, ..., N. \end{cases}$$

The solutions of the equations for each $k = 1, ..., N$ has the form

$$w_k^{(N)}(t) = \beta_k \int_0^t \exp(-\lambda_k(t-\tau))q^{(N)}(\tau)\, d\tau.$$

Thus, we get that

$$v^{(N)}(x',t) = \sum_{k=1}^{N} \beta_k \Big(\int_0^t \exp(-\lambda_k(t-\tau)) q^{(N)}(\tau)\, d\tau \Big) u_k(x'). \qquad (2.14)$$

Now we choose the function $q^{(N)}(t)$ in order to satisfy the condition

$$\int_\sigma v^{(N)}(x',t)\, dx' = F(t), \quad \forall t \in (0,T). \qquad (2.15)$$

Inserting (2.14) into (2.15), we obtain the equality

$$\int_\Omega v^{(N)}(x',t)\, dx' = \sum_{k=1}^{N} \beta_k \int_0^t \exp(-\lambda_k(t-\tau)) q^{(N)}(\tau)\, d\tau \int_\Omega u_k(x')\, dx' = F(t),$$

which is equivalent to the Volterra integral equation of the first kind for the function $q^{(N)}(t)$:

$$\sum_{k=1}^{N} \beta_k^2 \int_0^t \exp(-\lambda_k(t-\tau)) q^{(N)}(\tau)\, d\tau = F(t). \qquad (2.16)$$

Suppose that the derivative of $F(t)$ exists. Differentiating (2.16), we deduce that

$$q^{(N)}(t) - \frac{1}{\varkappa_N} \sum_{k=1}^{N} \beta_k^2 \lambda_k \int_0^t \exp(-\lambda_k(t-\tau)) q^{(N)}(\tau)\, d\tau = \varphi^{(N)}(t), \qquad (2.17)$$

where $\varkappa_N = \sum\limits_{k=1}^{N} \beta_k^2$ and $\varphi^{(N)}(t) = F'(t)/\varkappa_N$. This is a Volterra equation of the second kind with the kernel

$$K^{(N)}(t,\tau) = \varkappa_N^{-1} \sum_{k=1}^{N} \beta_k^2 \lambda_k \exp(-\lambda_k(t-\tau)).$$

The solvability of such equations is well known; see [34] for instance. However, it is not possible to pass to the limit as $N \to \infty$ directly in (2.17), since the series $\varkappa_\infty^{-1} \sum\limits_{k=1}^{\infty} \beta_k^2 \lambda_k$ that defines the limit kernel $K^{(\infty)}(t,t)$ at $t=\tau$ diverges.

We will prove below that under natural compatibility conditions the sequence $\{(v^{(N)}(x',t), q^{(N)}(t))\}$ converges as $N \to \infty$ in the form $C^{2l+2\delta,\, l+\delta}(\overline{Q} \times (0,T)) \times$

$C^{l+\delta}(0,T)$ to a solution $(v(x',t),q(t))$ of problem (2.11).

2.3 Uniform estimates for the solution to integral equation (2.17)

In this section we prove uniform estimates in the Hölder space for the solution $q^{(N)}(t)$ of the integral equation (2.17). Consider first the auxiliary integral equation

$$f^{(N)}(t)-\frac{1}{\varkappa_N}\sum_{k=2}^{N}\beta_k^2\lambda_k\int\limits_0^t\exp(-\lambda_k(t-\tau))f^{(N)}(\tau)\,d\tau=g(t). \qquad (2.18)$$

Note that differently from (2.17), the sum in (2.18) starts from $k=2$.

Lemma 2.1. *Suppose that $g\in C^\delta(0,T)$ and $g(0)=0$. Then there exists a unique solution $f^{(N)}\in C^\delta(0,T)$ to* (2.18). *Moreover, $f^{(N)}(0)=0$ and*

$$|f^{(N)}(t)|\le\frac{\varkappa_N}{\beta_1^2}\sup_{\tau\in[0,T]}|g(\tau)|,\quad \forall t\in[0,T], \qquad (2.19)$$

$$\|f^{(N)};C^\delta(0,T)\|\le 2\frac{\varkappa_N}{\beta_1^2}\|g;\ C^\delta(0,T)\|. \qquad (2.20)$$

Proof. Put

$$f_0^{(N)}(t)=g(t),\dots,$$

$$f_n^{(N)}(t)=\frac{1}{\varkappa_N}\sum_{k=2}^{N}\beta_k^2\lambda_k\int\limits_0^t\exp(-\lambda_k(t-\tau))f_{n-1}^{(N)}(\tau)\,d\tau,\dots,$$

$$f^{(N)}(t)=\sum_{n=1}^{\infty}f_n^{(N)}(t). \qquad (2.21)$$

Then

$$|f_0^{(N)}(t)|\le\sup_{\tau\in[0,t]}|g(\tau)|,$$

$$|f_1^{(N)}(t)|\le\frac{1}{\varkappa_N}\sup_{\tau\in\overline{[0,t]}}|g(\tau)|\sum_{k=2}^{N}\beta_k^2\lambda_k\int\limits_0^t\exp(-\lambda_k(t-\tau))\,d\tau\le$$

$$\leq \frac{1}{\varkappa_N} \sup_{\tau\in[0,t]} |g(\tau)| \sum_{k=2}^{N} \beta_k^2 (1-\exp(-\lambda_k t)) \leq \frac{\gamma_N}{\varkappa_N} \sup_{\tau\in[0,t]} |g(\tau)|, ...,$$

$$|f_n^{(N)}(t)| \leq \left(\frac{\gamma_N}{\varkappa_N}\right)^n \sup_{\tau\in[0,t]} |g(\tau)|, ...,$$

where $\gamma_N = \sum\limits_{k=2}^{N} \beta_k^2 = \varkappa_N - \beta_1^2$. Since $\gamma_N/\varkappa_N < 1$, the series (2.21) that determines a solution to (2.18) converges absolutely and uniformly on the interval $t \in [0,\, T]$. Moreover,

$$|f^{(N)}(t)| \leq \sup_{\tau\in[0,t]} |g(\tau)| \sum_{n=0}^{\infty} \left(\frac{\gamma_N}{\varkappa_N}\right)^n =$$

$$= \frac{\varkappa_N}{\varkappa_N - \gamma_N} \sup_{\tau\in[0,t]} |g(\tau)| = \frac{\varkappa_N}{\beta_1^2} \sup_{\tau\in[0,t]} |g(\tau)|,$$

and (2.18) implies that $f^{(N)}(0) = 0$.

Estimate now $f^{(N)}(t)$ in the Hölder norm. Since $f_{n-1}^{(N)}(0) = 0$, we have

$$|f_n^{(N)}(t+h) - f_n^{(N)}(t)| = \Bigg| \frac{1}{\varkappa_N} \sum_{k=2}^{N} \beta_k^2 \lambda_k \int\limits_0^{t+h} \exp(-\lambda_k(t+h-\tau)) f_{n-1}^{(N)}(\tau) d\tau -$$

$$- \frac{1}{\varkappa_N} \sum_{k=2}^{N} \beta_k^2 \lambda_k \int\limits_0^{t} \exp(-\lambda_k(t-\tau)) f_{n-1}^{(N)}(\tau)\, d\tau \Bigg| \leq$$

$$\leq \frac{1}{\varkappa_N} \sum_{k=2}^{N} \beta_k^2 \lambda_k \int\limits_0^{t} \exp(-\lambda_k(t-\tau)) |f_{n-1}^{(N)}(\tau+h) - f_{n-1}^{(N)}(\tau)|\, d\tau +$$

$$+ \frac{1}{\varkappa_N} \sum_{k=2}^{N} \beta_k^2 \lambda_k \int\limits_0^{h} \exp(-\lambda_k(t+h-\tau)) |f_{n-1}^{(N)}(\tau) - f_{n-1}^{(N)}(0)|\, d\tau, \quad n = 1, 2, \ldots.$$

Estimating the right-hand sides here, we find that

$$|f_0^{(N)}(t+h) - f_0^{(N)}(t)| = |g(t+h) - g(t)| \leq h^\delta \|g;\ C^\delta(0,T)\|,$$

$$|f_1^{(N)}(t+h) - f_1^{(N)}(t)| \leq 2h^\delta \|g;\ C^\delta(0,T)\| \frac{\gamma_N}{\varkappa_N}, ...,$$

$$|f_n^{(N)}(t+h) - f_n^{(N)}(t)| \leq 2h^\delta \|g;\ C^\delta(0,T)\| \left(\frac{\gamma_N}{\varkappa_N}\right)^n, \ldots.$$

for all $\forall t, t+h \in [0,T]$. Hence,

$$\|f_n^{(N)};\ C^\delta(0,T)\| \leq 2\|g;\ C^\delta(0,T)\| \left(\frac{\gamma_N}{\varkappa_N}\right)^n$$

and consequently, the estimate (2.20) holds for $f^{(N)}(t) = \sum\limits_{n=1}^{\infty} f_n^{(N)}(t)$. The lemma is proved.

Define now an operator B_N on the space $\widehat{C}^\delta(0,T) = \{\Phi(t) \in C^\delta(0,T) : \Phi(0) = 0\}$ by the formula

$$B_N\Phi(t) = \Phi(t) - \frac{1}{\varkappa_N} \sum_{k=2}^{N} \beta_k^2 \lambda_k \int\limits_0^t \exp(-\lambda_k(t-\tau))\Phi(\tau)\, d\tau.$$

Lemma 2.1 implies that there exists a bounded inverse operator
$B_N^{-1} : \widehat{C}^\delta(0,T) \to \widehat{C}^\delta(0,T)$ and that

$$\|B_N^{-1}; \widehat{C}^\delta(0,T) \to \widehat{C}^\delta(0,T)\| \leq 2\frac{\varkappa_N}{\beta_1^2}. \tag{2.22}$$

In addition,

$$|B_N^{-1} g(t)| \leq \frac{\varkappa_N}{\beta_1^2} \sup_{\tau\in[0,t]} |g(\tau)|. \tag{2.23}$$

Consider the full equation (2.17), rewriting it in the form

$$B_N q^{(N)}(t) = \frac{1}{\varkappa_N}\beta_1^2\lambda_1 \int\limits_0^t \exp(-\lambda_1(t-\tau)) q^{(N)}(\tau)\, d\tau + \varphi^{(N)}(t). \tag{2.24}$$

Introducing the notation $g^{(N)} = B_N q^{(N)}$ and $q^{(N)} = B_N^{-1} g_N$, we obtain from (2.24) the integral equation

$$g^{(N)}(t) - \frac{1}{\varkappa_N}\beta_1^2\lambda_1 \int\limits_0^t \exp(-\lambda_1(t-\tau))(B_N^{-1} g^{(N)})(\tau)\, d\tau = \varphi^{(N)}(t). \tag{2.25}$$

Lemma 2.2. *Let* $\varphi^{(N)}(t) \in \widehat{C}^\delta(0,T)$. *Then there exists a unique solution* $g^{(N)} \in$

$\widehat{C}^{\delta}(0,T)$ *to* (2.25) *satisfying the estimates*

$$|g^{(N)}(t)| \le \exp(\lambda_1 t) \sup_{\tau\in[0,t]} |\varphi^{(N)}(\tau)|, \tag{2.26}$$

$$\|g^{(N)}; C^{\delta}(0,T)\| \le c \exp(2\lambda_1 T)\|\varphi^{(N)};\ C^{\delta}(0,T)\|. \tag{2.27}$$

Proof. Put

$$g^{(N)}(t) = \sum_{n=0}^{\infty} g_n^{(N)}(t), \tag{2.28}$$

where

$$g_0^{(N)}(t) = \varphi^{(N)}(t),$$

$$g_1^{(N)}(t) = \frac{1}{\varkappa_N}\beta_1^2\lambda_1 \int_0^t \exp(-\lambda_1(t-\tau))(B_N^{-1}g_0^{(N)})(\tau)\,d\tau, \dots\ ,$$

$$g_{n+1}(t) = \frac{1}{\varkappa_N}\beta_1^2\lambda_1 \int_0^t \exp(-\lambda_1(t-\tau))(B_N^{-1}g_n^{(N)})(\tau)\,d\tau, \dots\ .$$

Using (2.23), we infer by induction that

$$|g_1^{(N)}(t)| \le \frac{\varkappa_N}{\beta_1^2} \sup_{\tau\in[0,t]} |\varphi^{(N)}(\tau)| \frac{1}{\varkappa_N}\beta_1^2\lambda_1 \int_0^t \exp(-\lambda_1(t-\tau))\,d\tau \le$$

$$\le \lambda_1 t \sup_{\tau\in[0,t]} |\varphi^{(N)}(\tau)|, \dots\ ,$$

$$|g_{n+1}^{(N)}(t)| \le \frac{1}{\varkappa_N}\beta_1^2\lambda_1 \int_0^t \exp(-\lambda_1(t-\tau))\,\frac{\varkappa_N}{\beta_1^2} \sup_{s\in[0,t]} |g_n^{(N)}(s)|\,d\tau \le$$

$$\le \frac{\lambda_1^{n+1}}{n!} \sup_{\tau\in\overline{\Delta^t}} |\varphi^{(N)}(\tau)| \int_0^t \tau^n\,d\tau = \frac{(\lambda_1 t)^{n+1}}{(n+1)!} \sup_{\tau\in[0,t]} |\varphi^{(N)}(\tau)|, \dots\ .$$

Thus,

$$\sum_{n=0}^{\infty} |g_n^{(N)}(t)| \le \sup_{\tau\in[0,t]} |\varphi^{(N)}(\tau)| \sum_{n=0}^{\infty} \frac{(\lambda_1 t)^n}{n!} = \exp(\lambda_1 t) \sup_{\tau\in[0,t]} |\varphi^{(N)}(\tau)| \tag{2.29}$$

and hence series (2.28), determining the solution to integral equation (2.25) converges absolutely and uniformly on each finite interval $[0,\ T]$. Moreover,

$$|g^{(N)}(t)| \leq \exp(\lambda_1 t) \sup_{\tau\in[0,t]} |\varphi^{(N)}(\tau)| \leq \exp(\lambda_1 T) \sup_{t\in[0,t]} |\varphi^{(N)}(t)|. \qquad (2.30)$$

Using (2.22), it is easy to show that for the difference $|g^{(N)}(t+h) - g^{(N)}(t)|$ holds the inequality

$$|g^{(N)}(t+h) - g^{(N)}(t)| \leq ch^\delta \exp(2\lambda_1 T)\|g^{(N)};\ \widehat{C}^\delta(0,T)\|, \quad \forall t, t+h \in [0,\ T].$$

Therefore, $g^{(N)} \in \widehat{C}^\delta(0,T)$ and (2.27) holds. The lemma is proved.

Lemma 2.3. *Let $\varphi^{(N)} \in \widehat{C}^\delta(0,T)$. Then there exists a unique solution $q^{(N)} \in \widehat{C}^\delta(0,T)$ to (2.17) and*

$$|q^{(N)}(t)| \leq \frac{\varkappa_N}{\beta_1^2} \exp(\lambda_1 t) \sup_{\tau\in[0,t]} |\varphi^{(N)}(\tau)|, \qquad (2.31)$$

$$\|q^{(N)}; C^\delta(0,T)\| \leq c \exp(2\lambda_1 T)\|\varphi^{(N)};\ C^\delta(0,T)\|. \qquad (2.32)$$

To prove this lemma it suffices to take $q^{(N)} = B_N^{-1} g^{(N)}$ and to apply (2.22), (2.23), (2.26), (2.27).

Denote by $\widehat{C}^{l+\delta}(0,T)$ the subspace of all functions in $C^{l+\delta}(0,T)$, satisfying the conditions

$$h(0) = 0, \quad \frac{\mathrm{d}}{\mathrm{d}t}h(0) = 0, \quad \ldots \quad , \quad \frac{\mathrm{d}^l}{\mathrm{d}t^l}h(0) = 0. \qquad (2.33)$$

Lemma 2.4. *Let $\varphi^{(N)} \in \widehat{C}^{l+\delta}(0,T), l \geq 0$. Then there exists a unique solution $q^{(N)} \in \widehat{C}^{l+\delta}(0,T)$ to (2.17) and*

$$\Big|\frac{\mathrm{d}^s}{\mathrm{d}t^s} q^{(N)}(t)\Big| \leq \frac{\varkappa_N}{\beta_1^2} \exp(\lambda_1 t) \sup_{\tau\in(0,t)} \Big|\frac{\mathrm{d}^s}{\mathrm{d}\tau^s}\varphi^{(N)}(\tau)\Big|, \quad s = 0,\ldots,l, \qquad (2.34)$$

$$\|q^{(N)}; C^{l+\delta}(0,T)\| \leq c \exp(2\lambda_1 T)\|\varphi^{(N)};\ C^{l+\delta}(0,T)\|. \qquad (2.35)$$

Proof. Differentiating (2.17) we find that,

$$q^{(N)\prime}(t) - \frac{1}{\varkappa_N}\sum_{k=1}^{N}\beta_k^2\lambda_k q^{(N)}(t) -$$

$$-\frac{1}{\varkappa_N}\sum_{k=1}^{N}\beta_k^2\lambda_k\int\limits_0^t \frac{\mathrm{d}}{\mathrm{dt}}\big(\exp(-\lambda_k(t-\tau))\big)q^{(N)}(\tau)\,d\tau = \varphi^{(N)\prime}(t).$$

Integrating by parts in the third term and using the condition $q^{(N)}(0) = \varphi^{(N)}(0) = 0$, we rewrite this equation as

$$q^{(N)\prime}(t) - \frac{1}{\varkappa_N}\sum_{k=1}^{N}\beta_k^2\lambda_k\int\limits_0^t \exp(-\lambda_k(t-\tau))q^{(N)\prime}(\tau)\,d\tau = \varphi^{(N)\prime}(t).$$

Thus, for $q^{(N)\prime}(t)$ we obtain an equation analogous to (2.17). Since $\varphi^{(N)}(t) \in \widehat{C}^{l+\delta}(0,T)$, i. e., $\frac{\mathrm{d}^s}{\mathrm{d}t^s}\varphi^{(N)}(0) = 0, s = 0,\dots,l$, the derivatives $\frac{\mathrm{d}^s}{\mathrm{d}t^s}q^{(N)}(t), s = 1,\dots,l$, satisfy (2.17) with the right-hand side equal to $\frac{\mathrm{d}^s}{\mathrm{d}t^s}\varphi^{(N)}(t)$. Lemma 2.3 implies that $\frac{d^s}{dt^s}q^{(N)}(t) \in \widehat{C}^{l+\delta}(0,T)$, and because of (2.31) and (2.32), the estimates (2.34) and (2.35) holds. The lemma is proved.

2.4 Existence of a solution to problem (2.11)

Let us prove first that the sequence $q^{(N)}(t)$ converges in the norm of the space $C^{l+\delta}(0,T)$.

Lemma 2.5. *If $F(t) \in \widehat{C}^{l+1+\delta}(0,T)$ with, $l \geq 0$ and $\delta \in (0,1)$, then the sequence $q^{(N)}(t)$ converges in the norm of $C^{l+\delta}(0,T)$, and the limit function $q(t) \in \widehat{C}^{l+\delta}(0,T)$ satisfies the inequalities*

$$\Big|\frac{\mathrm{d}^s}{\mathrm{d}t^s}q(t)\Big| \leq \frac{|\sigma|}{\beta_1^2}\exp(\lambda_1 t)\sup_{\tau\in[0,t]}\Big|\frac{\mathrm{d}^{s+1}}{\mathrm{d}\tau^{s+1}}F(\tau)\Big|, \quad s = 0,\dots,l, \tag{2.36}$$

$$\|q; C^{l+\delta}(0,T)\| \leq c\exp(2\lambda_1 T)\|F;\ C^{l+1+\delta}(0,T)\|. \tag{2.37}$$

Proof. It is easy to see that the difference $\mathcal{Q}^{(N,M)}(t) = q^{(N+M)}(t) - q^{(N)}(t)$

satisfies the relation

$$\mathcal{Q}^{(N,M)}(t) - \frac{1}{\varkappa_N}\sum_{k=1}^{N}\beta_k^2\lambda_k\int\limits_0^t \exp(-\lambda_k(t-\tau))\mathcal{Q}^{(N,M)}(\tau)\,d\tau =$$

$$= (\frac{1}{\varkappa_{N+M}} - \frac{1}{\varkappa_N})\sum_{k=1}^{N}\beta_k^2\lambda_k\int\limits_0^t \exp(-\lambda_k(t-\tau))q^{(N+M)}(\tau)\,d\tau+$$

$$+\frac{1}{\varkappa_{N+M}}\sum_{k=N+1}^{N+M}\beta_k^2\lambda_k\int\limits_0^t \exp(-\lambda_k(t-\tau))q^{(N+M)}(\tau)\,d\tau+$$

$$+(\varphi^{(N+M)}(t) - \varphi^{(N)}(t)) \equiv I_1^{(N,M)}(t) + I_2^{(N,M)}(t) + I_3^{(N,M)}(t).$$

Since

$$\lim_{N\to\infty}\varkappa_N = \sum_{k=1}^{\infty}\beta_k^2 = |\sigma|,\ \ \varphi^{(N)}(t) = F'(t)/\varkappa_N,$$

we find using (2.35) that

$$\|I_1^{(N,M)}; C^{l+\delta}(0,T)\| \le c\Big|\frac{1}{\varkappa_{N+M}} - \frac{1}{\varkappa_N}\Big|\ \sum_{k=1}^{N}\beta_k^2\ \|q^{(N+M)};\ C^{l+\delta}(0,T)\| \le$$

$$\le c\,\Big|\frac{1}{\varkappa_{N+M}} - \frac{1}{\varkappa_N}\Big|\ \sum_{k=1}^{N}\beta_k^2\exp(2\lambda_1 T)\|\varphi^{(N+M)};\ C^{l+\delta}(0,T)\| \le$$

$$\le c\,\Big|\frac{1}{\varkappa_{N+M}} - \frac{1}{\varkappa_N}\Big|\ \exp(2\lambda_1 T)\|F';\ C^{l+\delta}(0,T)\| \to 0,$$

$$\|I_2^{(N,M)}; C^{l+\delta}(0,T)\| \le$$

$$\le \frac{c}{\varkappa_{N+M}}\exp(2\lambda_1 T)\|F';\ C^{l+\delta}(0,T)\|\sum_{k=N+1}^{N+M}\beta_k^2 \to 0,$$

$$\|I_3^{(N,M)}; C^{l+\delta}(0,T)\| = \Big|\frac{1}{\varkappa_{N+M}} - \frac{1}{\varkappa_N}\Big|\ \|F'; C^{l+\delta}(0,T)\| \to 0,$$

as $N, M \to \infty$. Since the function $\mathcal{Q}^{(N,M)}(t)$ is a solution to (2.17), by Lemma 2.4 (see the estimate (2.35)) we have

$$\|\mathcal{Q}^{(N,M)}; C^{l+\delta}(0,T)\| \le$$

$$\leq 2\frac{\varkappa_N}{\beta_1^2}\exp(2\lambda_1 T)\|(I_1^{(N,M)} + I_2^{(N,M)} + I_3^{(N,M)});\ C^{l+\delta}(0,T)\| \to 0$$

as $N, M \to \infty$. Thus, $\{q^N(t)\}$ is a Cauchy sequence in $\widehat{C}^{l+\delta}(0,T)$, and so there exists a limit function $q(t) \in \widehat{C}^{l+\delta}(0,T)$. Estimates (2.36) and (2.37) for $q(t)$ follow from the corresponding estimates for $q^N(t)$. The lemma is proved.

Now we shall prove the main result of the chapter.

Theorem 2.1. *Let* $\partial\sigma \in C^{2l+2+2\delta}$, $F(t) \in C^{l+1+\delta}(0,T)$ *and* $a(x) \in C^{2l+2+2\delta}(0,T)$ *with* $l \geq 0$, $\delta \in (0,\ 1/2)$. *Suppose the compatibility conditions of order* $l+1$

$$\big(\Delta^m a(x')\big)|_{\partial\sigma} = 0, \quad \frac{\mathrm{d}^m}{\mathrm{d}t^m}F(0) = \int\limits_\sigma \Delta^m a(x')dx, \quad m = 0,\ldots,l+1, \tag{2.38}$$

are valid. Then there exists a unique solution

$$(v(x',t), q(t)) \in C^{2l+2+2\delta, l+1+\delta}(\sigma \times (0,T)) \times \widehat{C}^{l+\delta}(0,T)$$

to problem (2.11)*, and*

$$\|v;\ C^{2l+2+2\delta,l+1+\delta}(0,T)\| + \|q;\ C^{l+\delta}(0,T)\| \leq c(T)\|F;\ C^{l+1+\delta}(0,T)\|. \tag{2.39}$$

Proof. Consider first the case $a(x') \equiv 0$. Suppose that $F \in \widehat{C}^{l+1+\delta}(0,T)$. By construction, the function

$$v^{(N)}(x',t) = \sum_{k=1}^N \beta_k \Bigg(\int\limits_0^t \exp(-\lambda_k(t-\tau))q^{(N)}(\tau)\,d\tau \Bigg) u_k(x')$$

solves the initial-boundary value problem:

$$\begin{cases} v_t^{(N)}(x',t) - \nu\Delta v^{(N)}(x',t) = f^{(N)}(x',t), \\ v^{(N)}(x',t)|_{\partial\sigma} = 0, \\ v^{(N)}(x',0) = 0, \end{cases} \tag{2.40}$$

where $f^{(N)}(x',t) = q^{(N)}(t)\sum\limits_{k=1}^N \beta_k u_k(x')$. Thus (see for instance [32], [75]),

$$\|v^{(N)};\ W_2^{1,1}(\sigma \times (0,T))\| \leq c\|f^{(N)};\ L_2(\sigma \times (0,T))\|, \tag{2.41}$$

and the constant c is independent of N. Using Lemma 2.5, it is easy to show that the sequence

$$\{f^{(N)}(x',t)\} = \left\{q^{(N)}(t)\sum_{k=1}^{N}\beta_k u_k(x')\right\}$$

converges in the norm of $L_2(\sigma\times(0,T))$ to $q(t)$. Since the problem (2.40) is linear, the difference

$$V^{(N,M)}(x',t) = v^{(N+M)}(x',t) - v^{(N)}(x',t)$$

is a solution to the analogous problem with the right-hand side $f^{(N+M)}(x',t) - f^{(N)}(x',t)$. By (2.41) we conclude that $v^{(N)}(t)$ is a Cauchy sequence in $W_2^{1,1}(\sigma\times(0,T))$, and so the limit

$$\lim_{N\to\infty} v^{(N)}(x',t) = v(x',t) \in W_2^{1,1}(\sigma\times(0,T))$$

exists, yielding a generalized solution to

$$\begin{cases} v_t(x',t) - \nu\Delta v(x',t) = q(t), \\ v(x',t)|_{\partial\sigma} = 0, \\ v(x',0) = 0. \end{cases} \tag{2.42}$$

The right-hand side $q(t)\in C^{l+\delta}(0,T)$ of (2.42) depends only on t. Thus, $q(t)\in C^{2l+2\delta,l+\delta}(\sigma\times(0,T))$, and

$$\|q;\ C^{2l+2\delta,l+\delta}(\sigma\times(0,T))\| \le c\|q;\ C^{l+\delta}(0,T)\|.$$

Furthermore, $q(t)$ satisfies the compatibility conditions

$$q(0) = \ldots = \frac{\mathrm{d}^l}{\mathrm{d}t^l}q(0) = 0.$$

Consequently (see [32], [75]), $v\in C^{2l+2+2\delta,l+1+\delta}(\sigma\times(0,T))$ and

$$\|v;\ C^{2l+2+2\delta,l+1+\delta}(\sigma\times(0,T))\| \le c\|q;\ C^{2l+2\delta,l+\delta}(\sigma\times(0,T))\| \le$$

$$\le c\|q;\ C^{l+\delta}(0,T)\| \le c\exp(2\lambda_1 T)\|F;\ C^{l+1+\delta}(0,T)\|.$$

By construction, for all N the functions $v^{(N)}(x',t)$ satisfy the flux condition

$$\int\limits_{\sigma} v^{(N)}(x',t)dx = F(t), \quad \forall t \in [0,T],$$

which holds for the limit function $v(x',t)$ as well. Therefore, $(v(x',t), q(t))$ is a solution to the inverse problem (2.11) for $a(x') \equiv 0$ and estimate (2.39) holds. Consider now the case of an arbitrary $a(x') \in C^{l+2+\delta}(\overline{\sigma})$. Look for a solution of the problem (2.11) in the form of the sum $(v(x',t), q(t)) = (v_1(x',t), 0) + (v_2(x',t), q(t))$, where $v_1(x',t)$ is a solution to the initial-boundary value problem

$$\begin{cases} v_{1t}(x',t) - \nu\Delta v_1(x',t) = 0, \\ v_1(x',t)|_{\partial\sigma} = 0, \\ v_1(x',0) = a(x'). \end{cases} \tag{2.43}$$

Since $a(x')$ satisfies the compatibility conditions of order $l+1$ (see (2.38)), there exists a unique solution $v_1(x',t)$ to (2.43), satisfying the estimate

$$\|v_1;\ C^{2l+2+2\delta,l+1+\delta}(\sigma\times(0,T))\| \le c\|a;\ C^{2l+2+2\delta}(\overline{\sigma})\|. \tag{2.44}$$

For $(v_2(x',t), q(t))$ we obtain the inverse problem

$$\begin{cases} v_{2t}(x',t) - \nu\Delta v_2(x',t) = q(t), \\ v_2(x',t)|_{\partial\sigma} = 0, \\ v_2(x',0) = 0, \\ \int\limits_{\sigma} v_2(x')dx' = F(t) - \int\limits_{\sigma} v_1(x',t)dx' := \widetilde{F}(t). \end{cases} \tag{2.45}$$

Using the second equalities in the compatibility conditions (2.38), we see that

$$\frac{\mathrm{d}^m}{\mathrm{d}t^m}\widetilde{F}(0) = 0,\ m = 0,\dots,l+1.$$

Moreover, $\widetilde{F}(t) \in \widehat{C}^{l+1+\delta}(0,T)$ and

$$\|\widetilde{F};\ \widehat{C}^{l+1+\delta}(0,T)\| \le c\big(\|F;\ \widehat{C}^{l+1+\delta}(0,T)\| + \|v_1;\ C^{2l+2+2\delta,l+1+\delta}(\sigma\times(0,T))\|\big) \le$$

$$\leq c\big(\|F;\ \widehat{C}^{l+1+\delta}(0,T)\| + \|a;\ C^{2l+2+2\delta}(\overline{\sigma})\|\big).$$

Therefore, there exists a solution $(v_2(x',t), q(t)) \in C^{2l+2+2\delta, l+1+\delta}(\sigma \times (0,T)) \times \widehat{C}^{l+\delta}(0,T)$ to (2.45), satisfying the estimate

$$\|v_2;\ C^{2l+2+2\delta,l+1+\delta}(\sigma\times(0,T))\| + \|q;\ C^{l+\delta}(0,T)\| \leq c(T)\|\widetilde{F};\ C^{l+1+\delta}(0,T)\| \leq$$

$$\leq c(T)\Bigg(\|F;\ \widehat{C}^{l+1+\delta}(0,T)\| + \|a;\ C^{2l+2+2\delta}(\overline{\sigma})\|\Bigg).$$

Thus, $(v(x',t), q(t))$, where $v(x') = v_1(x',t) + v_2(x',t)$, is a solution to (2.11), and estimate (2.39) holds.

Let us prove the uniqueness of the solution. Take $F(t) \equiv 0$ and $a(x') \equiv 0$. Multiply (2.11) by $v(x',t)$ and then integrating by parts in σ we get:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\sigma} |v(x',t)|^2\, dx' + \nu \int\limits_{\sigma} |\nabla v(x',t)|^2\, dx' = q(t)\int\limits_{\sigma} v(x',t) dx' = 0.$$

Integrate the last equality with respect to t,

$$\frac{1}{2}\int\limits_{\sigma} |v(x',t)|^2\, dx' + \nu \int\limits_{0}^{t}\int\limits_{\sigma} |\nabla v(x',\tau)|^2\, dx' d\tau = 0.$$

Hence, $v(x',t) = 0$, and from (2.11) we find that $q(t) = 0$. The theorem is proved.

Let us consider in the infinite cylinder $\Pi = \{x \in \mathbb{R}^n, n = 2,3 : x' = (x_1, .., x_{n-1}) \in \omega, x_n \in \mathbb{R}\}$ the Navier-Stokes equations with time periodic conditions, i. e. we consider the following problem:

$$\begin{cases} \mathbf{u}_t(x,t) - \nu\Delta\mathbf{u}(x,t) + (\mathbf{u}(x,t)\cdot\nabla\mathbf{u}(x,t)) + \nabla p(x,t) = \mathbf{f}(x,t), \\ \mathrm{div}\mathbf{u} = 0, \\ \mathbf{u}(x,0) = \mathbf{u}(x,T), \\ \mathbf{u}(x,0)|_{\partial\Pi} = \mathbf{a}(x). \end{cases} \tag{2.46}$$

We look for a solution of problem (2.46) having a prescribed time periodic flux

$F(t)$ through the cross-section σ:

$$\int\limits_{\sigma} u_n(x',t)dx' = F(t), \;\; F(0) = F(T). \qquad (2.47)$$

We assume that the initial data has the form $\mathbf{a} = (0, .., 0, a_n)$, where $a_n = a_n(x')$ does not depend on x_n. We also suppose that external force $\mathbf{f}$ has the form $\mathbf{f} = (0, .., 0, f_n(x',t))$, where $f_n(x',t)$ is a time periodic function: $f_n(x',0) = f_n(x',T)$.

The Poiseuille solution of problem (2.46), (2.47) has the form

$$\mathbf{U}(x,t) = (0, .., 0, \, U_n(x',t)), \qquad P(x,t) = -q(t)x_n. \qquad (2.48)$$

Substituting (2.4) into (2.1), we get for $v(x',t)$ and $q(t)$ the following problem for the heat equation

$$\begin{cases} U_{nt}(x',t) - \nu\Delta' U_n(x',t) = q(t) + f_n(x',t), \\ U_n(x',t)\big|_{\partial\sigma} = 0, \\ U_n(x',0) = U_n(x',2\pi), \\ \int\limits_{\sigma} U_n(x',t)dx' = F(t). \end{cases} \qquad (2.49)$$

Theorem 2.2. *There exists the time periodic Poiseuille solution $\big(\mathbf{U}(x), P(x,t)\big)$, i. e. the exact solution of the time periodic Stokes (and Navier–Stokes) problem (2.46), having the form (2.48) and satisfying the flux condition*

$$\int\limits_{\sigma} U_n(x',t)dx' = F(t).$$

The pair of functions $\big(U_n(x',t), \, q(t)\big)$ is the solution of problem (2.49). For any periodic $F(t)$ the problem (2.49) admits a unique time periodic solution $\big(U_n(x',t), \, q(t)\big) \in \big(W_2^{2,1}(\sigma \times (0,2\pi)) \times L_2(0,2\pi)\big)$. Moreover, there holds the estimate

$$\|U_n; \, W_2^{2,1}(\sigma \times (0,2\pi))\|^2 + \|q; L_2(0,2\pi)\|^2 \le c\|F; W_2^1(0,2\pi)\|^2. \qquad (2.50)$$

Theorem 2.2 is proved in [8], [19].
The results obtained in this chapter is presented in [A1].

3

Time periodic Stokes problem

3.1 Formulation of the problem

In this chapter we study in the domain Ω with cylindrical outlets to infinity the linear time periodic [1] Stokes problem:

$$\begin{cases} \mathbf{u}_t(x,t) - \nu\Delta\mathbf{u}(x,t) + \nabla p(x,t) = \mathbf{f}(x,t), \\ \mathrm{div}\mathbf{u}(x,t) = 0, \\ \mathbf{u}(x,t)\big|_{\partial\Omega} = 0, \\ \mathbf{u}(x,0) = \mathbf{u}(x,2\pi), \\ \int\limits_{\sigma_j} \mathbf{u}(x,t)\cdot\mathbf{n}(x)\,ds = F_j(t), \qquad j = 1,\dots,J, \end{cases} \tag{3.1}$$

with prescribed fluxes $F_j(t)$ over cross–sections σ_j of outlets to infinity. We assume that the external force $\mathbf{f}(x,t)$ and the fluxes $F_j(t)$ are time periodic functions with the period 2π:

$$\mathbf{f}(x,0) = \mathbf{f}(x,2\pi), \quad F_j(0) = F_j(2\pi), \;\; j = 1,\dots,J.$$

[1] without loss of generality we assume the period is equal to 2π

Moreover, we suppose that there holds the necessary compatibility condition

$$\sum_{j=1}^{J} F_j(t) = 0, \qquad \forall t \in [0, 2\pi].$$

We shall prove that problem (3.1) admits a unique solution $(\mathbf{u}, \, p)$ that tends in each outlet to infinity to the time periodic Poiseuille flow corresponding to the cylinder Π_j and the flux $F_j(t)$.

3.2 Solvability of problem (3.1) with zero fluxes

First we consider in $\Omega \times (0, 2\pi)$ the time periodic Stokes problem assuming that all fluxes $F_j(t), j = 1, ..., J$, are equal to zero, i. e., consider the problem

$$\begin{cases} \mathbf{u}_t(x,t) - \nu\Delta\mathbf{u}(x,t) + \nabla p(x,t) = \mathbf{f}(x,t), \\ \mathrm{div}\mathbf{u}(x,t) = 0, \\ \mathbf{u}(x,t)|_{\partial\Omega} = 0, \\ \mathbf{u}(x,0) = \mathbf{u}(x,2\pi), \\ \int\limits_{\sigma_j} \mathbf{u}(x,t)\cdot\mathbf{n}(x)ds = 0, \ \ j = 1, ..., J. \end{cases} \tag{3.2}$$

Definition. *By the weak solution of problem (3.2) we call time periodic function* $\mathbf{u} \in \overset{\circ}{W}{}_2^{1,1}(\Omega \times (0, 2\pi))$ *satisfying the condition*

$$\mathrm{div}\mathbf{u}(x,t) = 0,$$

and the integral identity

$$\int\limits_0^{2\pi}\int\limits_\Omega \mathbf{u}_t(x,t)\cdot\boldsymbol{\eta}(x,t)dxdt + \nu\int\limits_0^{2\pi}\int\limits_\Omega \nabla\mathbf{u}(x,t)\cdot\nabla\boldsymbol{\eta}(x,t)dxdt$$

$$= \int_0^{2\pi} \int_\Omega \mathbf{f}(x,t) \cdot \boldsymbol{\eta}(x,t) dx dt, \tag{3.3}$$

for every divergence free time periodic $\boldsymbol{\eta}(x,t) \in \mathring{W}_2^{1,0}(\Omega \times (0, 2\pi))$.

Note that each divergence free vector-field $\mathbf{u}(x,t)$ equal to zero on $\partial\Omega$ and such $\nabla\mathbf{u}(\cdot, t) \in L_2(\Omega)$ has zero fluxes over all cross-sections σ_j, i. e.

$$\int_{\sigma_j} \mathbf{u}(x,t) \cdot \mathbf{n}(x) dS = 0, \quad j = 1, ..., J.$$

First, let us construct an approximate solution to problem (3.2). The right–hand side $\mathbf{f}(x,t)$ is time periodic and $\mathbf{f}(x,t) \in L_2(\Omega \times (0, 2\pi))$. Therefore $\mathbf{f}(x,t)$ may be represented as the Fourier series

$$\mathbf{f}(x,t) = \sum_{n=0}^{\infty} \Big(\mathbf{f}_{sn}(x) \sin(nt) + \mathbf{f}_{cn}(x) \cos(nt) \Big),$$

where $\mathbf{f}_{sn}(x),\ \mathbf{f}_{cn}(x) \in L_2(\Omega),\ n = 0, 1, 2, \dots .$

We look for an approximate solution $\mathbf{u}_N(x,t)$ of problem (3.2) in the form

$$\mathbf{u}_N(x,t) = \sum_{n=0}^{N} \Big(\mathbf{a}_n(x) \sin(nt) + \mathbf{b_n}(x) \cos(nt) \Big), \tag{3.4}$$

where the coefficients $\big(\mathbf{a}_n,\ \mathbf{b}_n\big) \in \mathring{W}_2^1(\Omega)$ are found as divergence free solutions to the system of the integral identities

$$\begin{cases} -n \int_\Omega \mathbf{b}_n(x) \cdot \boldsymbol{\eta}(x) dx + \nu \int_\Omega \nabla \mathbf{a}_n(x) \cdot \nabla \boldsymbol{\eta}(x) dx = \int_\Omega \mathbf{f}_{sn}(x) \cdot \boldsymbol{\eta}(x) dx, \\ \\ n \int_\Omega \mathbf{a}_n(x) \cdot \boldsymbol{\xi}(x) dx + \nu \int_\Omega \nabla \mathbf{b}_n(x) \cdot \nabla \boldsymbol{\xi}(x) dx = \int_\Omega \mathbf{f}_{cn}(x) \cdot \boldsymbol{\xi}(x) dx, \end{cases} \tag{3.5}$$

where $\boldsymbol{\eta}(x)$ and $\boldsymbol{\xi}(x)$ are arbitrary divergence free functions in $\mathring{W}_2^1(\Omega)$, i. e., $\big(\mathbf{a}_n(x),\ \mathbf{b}_n(x)\big), n = 0, 1, 2, \dots,$ are weak solutions of the following stationary

"Stokes type" problems

$$\begin{cases} -n\mathbf{b}_n(x) - \nu\Delta\mathbf{a}_n(x) + \nabla p_{sn}(x) = \mathbf{f}_{sn}(x), \\ \\ n\mathbf{a}_n(x) - \nu\Delta\mathbf{b}_n(x) + \nabla p_{cn}(x) = \mathbf{f}_{cn}(x), \\ \\ \mathrm{div}\mathbf{a}_n(x) = 0, \quad \mathrm{div}\mathbf{b}_n(x) = 0, \\ \\ \mathbf{a}_n(x)\Big|_{\partial\Omega} = 0, \quad \mathbf{b}_n(x)\Big|_{\partial\Omega} = 0. \end{cases} \tag{3.6}$$

The solution of problem (3.6) is unique. Indeed, let $\mathbf{f}_{sn}(x) = \mathbf{f}_{cn}(x) = 0$. Take in (3.5_1) $\boldsymbol{\eta}(x) = \mathbf{a}_n(x)$ and in (3.5_2) $\boldsymbol{\xi}(x) = \mathbf{b}_n(x)$. Summing the obtained inequalities, we derive

$$\nu\int\limits_{\Omega}|\nabla\mathbf{a}_n^k(x)|^2dx + \nu\int\limits_{\Omega}|\nabla\mathbf{b}_n^k(x)|^2dx = 0$$

and, hence, $\mathbf{a}_n^k(x) = \mathbf{b}_n^k(x) = 0$. Moreover, it is straightforward to verify the following a priori estimate

$$\|\mathbf{a}_n; W_2^1(\Omega)\|^2 + \|\mathbf{b}_n; W_2^1(\Omega)\|^2$$
$$\leq c\Big(\|\mathbf{f}_{sn}; L_2(\Omega)\|^2 + \|\mathbf{f}_{cn}; L_2(\Omega)\|^2\Big) \ \ \forall n = 0, 1, ... \tag{3.7}$$

with the constant c independent of n. Therefore, using standard for elliptic problems arguments(e. g. [6]), we conclude

Lemma 3.1. *Let* $\mathbf{f}_{cn}(x), \mathbf{f}_{sn}(x) \in L_2(\Omega)$. *Then problem* (3.5) *admits a unique weak solution* $\big(\mathbf{a}_n(x), \mathbf{b}_n(x)\big) \in \mathring{W}_2^1(\Omega)$ *satisfying the estimate* (3.7).

Let us prove that the approximate solutions $\mathbf{u}_N(x,t)$ defined by (3.4) converge to a weak solution of problem (3.2).

Theorem 3.1. *Let* $\mathbf{f}(x,t) \in L_2(\Omega\times(0,2\pi))$ *be time periodic function. Then problem (3.2) admits a unique time periodic weak solution* $\mathbf{u}(x,t) \in W_2^{1,1}(\Omega\times(0,2\pi))$ *and there holds the estimate*

$$\sup_{t\in[0,2\pi]}\|\mathbf{u}(\cdot,t); \mathring{W}_2^1(\Omega)\|^2 + \|\mathbf{u}; W_2^{1,1}(\Omega\times(0,2\pi))\|^2$$

$$\leq c\|\mathbf{f}; L_2(\Omega\times(0,2\pi))\|^2. \qquad (3.8)$$

Proof. From the definition of $\mathbf{u}_N(x,t)$ and $\mathbf{a}_n(x)$, $\mathbf{b}_n(x)$ (see (3.4) and (3.5)) it follows that there holds the integral identity

$$\int\limits_\Omega \mathbf{u}_{Nt}(x,t)\cdot\boldsymbol{\eta}(x)dx+\nu\int\limits_\Omega \nabla\mathbf{u}_N(x,t)\cdot\nabla\boldsymbol{\eta}(x)dx=\int\limits_\Omega \mathbf{f}_N(x,t)\cdot\boldsymbol{\eta}(x)dx, \quad (3.9)$$

where

$$\mathbf{f}_N(x,t)=\sum_{n=0}^{N}\Big(\mathbf{f}_{sn}(x)\sin(nt)+\mathbf{f}_{cn}(x)\cos(nt)\Big)$$

and $\boldsymbol{\eta}(x)$ is an arbitrary divergence free function in $\overset{\circ}{W}{}^1_2(\Omega)$. In order to prove (3.9), it is sufficient to take in (3.5) $\boldsymbol{\eta}(x)=\boldsymbol{\xi}(x)$, to multiply (3.5_1) by $\sin(nt)$, (3.5_2) by $\cos(nt)$ and to sum the obtained relations. Taking in (3.9) $\boldsymbol{\eta}=\mathbf{u}_N(x,t)$ and using Young and Poincaré inequalities we derive

$$\frac{1}{2}\frac{d}{dt}\int\limits_\Omega |\mathbf{u}_N(x,t)|^2dx+\frac{\nu}{2}\int\limits_\Omega |\nabla\mathbf{u}_N(x,t)|^2dx\leq c\int\limits_\Omega |\mathbf{f}_N(x,t)|^2dx. \qquad (3.10)$$

Integrating (3.10) with respect to t over the interval $(0,2\pi)$, we get, in view of the periodicity of $\mathbf{u}_N(x,t)$, that

$$\int\limits_0^{2\pi}\int\limits_\Omega |\nabla\mathbf{u}_N(x,t)|^2dxdt\leq c\int\limits_0^{2\pi}\int\limits_\Omega |\mathbf{f}_N(x,t)|^2dxdt. \qquad (3.11)$$

Let us take now in (3.9) $\boldsymbol{\eta}=t\,\mathbf{u}_N(x,t)$:

$$\frac{1}{2}\frac{d}{dt}\int\limits_\Omega t|\mathbf{u}_N(x,t)|^2dx+\nu t\int\limits_\Omega |\nabla\mathbf{u}_N(x,t)|^2dx=t\int\limits_\Omega \mathbf{f}_N(x,t)\cdot\mathbf{u}_N(x,t)dx$$

$$+\frac{1}{2}\int\limits_\Omega |\mathbf{u}_N(x,t)|^2dx\leq\frac{t^2}{2}\int\limits_\Omega |\mathbf{f}_N(x,t)|^2dx+\int\limits_\Omega |\mathbf{u}_N(x,t)|^2dx.$$

Integrating the last inequality over the interval $(0,2\pi)$, we get using Poincaré inequality and (3.11)

$$\pi\int\limits_\Omega |\mathbf{u}_N(x,2\pi)|^2dx\leq 2\pi^2\int\limits_0^{2\pi}\int\limits_\Omega |\mathbf{f}_N(x,t)|^2dxdt+\int\limits_0^{2\pi}\int\limits_\Omega |\mathbf{u}_N(x,t)|^2dxdt$$

$$\leq 2\pi^2 \int\limits_0^{2\pi} \int\limits_\Omega |\mathbf{f}_N(x,t)|^2 dxdt + c \int\limits_0^{2\pi} \int\limits_\Omega |\nabla \mathbf{u}_N(x,t)|^2 dxdt \leq c \int\limits_0^{2\pi} \int\limits_\Omega |\mathbf{f}_N(x,t)|^2 dxdt$$

and, since $\mathbf{u}_N(x,0) = \mathbf{u}_N(x,2\pi)$, we have

$$\int\limits_\Omega |\mathbf{u}_N(x,0)|^2 dx \leq c \int\limits_0^{2\pi} \int\limits_\Omega |\mathbf{f}_N(x,t)|^2 dxdt.$$

Let us take in (3.9) $\boldsymbol{\eta} = t\mathbf{u}_{Nt}(x,t)$:

$$t \int\limits_\Omega |\mathbf{u}_{Nt}(x,t)|^2 dx + \frac{\nu}{2} \frac{d}{dt} \int\limits_\Omega t |\nabla \mathbf{u}_N(x,t)|^2 dx = t \int\limits_\Omega \mathbf{f}_N(x,t) \cdot \mathbf{u}_{Nt}(x,t) dx$$

$$+ \frac{\nu}{2} \int\limits_\Omega |\nabla \mathbf{u}_N(x,t)|^2 dx \leq \frac{t}{2} \int\limits_\Omega |\mathbf{f}_N(x,t)|^2 dxdt + \frac{t}{2} \int\limits_\Omega |\mathbf{u}_{Nt}(x,t)|^2 dxdt$$

$$+ \frac{\nu}{2} \int\limits_\Omega |\nabla \mathbf{u}_N(x,t)|^2 dx.$$

Therefore,

$$\frac{\nu}{2} \frac{d}{dt} \int\limits_\Omega t |\nabla \mathbf{u}_N(x,t)|^2 dx \leq \frac{t}{2} \int\limits_\Omega |\mathbf{f}_N(x,t)|^2 dxdt + \frac{\nu}{2} \int\limits_\Omega |\nabla \mathbf{u}_N(x,t)|^2 dxdt.$$

Integrating over $(0,2\pi)$, using the periodicity condition $\nabla \mathbf{u}_N(x,0) = \nabla \mathbf{u}_N(x,2\pi)$ and (3.11), we conclude

$$\int\limits_\Omega |\nabla \mathbf{u}_N(x,0)|^2 dx \leq c \int\limits_0^{2\pi} \int\limits_\Omega |\mathbf{f}_N(x,t)|^2 dxdt.$$

Thus, $\mathbf{u}_N(\cdot,0) \in \mathring{W}_2^1(\Omega)$ and

$$\|\mathbf{u}_N(\cdot,0);\, \mathring{W}_2^1(\Omega)\|^2 \leq c \int\limits_0^{2\pi} \|\mathbf{f}_N(\cdot,t);\, L_2(\Omega)\|^2 dt$$

$$\leq c\int\limits_0^{2\pi}\|\mathbf{f}(\cdot,t);\,L_2(\Omega)\|^2dt. \tag{3.12}$$

Now arguing in the usual way (as in the case of initial–boundary value problem, e. g. [27]), we derive the estimate

$$\sup_{t\in[0,2\pi]}\|\mathbf{u}_N(\cdot,t);\,\mathring{W}_2^1(\Omega)\|^2+\int\limits_0^{2\pi}\|\mathbf{u}_{Nt}(\cdot,t);\,\mathring{W}_2^1(\Omega)\|^2dt\leq c\Big(\|\mathbf{u}_N(\cdot,0);\,\mathring{W}_2^1(\Omega)\|^2$$

$$+\int\limits_0^{2\pi}\|\mathbf{f}_N(\cdot,t);\,L_2(\Omega)\|^2dt\Big)\leq c\int\limits_0^{2\pi}\|\mathbf{f}(\cdot,t);\,L_2(\Omega)\|^2dt. \tag{3.13}$$

Thus, the sequence $\big\{\mathbf{u}_N(x,t)\big\}$ is bounded in the norm of the space $W_2^{1,1}(\Omega\times(0,2\pi))$ and there exists a weakly convergent in $W_2^{1,1}(\Omega\times(0,2\pi))$ subsequence $\big\{\mathbf{u}_{N_l}(x,t)\big\}$. It is easy to conclude from (3.9) that every $\mathbf{u}_{N_l}(x,t)$ satisfies the integral identity

$$\int\limits_0^{2\pi}\int\limits_\Omega\mathbf{u}_{N_lt}(x,t)\cdot\boldsymbol{\eta}(x,t)dxdt+\int\limits_0^{2\pi}\int\limits_\Omega\nabla\mathbf{u}_{N_l}(x,t)\cdot\nabla\boldsymbol{\eta}(x,t)dxdt$$

$$=\int\limits_0^{2\pi}\int\limits_\Omega\mathbf{f}_{N_l}(x,t)\cdot\boldsymbol{\eta}(x,t)dxdt \tag{3.14}$$

with any $\boldsymbol{\eta}(x,t)$ having the form

$$\boldsymbol{\eta}(x,t)=\sum_{n=0}^{L}\Big(\boldsymbol{\eta}_{sn}(x)\sin(nt)+\boldsymbol{\eta}_{cn}(x)\cos(nt)\Big). \tag{3.15}$$

In (3.15) the functions $\boldsymbol{\eta}_{sn}(x),\ \boldsymbol{\eta}_{cn}(x)\in\mathring{W}_2^1(\Omega),\ n=0,1,2,\dots$ are divergence free. Passing $N_l\to\infty$ in (3.14), we find that the limit function $\mathbf{u}(x,t)$ satisfies the integral identity (3.3) for every $\boldsymbol{\eta}(x,t)$ having the representation (3.15). Any time periodic divergence free function belonging to the space $\mathring{W}_2^{1,0}(\Omega\times(0,2\pi))$ could be decomposed in a Fourier series with coefficients in $\mathring{W}_2^1(\Omega)$. Therefore, the sums (3.15) are dense in the space of time periodic divergence free functions from $\mathring{W}_2^{1,0}(\Omega\times(0,2\pi))$. Hence, $\mathbf{u}(x,t)$ satisfies the integral identity (3.3) for any such function $\boldsymbol{\eta}(x,t)$. Obviously, for the limit function $\mathbf{u}(x,t)$ the estimate (3.13) remains valid. The theorem is proved.

3.3 Estimates of the solution in weighted function spaces

In this section we shall prove that the solution of problem (3.2) belongs to the weighted function space provided that the right–hand side $\mathbf{f}(x,t)$ belongs to the corresponding weighted space.

Theorem 3.2. *Let* $\mathbf{f}(x,t) \in \mathcal{L}_{2,\boldsymbol{\beta}}(\Omega \times (0,2\pi)), \beta_j \geq 0, j = 1,...,J$, *be time periodic function. If the number* γ_* *in inequality* (1.3_3) *is sufficiently small, then the time periodic weak solution* $\mathbf{u}(x,t)$ *of the problem (3.2) belongs to the space* $\mathcal{W}^{1,1}_{2,\boldsymbol{\beta}}(\Omega \times (0,2\pi))$ *and there holds the estimate*

$$\sup_{t\in[0,2\pi]} \|\mathbf{u}(\cdot,t); \mathcal{W}^{1}_{2,\boldsymbol{\beta}}(\Omega)\|^2 + \|\mathbf{u}; \mathcal{W}^{1,1}_{2,\boldsymbol{\beta}}(\Omega \times (0,2\pi))\|^2$$

$$\leq c\|\mathbf{f}; \mathcal{L}_{2,\boldsymbol{\beta}}(\Omega \times (0,2\pi))\|^2. \tag{3.16}$$

Moreover, if $\partial\Omega \in C^2$*, then* $\mathbf{u} \in \mathcal{W}^{2,1}_{2,\boldsymbol{\beta}}(\Omega \times (0,2\pi)),\ \nabla p \in \mathcal{L}_{2,\boldsymbol{\beta}}(\Omega \times (0,2\pi))$ *and there holds the estimate*

$$\|\mathbf{u}; \mathcal{W}^{2,1}_{2,\boldsymbol{\beta}}(\Omega \times (0,2\pi))\|^2 + \|\nabla p; \mathcal{L}_{2,\boldsymbol{\beta}}(\Omega \times (0,2\pi))\|^2$$

$$\leq c\|\mathbf{f}; \mathcal{L}_{2,\boldsymbol{\beta}}(\Omega \times (0,2\pi))\|^2. \tag{3.17}$$

Proof. Let us take in the integral identity (3.3) $\boldsymbol{\eta}(x,t) = E^{(k)}_{\boldsymbol{\beta}}(x)\mathbf{u}(x,t) + \mathbf{W}^{(k)}(x,t)$, where $E^{(k)}_{\boldsymbol{\beta}}(x)$ is the "step" weight function (1.5) and $\mathbf{W}^{(k)}(x,t)$ is the vector-field from Lemma 1.7. This gives

$$\int_0^{2\pi}\int_\Omega E^{(k)}_{\boldsymbol{\beta}}(x)\mathbf{u}_t(x,t)\cdot\mathbf{u}(x,t)dxdt + \int_0^{2\pi}\int_\Omega \mathbf{u}_t(x,t)\cdot\mathbf{W}^{(k)}(x,t)dxdt$$

$$+\nu\int_0^{2\pi}\int_\Omega \nabla\mathbf{u}(x,t)\cdot\nabla\big(E^{(k)}_{\boldsymbol{\beta}}(x)\mathbf{u}(x,t) + \mathbf{W}^{(k)}(x,t)\big)dxdt$$

$$= \int_0^{2\pi}\int_\Omega \mathbf{f}(x,t)\cdot\big(E^{(k)}_{\boldsymbol\beta}(x)\mathbf{u}(x,t)+\mathbf{W}^{(k)}(x,t)\big)dxdt \tag{3.18}$$

(note that $\boldsymbol\eta\in \mathring{W}^{1,0}_2(\Omega\times(0,2\pi))$ and $\operatorname{div}\boldsymbol\eta(x,t)=0,\ \ \forall t\in[0,2\pi]$). Since

$$\int_0^{2\pi}\int_\Omega E^{(k)}_{\boldsymbol\beta}(x)\mathbf{u}_t(x,t)\cdot\mathbf{u}(x,t)dxdt=\frac12\int_\Omega E^{(k)}_{\boldsymbol\beta}(x)(|\mathbf{u}(x,2\pi)|^2-|\mathbf{u}(x,0)|^2)dx=0,$$

the relation (3.18) takes the form

$$\nu\int_0^{2\pi}\int_\Omega E^{(k)}_{\boldsymbol\beta}(x)|\nabla\mathbf{u}(x,t)|^2dxdt=-\int_0^{2\pi}\int_\Omega \mathbf{u}_t(x,t)\cdot\mathbf{W}^{(k)}(x,t)dxdt$$

$$-\nu\int_0^{2\pi}\int_\Omega \nabla\mathbf{u}(x,t)\cdot\big(\nabla E^{(k)}_{\boldsymbol\beta}(x)\mathbf{u}(x,t)+\nabla\mathbf{W}^{(k)}(x,t)\big)dxdt$$

$$+\int_0^{2\pi}\int_\Omega \mathbf{f}(x,t)\cdot(E^{(k)}_{\boldsymbol\beta}(x)\mathbf{u}(x,t)+\mathbf{W}^{(k)}(x,t)\big)dxdt=I_1+I_2+I_3. \tag{3.19}$$

We estimate the term I_1 using the Hölder inequality, weighted Poincaré inequality (1.12) and properties of the function $\mathbf{W}^{(k)}(x,t)$ (see (1.13) and (1.14)):

$$|I_1|\le c\Big(\int_0^{2\pi}\int_\Omega E^{(k)}_{-\boldsymbol\beta}(x)|\mathbf{W}^{(k)}(x,t)|^2dxdt\Big)^{1/2}\Big(\int_0^{2\pi}\int_\Omega E^{(k)}_{\boldsymbol\beta}(x)|\mathbf{u}_t(x,t)|^2dxdt\Big)^{1/2}$$

$$\le c\Big(\int_0^{2\pi}\int_\Omega E^{(k)}_{-\boldsymbol\beta}(x)|\nabla\mathbf{W}^{(k)}(x,t)|^2dxdt\Big)^{1/2}\Big(\int_0^{2\pi}\int_\Omega E^{(k)}_{\boldsymbol\beta}(x)|\mathbf{u}_t(x,t)|^2dxdt\Big)^{1/2}$$

$$\le c_1\gamma_*\Big[\int_0^{2\pi}\int_\Omega E^{(k)}_{\boldsymbol\beta}(x)|\nabla\mathbf{u}(x,t)|^2dxdt+\int_0^{2\pi}\int_\Omega E^{(k)}_{\boldsymbol\beta}(x)|\mathbf{u}_t(x,t)|^2dxdt\Big]. \tag{3.20}$$

Arguing analogously and using the property (1.3_3) of the weight-function $E^{(k)}_{\boldsymbol\beta}(x)$, we get

$$|I_2| \le c\gamma_* \Big(\int_0^{2\pi} \int_\Omega E_{\boldsymbol\beta}^{(k)}(x) |\nabla \mathbf{u}(x,t)|^2 dx dt \Big)^{1/2} \Big(\int_0^{2\pi} \int_\Omega E_{\boldsymbol\beta}^{(k)}(x) |\mathbf{u}(x,t)|^2 dx dt \Big)^{1/2}$$

$$+c \Big(\int_0^{2\pi} \int_\Omega E_{-\boldsymbol\beta}^{(k)}(x) |\nabla \mathbf{W}^{(k)}(x,t)|^2 dx dt \Big)^{1/2} \Big(\int_0^{2\pi} \int_\Omega E_{\boldsymbol\beta}^{(k)}(x) |\nabla \mathbf{u}(x,t)|^2 dx dt \Big)^{1/2}$$

$$\le c_2\gamma_* \int_0^{2\pi} \int_\Omega E_{\boldsymbol\beta}^{(k)}(x) |\nabla \mathbf{u}(x,t)|^2 dx dt. \qquad (3.21)$$

Finally,

$$|I_3| \le c_\varepsilon \int_0^{2\pi} \int_\Omega E_{\boldsymbol\beta}^{(k)}(x) |\mathbf{f}(x,t)|^2 dx dt + \varepsilon \Big(\int_0^{2\pi} \int_\Omega E_{\boldsymbol\beta}^{(k)}(x) |\mathbf{u}(x,t)|^2 dx dt$$

$$+ \int_0^{2\pi} \int_\Omega E_{-\boldsymbol\beta}^{(k)}(x) |\mathbf{W}^{(k)}(x,t)|^2 dx dt \Big)$$

$$\le c_\varepsilon \int_0^{2\pi} \int_\Omega E_{\boldsymbol\beta}^{(k)}(x) |\mathbf{f}(x,t)|^2 dx dt + c_3\varepsilon \int_0^{2\pi} \int_\Omega E_{\boldsymbol\beta}^{(k)}(x) |\nabla \mathbf{u}(x,t)|^2 dx dt. \qquad (3.22)$$

From (3.19)–(3.22) it follows that

$$\nu \int_0^{2\pi} \int_\Omega E_{\boldsymbol\beta}^{(k)}(x) |\mathbf{u}(x,t)|^2 dx dt \le c_4(\varepsilon + \gamma_*) \int_0^{2\pi} \int_\Omega E_{\boldsymbol\beta}^{(k)}(x) |\nabla \mathbf{u}(x,t)|^2 dx dt$$

$$+c_1\gamma_* \int_0^{2\pi} \int_\Omega E_{\boldsymbol\beta}^{(k)}(x) |\mathbf{u}_t(x,t)|^2 dx dt + c_\varepsilon \int_0^{2\pi} \int_\Omega E_{\boldsymbol\beta}^{(k)}(x) |\mathbf{f}(x,t)|^2 dx dt. \qquad (3.23)$$

Next we take in the integral identity (3.3) $\boldsymbol\eta(x,t) = E_{\boldsymbol\beta}^{(k)}(x)\mathbf{u}_t(x,t) + \mathbf{W}_t^{(k)}(x,t)$.

This gives

$$\int_0^{2\pi}\int_\Omega E_{\boldsymbol\beta}^{(k)}(x)|\mathbf{u}_t(x,t)|^2dxdt+\frac{\nu}{2}\int_\Omega E_{\boldsymbol\beta}^{(k)}(x)(|\nabla\mathbf{u}(x,2\pi)|^2-|\nabla\mathbf{u}(x,0)|^2)dx$$

$$=-\int_0^{2\pi}\int_\Omega \mathbf{u}_t(x,t)\cdot\mathbf{W}_t^{(k)}(x,t)dxdt$$

$$-\nu\int_0^{2\pi}\int_\Omega \nabla\mathbf{u}(x,t)\cdot\big(\nabla E_{\boldsymbol\beta}^{(k)}(x)\mathbf{u}_t(x,t)+\nabla\mathbf{W}_t^{(k)}(x,t)\big)dxdt$$

$$+\int_0^{2\pi}\int_\Omega \mathbf{f}(x,t)\cdot\big(\nabla E_{\boldsymbol\beta}^{(k)}(x)\mathbf{u}_t(x,t)+\mathbf{W}_t^{(k)}(x,t)\big)dxdt \tag{3.24}$$

Because of the periodicity condition the second term in the left–hand side of (3.24) is equal to zero. Estimating the right-hand side of (3.24) as above and using (1.14), we derive

$$\int_0^{2\pi}\int_\Omega E_{\boldsymbol\beta}^{(k)}(x)|\mathbf{u}_t(x,t)|^2dxdt\le c_5(\varepsilon+\gamma_*)\int_0^{2\pi}\int_\Omega E_{\boldsymbol\beta}^{(k)}(x)|\mathbf{u}_t(x,t)|^2dxdt$$

$$+c_6\gamma_*\int_0^{2\pi}\int_\Omega E_{\boldsymbol\beta}^{(k)}(x)|\nabla\mathbf{u}(x,t)|^2dxdt+c_\varepsilon\int_0^{2\pi}\int_\Omega E_{\boldsymbol\beta}^{(k)}(x)|\mathbf{f}(x,t)|^2dxdt. \tag{3.25}$$

Summing the inequalities (3.23) and (3.25) yields

$$\int_0^{2\pi}\int_\Omega E_{\boldsymbol\beta}^{(k)}(x)\big(|\mathbf{u}_t(x,t)|^2+|\nabla\mathbf{u}(x,t)|^2dxdt$$

$$\le c_7(\varepsilon+\gamma_*)\int_0^{2\pi}\int_\Omega E_{\boldsymbol\beta}^{(k)}(x)(|\mathbf{u}_t(x,t)|^2+|\nabla\mathbf{u}(x,t)|^2)dxdt$$

$$+c_\varepsilon\int_0^{2\pi}\int_\Omega E_{\boldsymbol\beta}^{(k)}(x)|\mathbf{f}(x,t)|^2dxdt. \tag{3.26}$$

Let us take $\varepsilon = \gamma_*$. If the number γ_* is sufficiently small, i. e. if

$$\gamma_* \le \frac{1}{4c_7}\min\{\nu, 1\},$$

then from (3.26) it follows that

$$\begin{aligned} &\|\mathbf{u}_t; \mathcal{L}_{2,\boldsymbol{\beta}}(\Omega \times (0,2\pi))\|^2 + \|\nabla \mathbf{u}; \mathcal{L}_{2,\boldsymbol{\beta}}(\Omega \times (0,2\pi))\|^2 \\ &\qquad \le c\|\mathbf{f}; \mathcal{L}_{2,\boldsymbol{\beta}}(\Omega \times (0,2\pi))\|^2. \end{aligned} \tag{3.27}$$

Now, let us take in the integral identity (3.3) $\boldsymbol{\eta}(x,t) = tE^{(k)}_{\boldsymbol{\beta}}(x)\mathbf{u}(x,t) + t\mathbf{W}^{(k)}(x,t)$. Since

$$\int\limits_0^{2\pi}\int\limits_\Omega tE^{(k)}_{\boldsymbol{\beta}}(x)\mathbf{u}_t(x,t)\cdot\mathbf{u}(x,t)dxdt = \pi\int\limits_\Omega E^{(k)}_{\boldsymbol{\beta}}(x)|\mathbf{u}(x,2\pi)|^2dx$$

$$-\frac{1}{2}\int\limits_0^{2\pi}\int\limits_\Omega E^{(k)}_{\boldsymbol{\beta}}(x)|\mathbf{u}(x,t)|^2dxdt,$$

arguing as above and using weighted Poincaré inequality (1.12), we derive

$$\pi\int\limits_\Omega E^{(k)}_{\boldsymbol{\beta}}(x)|\mathbf{u}(x,2\pi)|^2dx \le c\int\limits_0^{2\pi}\int\limits_\Omega E^{(k)}_{\boldsymbol{\beta}}(x)\big(|\mathbf{u}_t(x,t)|^2 + |\nabla\mathbf{u}(x,t)|^2\big)dxdt$$

$$+\int\limits_0^{2\pi}\int\limits_\Omega E^{(k)}_{\boldsymbol{\beta}}(x)|\mathbf{f}(x,t)|^2dxdt$$

and, from (3.27) we conclude

$$\begin{aligned} \int\limits_\Omega E^{(k)}_{\boldsymbol{\beta}}(x)|\mathbf{u}(x,0)|^2dx &= \int\limits_\Omega E^{(k)}_{\boldsymbol{\beta}}(x)|\mathbf{u}(x,2\pi)|^2dx \\ &\le c\int\limits_0^{2\pi}\int\limits_\Omega E^{(k)}_{\boldsymbol{\beta}}(x)|\mathbf{f}(x,t)|^2dxdt. \end{aligned} \tag{3.28}$$

Analogously, taking in (3.3) $\boldsymbol{\eta}(x,t) = tE^{(k)}_{\boldsymbol{\beta}}(x)\mathbf{u}_t(x,t) + t\mathbf{W}^{(k)}_t(x,t)$, we get

$$\int\limits_\Omega E^{(k)}_{\boldsymbol{\beta}}(x)|\nabla \mathbf{u}(x,0)|^2dx = \int\limits_\Omega E^{(k)}_{\boldsymbol{\beta}}(x)|\nabla \mathbf{u}(x,2\pi)|^2dx$$

$$\le c\int\limits_0^{2\pi}\int\limits_\Omega E^{(k)}_{\boldsymbol{\beta}}(x)|\mathbf{f}(x,t)|^2dxdt. \tag{3.29}$$

Now, repeating the arguments from paper [66] (see Chapter 4, Theorem 3.2), we derive the estimate

$$\int\limits_\Omega E^{(k)}_{\boldsymbol{\beta}}(x)\big(|\mathbf{u}(x,t)|^2 + |\nabla \mathbf{u}(x,t)|^2\big)dx$$

$$+\int\limits_0^t\int\limits_\Omega E^{(k)}_{\boldsymbol{\beta}}(x)\big(|\mathbf{u}_\tau(x,\tau)|^2 + \nu|\nabla \mathbf{u}(x,\tau)|^2\big)dxd\tau \le c\Big(\int\limits_\Omega E^{(k)}_{\boldsymbol{\beta}}(x)|\mathbf{u}(x,0)|^2dx$$

$$+\int\limits_0^t\int\limits_\Omega E^{(k)}_{\boldsymbol{\beta}}(x)|\mathbf{f}(x,\tau)|^2dxd\tau\Big) \le c\int\limits_0^t\int\limits_\Omega E_{\boldsymbol{\beta}}(x)|\mathbf{f}(x,\tau)|^2dxd\tau \tag{3.30}$$

for $\forall t \in [0,2\pi]$. The right-hand side of (3.30) does not depend on k. Therefore, passing in (3.30) to a limit as $k \to \infty$, we get the estimate

$$\sup_{t\in[0,2\pi]} \|\mathbf{u}(\cdot,t);\mathcal{W}^1_{2,\boldsymbol{\beta}}(\Omega)\|^2 + \|\mathbf{u};\mathcal{W}^{1,1}_{2,\boldsymbol{\beta}}(\Omega\times(0,2\pi))\|^2$$

$$\le c\|\mathbf{f};\mathcal{L}_{2,\boldsymbol{\beta}}(\Omega\times(0,2\pi))\|^2. \tag{3.31}$$

Let $\partial\Omega \in C^2$. From integral identity (3.3) it follows that for almost all $t \in [0,T]$ there holds the identity

$$\nu\int\limits_\Omega \nabla\mathbf{u}(x,t)\cdot\nabla\boldsymbol{\eta}(x)dx = \int\limits_\Omega (\mathbf{f}(x,t) - \mathbf{u}_t(x,t))\cdot\boldsymbol{\eta}(x)dx$$

where $\boldsymbol{\eta}(x)$ is arbitrary divergence-free function belonging to the space $\mathring{W}^1_2(\Omega)$. Hence, $\mathbf{u}(\cdot,t) \in \mathring{W}^1_2(\Omega)$ could be considered as a weak solution of the stationary

Stokes problem with the right-hand side equal to $\mathbf{f}(x,t) - \mathbf{u}_t(x,t)$:

$$\begin{cases} -\nu\Delta\mathbf{u}(x,t) + \nabla p(x,t) = \mathbf{f}(x,t) - \mathbf{u}_t(x,t), \\ \operatorname{div}\mathbf{u}(x,t) = 0, \\ \mathbf{u}(x,t)|_{\partial\Omega} = 0, \\ \int\limits_{\sigma_j} \mathbf{u}(x,t)\cdot\mathbf{n}(x)ds = 0, j = 1,...,J. \end{cases} \tag{3.32}$$

We have $\mathbf{f}(\cdot,t) - \mathbf{u}_t(\cdot,t) \in \mathcal{L}_{2,\boldsymbol{\beta}}(\Omega)$ for almost all $t \in [0, 2\pi]$. If γ_* is sufficiently small, then the weak solution $\mathbf{u}(x,t)$ of (3.32) belongs to the space $\mathcal{W}^2_{2,\boldsymbol{\beta}}(\Omega)$. Moreover, there exists a function $p(x,t)$ with $\nabla p(\cdot,t) \in \mathcal{L}_{2,\boldsymbol{\beta}}(\Omega)$ such that $(\mathbf{u}(x,t), p(x,t))$ satisfy equation (3.32) almost everywhere in Ω and there holds the estimate

$$\|\mathbf{u}(\cdot,t); \mathcal{W}^2_{2,\boldsymbol{\beta}}(\Omega)\|^2 + \|\nabla p(\cdot,t); \mathcal{L}_{2,\boldsymbol{\beta}}(\Omega)\|^2 \le c\|(\mathbf{f}(\cdot,t) - \mathbf{u}_t(\cdot,t)); \mathcal{L}_{2,\boldsymbol{\beta}}(\Omega)\|^2$$
$$\le c\Big(\|\mathbf{f}(\cdot,t); \mathcal{L}_{2,\boldsymbol{\beta}}(\Omega)\|^2 + \|\mathbf{u}_t(\cdot,t); \mathcal{L}_{2,\boldsymbol{\beta}}(\Omega)\|^2\Big). \tag{3.33}$$

Integrating inequality (3.33) with respect to t and using (3.31) we derive

$$\int\limits_0^{2\pi}\int\limits_\Omega E_{\boldsymbol{\beta}}(x)\big(|\mathbf{u}(x,\tau)|^2 + |\nabla\mathbf{u}(x,\tau)|^2 + \sum_{|\alpha|=2}|D^\alpha_x\mathbf{u}(x,\tau)|^2\big)dxd\tau$$
$$+ \int\limits_0^{2\pi}\int\limits_\Omega E_{\boldsymbol{\beta}}(x)|\nabla p(x,\tau)|^2 dxd\tau \le c\|\mathbf{f}; \mathcal{L}_{2,\boldsymbol{\beta}}(\Omega\times(0,2\pi))\|^2.$$

The last estimate together with (3.31) is equivalent to (3.17). The theorem is proved.

3.4 Time periodic Stokes problem with nonzero fluxes

In this section we study the problem (3.1) in the case of nonzero periodic fluxes $F_j(t), j = 1, 2, \dots, J$. First we construct the divergence free vector function $\mathbf{V}(x,t)$ satisfying the boundary condition (3.1_3) and the flux condition (3.1_4) (a flux carrier) and then reduce the problem (3.1) to the problem (3.2). According to Theorem 2.2 (see also papers [8], [19]), in each cylinder Π_j there exists the time periodic Poiseuille flow $\big(\mathbf{U}^{(j)}(x^{(j)}), P^{(j)}(x^{(j)},t)\big)$, i. e. the exact solution of the time periodic Stokes (and Navier–Stokes) problem in the cylinder Π_j, having the form

$$\mathbf{U}^{(j)}(x^{(j)},t)=\big(0,...,0,U_n^{(j)}(x^{(j)\prime},t)\big),\quad P^{(j)}(x^{(j)},t)=-q^{(j)}(t)x_n^{(j)}, \quad (3.34)$$

and satisfying the flux condition

$$\int\limits_{\sigma_j} U_n^{(j)}(x^{(j)\prime},t)dx^{(j)\prime}=F_j(t).$$

The pair of functions $\big(U_n^{(j)}(x^{(j)\prime},t),\, q^{(j)}(t)\big)$ is the solution of the parabolic inverse problem

$$\begin{cases} U_{nt}^{(j)}(x^{(j)\prime},t)-\nu\Delta' U_n^{(j)}(x^{(j)\prime},t)=q^{(j)}(t), \\ U_n^{(j)}(x^{(j)\prime},t)\big|_{\partial\sigma_j}=0, \\ U_n^{(j)}(x^{(j)\prime},0)=U_{nt}^{(j)}(x^{(j)\prime},2\pi), \\ \int\limits_{\sigma_j} U_n^{(j)}(x^{(j)\prime},t)dx^{(j)\prime}=F_j(t), \end{cases} \quad (3.35)$$

where Δ' is the Laplace operator with respect to variables $x^{(j)\prime}$. It is proved in [8], [19] that for any periodic $F_j(t)\in W_2^1(0,2\pi)$ the problem (3.35) admits a unique time periodic solution $\big(U_n^{(j)}(x^{(j)\prime},t),\, q^{(j)}(t)\big)\in W_2^{2,1}(\sigma_j\times(0,2\pi))\times L_2(0,2\pi)$. Moreover, there holds the estimate

$$\|U_n^{(j)};\, W_2^{2,1}(\sigma_j\times(0,2\pi))\|^2+\|q^{(j)};L_2(0,2\pi)\|^2\leq c\|F_j;W_2^1(0,2\pi)\|^2. \quad (3.36)$$

Define

$$\mathbf{U}(x,t)=\sum_{j=1}^{J}\zeta(x_n^{(j)})\mathbf{U}^{(j)}(x^{(j)\prime},t),\ \ P(x,t)=\sum_{j=1}^{J}\zeta(x_n^{(j)})P^{(j)}(x^{(j)\prime},t), \tag{3.37}$$

where $\zeta(\tau)$ is smooth cut-off function with $\zeta(\tau)=0$ for $\tau\leq 1$ and $\zeta(\tau)=1$ for $\tau\geq 2$. Let

$$g(x,t)=-\mathrm{div}\mathbf{U}(x,t)=-\sum_{j=1}^{J}\frac{d\zeta(x_n^{(j)})}{dx_n^{(j)}}U_n^{(j)}(x^{(j)\prime},t).$$

Then

$$\mathrm{supp}_x g(x,t)\subset\overline{\Omega_{(2)}\backslash\Omega_{(1)}}$$

and, therefore, $g(\cdot,t)\in\mathring{W}_2^1(\Omega_{(3)})$. Moreover, from condition $\sum\limits_{j=1}^{J}F_j(t)=0$ it follows that

$$\int\limits_{\Omega_{(2)}}g(x,t)dx=-\int\limits_{\Omega_{(2)}}\mathrm{div}\mathbf{U}(x,t)dx=-\sum_{j=1}^{J}\int\limits_{\sigma_j}U_n^{(j)}(x^{(j)\prime},t)dx^{(j)\prime}$$

$$=-\sum_{j=1}^{J}F_j(t)=0,\ \ \forall t\in[0,T].$$

Therefore, by lemma 1.8 there exists a time periodic vector function $\mathbf{W}(\cdot,t)\in\mathring{W}_2^2(\Omega_{(3)})$ with $\mathbf{W}_t(\cdot,t)\in\mathring{W}_2^1(\Omega_{(3)})$ such that

$$\mathrm{div}\mathbf{W}(x,t)=g(x,t),\ \ \mathrm{div}\mathbf{W}_t(x,t)=g_t(x,t),$$

and there holds the estimate

$$\int\limits_0^{2\pi}\|\mathbf{W}(\cdot,t);W_2^2(\Omega_{(3)})\|^2dt+\int\limits_0^{2\pi}\|\mathbf{W}_t(\cdot,t);W_2^1(\Omega_{(3)})\|^2dt$$

$$\leq c\Big(\int\limits_0^{2\pi}\|g(\cdot,t);W_2^1(\Omega_{(3)})\|^2dt+\int\limits_0^{2\pi}\|g_t(\cdot,t);L_2(\Omega_{(3)})\|^2dt\Big)$$

$$\leq c \sum_{j=1}^{J} \|F_j; W_2^1(0, 2\pi)\|^2. \tag{3.38}$$

Define

$$\mathbf{V}(x,t) = \mathbf{U}(x,t) + \mathbf{W}(x,t). \tag{3.39}$$

Obviously,

$$\text{div}\mathbf{V}(x,t) = 0, \ \ \mathbf{V}(x,t)\big|_{\partial\Omega} = 0, \ \int\limits_{\sigma_j} \mathbf{V}(x,t) \cdot \mathbf{n}(x) ds = F_j(t), j = 1, ..., J,$$

and for $x \in \Omega_j \backslash \Omega_{j3}, j = 1, ..., J$, the vector function $\mathbf{V}(x,t)$ coincides with the velocity part $\mathbf{U}^{(j)}(x^{(j)\prime}, t)$ of the corresponding time periodic Poiseuille flow.

Now we are in a position to prove the main result of the chapter.

Theorem 3.3. *Let $\partial\Omega \in C^2$. Suppose that $F_j(t) \in W_2^1(0, 2\pi), j = 1, 2, \ldots, J$, and $\mathbf{f}(x,t) \in \mathcal{L}_{2,\boldsymbol{\beta}}(\Omega \times (0, 2\pi))$ are time periodic functions. If the number γ_* in the inequality (1.3_3) is sufficiently small, then problem (3.1) has a unique time periodic solution $\big(\mathbf{u}(x,t), p(x,t)\big)$ admitting the asymptotic representation*

$$\mathbf{u}(x,t) = \mathbf{V}(x,t) + \mathbf{v}(x,t), \ \ p(x,t) = P(x,t) + \widetilde{p}(x,t), \tag{3.40}$$

where $\mathbf{V}(x,t)$ is the flux carrier defined by (3.39) and $P(x,t)$ is the corresponding pressure function defined by (3.37), $\mathbf{v}(x,t) \in \mathcal{W}^{2,1}_{2,\boldsymbol{\beta}}(\Omega \times (0, 2\pi))$, $\nabla\widetilde{p}(x,t) \in \mathcal{L}_{2,\boldsymbol{\beta}}(\Omega \times (0, 2\pi))$. There holds the estimate

$$\|\mathbf{v}; \mathcal{W}^{2,1}_{2,\boldsymbol{\beta}}(\Omega \times (0, 2\pi))\| + \|\nabla\widetilde{p}; \mathcal{L}_{2,\boldsymbol{\beta}}(\Omega \times (0, 2\pi))\|$$

$$\leq c\Big(\|\mathbf{f}; \mathcal{L}_{2,\boldsymbol{\beta}}(\Omega \times (0, 2\pi))\| + \sum_{j=1}^{J} \|F_j; W_2^1(0, 2\pi)\|\Big). \tag{3.41}$$

Proof. We look for the solution $\big(\mathbf{u}(x,t), p(x,t)\big)$ of problem (3.1) in the form (3.40). Then for $\big(\mathbf{v}(x,t), \widetilde{p}(x,t)\big)$ we get the problem (3.2), i. e. the time periodic Stokes problem with zero fluxes and the new external force $\widetilde{\mathbf{f}}(x,t) = \mathbf{f}(x,t) + \mathbf{f}_{(1)}(x,t) + \mathbf{f}_{(2)}(x,t)$, where

$$\mathbf{f}_{(1)}(x,t) = -\mathbf{U}_t(x,t) + \nu\Delta\mathbf{U}(x,t) - \nabla P(x,t),$$

$$\mathbf{f}_{(2)}(x,t) = -\mathbf{W}_t(x,t) + \nu\Delta\mathbf{W}(x,t).$$

Since for $x \in \Omega_j \backslash \Omega_{j3}$ the pair $\big(\mathbf{V}(x,t), P(x,t)\big)$ coincides with the time periodic Poiseuille solution $\big(\mathbf{U}^{(j)}(x^{(j)\prime}, t), P^{(j)}(x,t)\big)$, $j = 1, ..., J$, which is the exact solutions of Stokes system in a cylinder Π_j, we conclude that

$$\mathrm{supp}_x(\mathbf{f}_{(1)}(x,t) + \mathbf{f}_{(2)}(x,t)) \subset \overline{\Omega}_{(3)}.$$

From the construction of $(\mathbf{U}(x,t), P(x,t))$ and $\mathbf{W}(x,t)$ (see (3.37), (3.39)) and from estimates (3.36), (3.38) it follows that

$$\|\mathbf{f}_{(1)} + \mathbf{f}_{(2)}; \mathcal{L}_{2,\boldsymbol{\beta}}(\Omega \times (0, 2\pi))\| \leq c\|\mathbf{f}_{(1)} + \mathbf{f}_{(2)}; L_2(\Omega \times (0, 2\pi))\|$$

$$\leq c \sum_{j=1}^{J} \|F_j;\, W_2^1(0, 2\pi)\|.$$

Therefore, by Theorems 3.1 and 3.2, there exists a unique solution $(\mathbf{v}(x,t), \widetilde{p}(x,t))$ of the problem (3.2) with the right–hand side $\widetilde{\mathbf{f}}(x,t)$. Moreover, there holds the estimate:

$$\|\mathbf{v}; \mathcal{W}^{2,1}_{2,\boldsymbol{\beta}}(\Omega \times (0, 2\pi))\| + \|\nabla \widetilde{p}; \mathcal{L}_{2,\boldsymbol{\beta}}(\Omega \times (0, 2\pi))\| \leq \|\widetilde{\mathbf{f}}; \mathcal{L}_{2,\boldsymbol{\beta}}(\Omega \times (0, 2\pi))\|$$

$$\leq c\Big(\sum_{j=1}^{J} \|F_j; W_2^1(0, 2\pi)\| + \|\mathbf{f}; \mathcal{L}_{2,\boldsymbol{\beta}}(\Omega \times (0, 2\pi))\|\Big).$$

The theorem is proved.
The results obtained in this chapter is presented in [A2].

4

Two-dimensional time periodic Navier-Stokes problem

4.1 Formulation of the problem

In this chapter we consider time periodic Navier-Stokes problem

$$\begin{cases} \mathbf{u}_t(x,t) - \nu\Delta\mathbf{u}(x,t) + (\mathbf{u}(x,t)\cdot\nabla)\mathbf{u}(x,t) + \nabla p(x,t) = \mathbf{f}(x,t), \\ \mathrm{div}\mathbf{u}(x,t) = 0, \\ \mathbf{u}(x,t)|_{\partial\Omega} = 0, \\ \mathbf{u}(x,0) = \mathbf{u}(x,2\pi), \\ \int\limits_{\sigma_j} \mathbf{u}(x,t)\cdot\mathbf{n}(x)ds = F_j(t), \quad j = 1,...,J. \end{cases} \tag{4.1}$$

in two-dimensional domain Ω with strip-like outlets to infinity. Assume that the external force $\mathbf{f}(x,t)$ admits the representation

$$\mathbf{f}(x,t) = \sum_{j=1}^{J} \zeta(x_2^{(j)})(0, f_2^{(j)}(x_1^{(j)}, t)) + \widehat{\mathbf{f}}(x), \tag{4.2}$$

where $f_2^{(j)} \in L_2(\sigma_j \times (0, 2\pi))$, $\widehat{\mathbf{f}} \in \mathcal{L}_{2,\boldsymbol{\beta}}(\Omega \times (0, 2\pi))$. $\zeta(\tau)$ is a smooth cut-off function with $\zeta(\tau) = 0$, for $\tau \leq 1$ and $\zeta(\tau) = 1$, for $\tau \geq 2$.

According to theorem 2.2 in each cylinder Π_j there exists the time periodic Poiseuille flow $(\mathbf{U}^{(j)}(x^{(j)\prime}, t), P^{(j)}(x^{(j)}, t))$ having the form

$$\mathbf{U}^{(j)}(x_1^{(j)}, t) = (0, U_2^{(j)}(x_1^{(j)}, t)), \;\; P^{(j)}(x_1^{(j)}, t) = -q^{(j)}(t)x_2^{(j)} + p_0(t)$$

and satisfying the estimates

$$\begin{aligned} &\|U_2^{(j)}; W_2^{2,1}(\sigma \times (0, 2\pi))\|^2 + \|q^{(j)}(t); L_2(0, 2\pi)\|^2 \\ &\leq c\left(\|f_2^{(j)}; L_2(\sigma_j \times 0, 2\pi)\|^2 + \|F_j; W_2^1(0, 2\pi)\|^2 \right). \end{aligned} \tag{4.3}$$

Set

$$\mathbf{U}(x,t) = \sum_{j=1}^{J} \zeta(x_2^{(j)})\mathbf{U}^{(j)}(x_1^{(j)}, t), \;\; P(x,t) = \sum_{j=1}^{J} \zeta(x_2^{(j)}) P^{(j)}(x_1^{(j)}, t). \tag{4.4}$$

Let $g(x,t) = -\mathrm{div}\mathbf{U}(x,t)$. Then $\mathrm{supp}_x g(x,t) \subset \overline{\Omega_{(3)}}$. From the condition $\sum\limits_{j=1}^{J} F_j(t) = 0$ we get that

$$\int\limits_{\Omega_{(3)}} g(x,t)dx = 0, \;\; \forall t \in [0, 2\pi].$$

Moreover, in virtue of Teorem 2.2

$$\begin{aligned} &\int\limits_0^{2\pi} \|g(\cdot,t); W_2^1(\Omega_{(3)})\|^2 dt + \int\limits_0^{2\pi} \|g_t(\cdot,t); L_2(\Omega_{(3)})\|^2 dt \\ &\leq c\sum_{j=1}^{J} \left(\int\limits_0^{2\pi} \|U_2^{(j)}(\cdot,t); W_2^2(\sigma_j)\|^2 dt + \int\limits_0^{2\pi} \|U_{2t}^{(j)}(\cdot,t); L_2(\sigma_j)\|^2 dt \right) \end{aligned}$$

$$\leq c\sum_{j=1}^{J}\Big(\|f_2^{(j)};L_2(\sigma_j\times(0,2\pi))\|^2+\|F_j;W_2^1(0,2\pi)\|^2\Big).$$

Hence $g(\cdot,t)\in \overset{\circ}{W}{}_2^1(\Omega_{(3)})$ and in virtue of lemma 1.8 there exists a vector-field $\mathbf{W}(\cdot,t)\in \overset{\circ}{W}{}_2^2(\Omega_{(3)})$ with $\mathbf{W}_t(\cdot,t)\in \overset{\circ}{W}{}_2^1(\Omega_{(3)})$ such that

$$\mathrm{div}\mathbf{W}(x,t)=g(x,t),\ \ \mathrm{div}\mathbf{W}_t(x,t)=g_t(x,t),$$

and there holds the estimate

$$\int_0^{2\pi}\|\mathbf{W}(\cdot,t);W_2^2(\Omega_{(3)})\|^2dt+\int_0^{2\pi}\|\mathbf{W}_t(\cdot,t);W_2^1(\Omega_{(3)})\|^2dt$$

$$\leq\Big(\int_0^{2\pi}\|g(\cdot,t);W_2^1(\Omega_{(3)})\|^2dt+\int_0^{2\pi}\|g_t(\cdot,t);L_2(\Omega_{(3)})\|^2dt\Big)$$

$$\leq c\sum_{j=1}^{J}\Big(\|f_2^{(j)};L_2(\sigma_j\times(0,2\pi))\|^2+\|F_j;W_2^1(0,2\pi)\|^2\Big).\tag{4.5}$$

Define

$$\mathbf{V}(x,t)=\mathbf{U}(x,t)+\mathbf{W}(x,t).\tag{4.6}$$

Then for $x\in\Omega_j\backslash\Omega_{j3}$, $j=1,...,J$, the vector-field $\mathbf{V}(x,t)$ coincides with the velocity part $\mathbf{U}^{(j)}(x^{(j)'},t)$ of the corresponding time-periodic Poiseuille flow. Moreover,

$$\mathrm{div}\mathbf{V}(x,t)=0,\ \ \mathbf{V}(x,t)|_{\partial\Omega}=0,\ \ \int_{\sigma_j}\mathbf{V}(x,t)\cdot\mathbf{n}(x)ds=F_j(t),\ \ j=1,...,J.$$

We look for the solution of problem (4.1) in the form

$$\mathbf{u}(x,t)=\mathbf{V}(x,t)+\mathbf{v}(x,t),\ \ p(x,t)=P(x,t)+\widetilde{p}(x,t).\tag{4.7}$$

Then for $(\mathbf{v}(x,t), \widetilde{p}(x,t))$ we derive the equation

$$\begin{cases} \mathbf{v}_t(x,t) - \nu\Delta\mathbf{v}(x,t) + (\mathbf{v}(x,t)\cdot\nabla)\mathbf{v}(x,t) + (\mathbf{V}(x,t)\cdot\nabla)\mathbf{v}(x,t) \\ \qquad +(\mathbf{v}(x,t)\cdot\nabla)\mathbf{V}(x,t) + \nabla\widetilde{p}(x,t) = \widetilde{\mathbf{f}}(x,t), \\ \mathrm{div}\mathbf{v} = 0, \\ \mathbf{v}(x,t)|_{\partial\Omega} = 0, \\ \mathbf{v}(x,0) = \mathbf{v}(x,2\pi), \\ \int\limits_{\sigma_j} \mathbf{v}(x,t)\cdot\mathbf{n}(x)ds = 0, \quad j = 1,...,J. \end{cases} \tag{4.8}$$

Where $\widetilde{\mathbf{f}}(x,t) = \widehat{\mathbf{f}}(x,t) + \mathbf{f}_{(1)}(x,t) + \mathbf{f}_{(2)}(x,t)$,

$$\mathbf{f}_{(1)}(x,t) = \sum_{j=1}^{J} \big(\nu \frac{d^2}{d{x_2^{(j)}}^2}\zeta(x_2^{(j)})(0, U_2^{(j)}(x_1^{(j)},t)) -$$

$$\zeta(x_2^{(j)})\frac{d}{dx_2^{(j)}}\zeta(x_2^{(j)})(0, |U_2^{(j)}(x_1^{(j)},t)|^2) - \frac{d}{dx_2^{(j)}}\zeta(x_2^{(j)})x_2^{(j)}(0, q^{(j)}(t))\big),$$

$$\mathbf{f}_{(2)}(x,t) = -\mathbf{W}_t(x,t) + \nu\Delta\mathbf{W}(x,t) - (\mathbf{W}(x,t)\cdot\nabla)\mathbf{W}(x,t)$$
$$-(\mathbf{U}(x,t)\cdot\nabla)\mathbf{W}(x,t) - (\mathbf{W}(x,t)\cdot\nabla)\mathbf{U}(x,t).$$

Note that $\mathrm{supp}_x[\mathbf{f}_{(1)}(x,t) + \mathbf{f}_{(2)}(x,t)] \subset \bar{\Omega}_{(3)}$.

Moreover, using Sobolev embedding theorem on the one-dimensional interval $(0, h_j)$, from (4.3) we obtain the estimate

$$\int\limits_0^{2\pi}\int\limits_{\Omega} |\mathbf{f}_{(1)}(x,t)|^2 dx d\tau$$

$$\leq c\sum_{j=1}^{J}\int\limits_0^{2\pi}\int\limits_{\Omega_j} (|U_2^{(j)}(x_1^{(j)},\tau)|^2 + |U_2^{(j)}(x_1^{(j)},\tau)|^4 + |q^{(j)}(\tau)|^2) dx d\tau$$

$$\leq c\sum_{j=1}^{J}\int\limits_0^{2\pi}\int\limits_0^{h_j}(\|U_2^{(j)}(\cdot,\tau);L_2(\sigma_j)\|^2+\|U_2^{(j)}(\cdot,\tau);W_2^1(\sigma_j)\|^4)d\tau$$

$$+c\int\limits_0^{2\pi}|q^{(j)}(\tau)|^2d\tau\leq cA_1(1+A_1), \tag{4.9}$$

where

$$A_1=\sum_{j=1}^{J}(\|f_2^{(j)};L_2(\sigma_j\times(0,2\pi))\|^2+\|F_j;W_2^1(0,2\pi)\|^2). \tag{4.10}$$

Analogously, using again Sobolev embedding theorem in the domain $\Omega_{(3)}$ and (4.3) we derive

$$\int\limits_0^{2\pi}\int\limits_{\Omega}|\mathbf{f}_{(2)}(x,\tau)|^2dxd\tau\leq c\int\limits_0^{2\pi}\int\limits_{\Omega_{(3)}}\Big(|\mathbf{W}_\tau(x,\tau)|^2+|\Delta\mathbf{W}(x,\tau)|^2$$

$$+(|\mathbf{W}(x,\tau)|^2+\mathbf{U}(x,\tau)|^2)|\nabla\mathbf{W}(x,\tau)|^2+|\mathbf{W}(x,\tau)|^2|\nabla\mathbf{U}(x,\tau)|^2\Big)dxd\tau$$

$$\leq cA_1+c\int\limits_0^{2\pi}\Big(\sup_{x\in\bar{\Omega}_{(3)}}(|\mathbf{W}(x,\tau)|^2+|\mathbf{U}(x,\tau)|^2)\times$$

$$\int\limits_{\Omega_{(3)}}(|\nabla\mathbf{W}(x,\tau)|^2+|\nabla\mathbf{U}(x,\tau)|^2)dx\Big)d\tau$$

$$\leq cA_1+c\left[\sup_{t\in[0,2\pi]}\|\mathbf{W}(\cdot,t);W_2^1(\Omega_{(3)})\|^2+\sum_{j=1}^{J}\|U_2^{(j)}(\cdot,t);W_2^1(\sigma_j)\|^2\right]$$

$$\times\int\limits_0^{2\pi}\Big(\|\mathbf{W}(\cdot,\tau);W_2^2(\Omega_{(3)})\|^2+\sum_{j=1}^{J}\|U_2^{(j)}(\cdot,\tau);W_2^1(\sigma_j)\|^2\Big)d\tau$$

$$\leq cA_1+c\Big(\|\mathbf{W};W_2^{2,1}(\Omega_{(3)}\times(0,2\pi))\|^2+\sum_{j=1}^{J}\|U_2^{(j)};W_2^{2,1}(\sigma\times(0,2\pi))\|^2\Big)$$

$$\leq cA_1(1+A_1). \tag{4.11}$$

Define,

$$A_2 = \int_0^{2\pi} \int_\Omega |\widehat{\mathbf{f}}(x,t)|^2 dx dt. \tag{4.12}$$

Then

$$\int_0^{2\pi} \int_\Omega |\widetilde{\mathbf{f}}(x,t)|^2 dx dt \le c(A_1 + A_1^2) + A_2. \tag{4.13}$$

4.2 A priori estimates of the solution

Lemma 4.1. *Let* $\widehat{\mathbf{f}} \in L_2(\Omega \times (0,2\pi))$*, and let* $\mathbf{v} \in W_2^{2,1}(\Omega \times (0,2\pi))$ *be a solution of problem (4.8). If*

$$a \equiv \frac{\nu}{c_0} - \frac{A_1}{\nu} > 0, \tag{4.14}$$

(the constant c_0 *is from Poincarè inequality) then there holds the estimate:*

$$\int_\Omega |\mathbf{v}(x,t)|^2 dx + \nu \int_0^t \int_\Omega |\nabla \mathbf{v}(x,\tau)|^2 dx d\tau$$

$$\le c\left((\frac{e^{-at}}{1-e^{-at}} A_1 + 1)(A_1 + A_1^2 + A_2) \right) := cB_1, \quad \forall t \in [0,2\pi], \tag{4.15}$$

where the constant A_1 *is defined by (4.10) and* A_2 *is defined by (4.12). The constant* c *in (4.15) does not depend on* $t \in (0,2\pi]$.

Proof. Multiply (4.8) by $\mathbf{v}(x,t)$ and integrate by parts in Ω:

$$\frac{1}{2}\frac{d}{dt} \int_\Omega |\mathbf{v}(x,t)|^2 dx + \nu \int_\Omega |\nabla \mathbf{v}(x,t)|^2 dx$$

$$= -\int_\Omega (\mathbf{v}(x,t)\cdot\nabla)\mathbf{V}(x,t)\cdot\mathbf{v}(x,t) dx + \int_\Omega \widetilde{\mathbf{f}}(x,t)\cdot\mathbf{v}(x,t) dx \tag{4.16}$$

Let us estimate the first integral on the right - hand side of (4.16). We have

$$\left| \int_\Omega (\mathbf{v}(x,t)\cdot\nabla)\mathbf{V}(x,t)\cdot\mathbf{v}(x,t) \right| \le \left| \int_{\Omega_{(3)}} (\mathbf{v}(x,t)\cdot\nabla)\mathbf{W}(x,t)\cdot\mathbf{v}(x,t) \right|$$

$$
+\sum_{j=1}^{J}\left|\int\limits_{\Omega_j}(\mathbf{v}(x,t)\cdot\nabla)\mathbf{U}(x,t)\cdot\mathbf{v}(x,t)\right| = J_1(t)+J_2(t). \tag{4.17}
$$

Using (1.17), Poincaré, Hölder, Young inequalities and multiplicative (1.9) inequalite the first integral at the right-hand side of (4.17) is estimated as follows

$$
|J_1(t)| \leq \left(\int\limits_{\Omega_{(3)}} |\mathbf{v}(x,t)|^4 dx\right)^{1/2} \left(\int\limits_{\Omega_{(3)}} |\nabla\mathbf{W}(x,t)|^2 dx\right)^{1/2}
$$

$$
\leq 2\left(\int\limits_{\Omega_{(3)}} |\mathbf{v}(x,t)|^2\right)^{1/2} \left(\int\limits_{\Omega_{(3)}} |\nabla\mathbf{v}(x,t)|^2 dx\right)^{1/2} \left(\int\limits_{\Omega_{(3)}} |\nabla\mathbf{W}(x,t)|^2 dx\right)^{1/2}
$$

$$
\leq \frac{6m_1(t)}{\nu}\int\limits_{\Omega} |\mathbf{v}(x,t)|^2 dx + \frac{\nu}{6}\int\limits_{\Omega} |\nabla\mathbf{v}(x,t)|^2 dx,
$$

where

$$
m_1(t) = \int\limits_{\Omega_{(3)}} |\nabla\mathbf{W}(x,t)|^2 dx.
$$

Analogously, the second term at the right-hand side of (4.17) admits the estimate

$$
|J_2(t)| = \sum_{j=1}^{J}\left|\int\limits_{\Omega_j}(\mathbf{v}(x,t)\cdot\nabla)\mathbf{U}(x,t)\cdot\mathbf{v}(x,t)\right|
$$

$$
\leq \sum_{j=1}^{J}\sum_{s=0}^{\infty}\left|\int\limits_{\omega_{js}}(\mathbf{v}(x,t)\cdot\nabla)\zeta(x_2^{(j)})\mathbf{U}^{(j)}(x_1^{(j)},t))\cdot\mathbf{v}(x,t)dx\right|
$$

$$
\leq 2\sum_{j=1}^{J}\sum_{s=0}^{\infty}\left(\int\limits_{\omega_{js}} |\mathbf{v}(x,t)|^2 dx\right)^{1/2} \left(\int\limits_{\omega_{js}} |\nabla\mathbf{v}(x,t)|^2 dx\right)^{1/2}
$$

$$
\times\left(\int\limits_{\omega_{js}} (|\mathbf{U}^{(j)}(x_1^{(j)},t)|^2 + |\nabla'\mathbf{U}^{(j)}(x_1^{(j)},t)|^2)dx\right)^{1/2}
$$

$$\le c\sum_{j=1}^{J}\sum_{s=0}^{\infty}\Bigg(\int\limits_{\sigma_j}|\nabla'\mathbf{U}^{(j)}(x_1^{(j)},t)|^2dx_1^{(j)}\Bigg)\Bigg(\int\limits_{\omega_{js}}|\mathbf{v}(x,t)|^2dx\Bigg)$$

$$+\frac{\nu}{6}\sum_{j=1}^{J}\sum_{s=0}^{\infty}\int\limits_{\omega_{js}}|\nabla\mathbf{v}(x,t)|^2dx\le\frac{6m_2(t)}{\nu}\int\limits_{\Omega}|\mathbf{v}(x,t)|^2dx+\frac{\nu}{6}\int\limits_{\Omega}|\nabla\mathbf{v}(x,t)|^2dx,$$

where

$$m_2(t)=c\sum_{j=1}^{J}\int\limits_{\sigma_j}|\nabla'\mathbf{U}^{(j)}(x_1^{(j)},t)|^2dx_1^{(j)}.$$

Therefore, from (4.16) it follows that

$$\frac{d}{dt}\int\limits_{\Omega}|\mathbf{v}(x,t)|^2dx+\nu\int\limits_{\Omega}|\nabla\mathbf{v}(x,t)|^2dx$$

$$\le\frac{\mathcal{M}(t)}{\nu}\int\limits_{\Omega}|\mathbf{v}(x,t)|^2dx+c\int\limits_{\Omega}|\widetilde{\mathbf{f}}(x,t)|^2dx \tag{4.18}$$

with $\sup\limits_{t\in[0,T]}\mathcal{M}(t)=12\Big(m_1(t)+m_2(t)\Big)$. Using (4.3) and (4.5) estimates we can conclude that $\mathcal{M}(t)\le A_1,\forall t\in[0,2\pi]$. Now, if $\dfrac{\nu}{c_0}-\dfrac{A_3}{\nu}=a>0$ then from (4.18) follows that

$$\frac{d}{dt}\int\limits_{\Omega}|\mathbf{v}(x,t)|^2dx+a\int\limits_{\Omega}|\mathbf{v}(x,t)|^2dx\le c\int\limits_{\Omega}|\widetilde{\mathbf{f}}(x,t)|^2dx. \tag{4.19}$$

Multiplying (4.19) inequality by e^{at} and integrating over t we derive

$$\int\limits_{\Omega}|\mathbf{v}(x,t)|^2dx$$

$$\le e^{-at}\Bigg[\int\limits_{\Omega}|\mathbf{v}(x,0)|^2dx+c\int\limits_{0}^{t}\int\limits_{\Omega}|e^{at}\widetilde{\mathbf{f}}(x,\tau)|^2dxd\tau\Bigg]. \tag{4.20}$$

Now using periodicity condition $v(x,0) = v(x,2\pi)$ from (4.20) follows that

$$\int_\Omega |\mathbf{v}(x,0)|^2 dx$$

$$\leq e^{-2\pi a}\left[\int_\Omega |\mathbf{v}(x,0)|^2 dx\right] + c\int_0^{2\pi}\int_\Omega |\widetilde{\mathbf{f}}(x,\tau)|^2 dx d\tau.$$

So from the last inequality we can conclude that

$$\int_\Omega |\mathbf{v}(x,0)|^2 dx \leq \frac{c}{1-e^{-2\pi a}}\int_0^{2\pi}\int_\Omega |\widetilde{\mathbf{f}}(x,t)|^2 dx dt. \tag{4.21}$$

Hence from (4.20) and (4.21) follows that

$$\int_\Omega |\mathbf{v}(x,t)|^2 dx \leq \frac{c}{1-e^{-2\pi a}}\int_0^{2\pi}\int_\Omega |\widetilde{\mathbf{f}}(x,t)|^2 dx dt. \tag{4.22}$$

Substituting (4.22) into (4.18) and integrating with respect to t we derive

$$\sup_{t\in[0,2\pi]}\int_\Omega |\mathbf{v}(x,t)|^2 dx + \nu\int_0^t\int_\Omega |\nabla\mathbf{v}(x,t)|^2 dx d\tau$$

$$\leq \frac{c}{1-e^{-2\pi a}}\mathcal{M}(t)\int_0^{2\pi}\int_\Omega |\widetilde{\mathbf{f}}(x,t)|^2 dx dt + c\int_0^t\int_\Omega |\widetilde{\mathbf{f}}(x,t)|^2 dx dt$$

$$\leq c\left(\frac{1}{1-e^{-2\pi a}}\mathcal{M}(t)+1\right)\int_0^{2\pi}\int_\Omega |\widetilde{\mathbf{f}}(x,t)|^2 dx dt$$

$$\leq c\left((c\frac{e^{2\pi a}}{e^{2\pi a}-1}A_3+1)(A_1+A_1^2+A_2)\right) := cB_1.$$

The lemma is proved.

4.3 Estimates of nonlinear terms

Consider problem (4.8) as the linear time dependent Stokes problem:

$$\begin{cases} \mathbf{v}_t(x,t) - \nu\Delta\mathbf{v}(x,t) + \nabla p(x,t) = \mathbf{g}(x,t,\mathbf{v}(x,t)), \\ \mathrm{div}\mathbf{v}(x,t) = 0, \\ \mathbf{v}(x,t)|_{\partial\Omega} = 0, \\ \mathbf{v}(x,0) = \mathbf{v}(x,2\pi), \\ \int\limits_{\sigma_j} \mathbf{v}(x,t)\cdot\mathbf{n}(x)ds = 0, \quad j=1,...,J, \end{cases} \tag{4.23}$$

with the right-hand side

$$\mathbf{g}(x,t,\mathbf{v}(x,t)) = \widetilde{\mathbf{f}}(x,t) - (\mathbf{v}(x,t)\cdot\nabla)\mathbf{v}(x,t)$$
$$-(\mathbf{V}(x,t)\cdot\nabla)\mathbf{v}(x,t) - (\mathbf{v}(x,t)\cdot\nabla)\mathbf{V}(x,t).$$

Let us estimate $\int\limits_0^t \|E^{(k)}_{\boldsymbol{\beta}}\mathbf{g}(\cdot,\tau,\mathbf{v}(\cdot,\tau)); L_2(\Omega)\|^2 d\tau$, where $E^{(k)}_{\boldsymbol{\beta}}(x)$ is a step weight-function (1.5). Estimates of this section analogous to those of papers [66], where the case of usual initial boundary value problem is studied. We have

$$\int\limits_0^t \|E^{(k)}_{\boldsymbol{\beta}}\mathbf{g}(\cdot,\tau,\mathbf{v}(\cdot,\tau)); L_2(\Omega)\|^2 d\tau \le c\bigg(\int\limits_0^t \|E^{(k)}_{\boldsymbol{\beta}}\widetilde{\mathbf{f}}(\cdot,\tau); L_2(\Omega)\|^2 d\tau$$
$$+\int\limits_0^t \|E^{(k)}_{\boldsymbol{\beta}}(\mathbf{v}(\cdot,\tau)\cdot\nabla)\mathbf{v}(\cdot,\tau); L_2(\Omega)\|^2 d\tau$$
$$+\int\limits_0^t \|E^{(k)}_{\boldsymbol{\beta}}(\mathbf{V}(\cdot,\tau)\cdot\nabla)\mathbf{v}(\cdot,\tau); L_2(\Omega)\|^2 d\tau$$

$$+\int\limits_0^t \|E_{\boldsymbol\beta}^{(k)}(\mathbf{v}(\cdot,\tau)\cdot\nabla)\mathbf{V}(\cdot,\tau);L_2(\Omega)\|^2 d\tau\Bigg)$$

$$= c\Bigg(\int\limits_0^t \|E_{\boldsymbol\beta}^{(k)}\widetilde{\mathbf{f}}(\cdot,\tau);L_2(\Omega)\|^2 d\tau + I_1(t)+I_2(t)+I_3(t)\Bigg).$$

Lemma 4.2. *Let* $\mathbf{v}(x,t)\in W_2^{2,1}(\Omega\times(0,2\pi))$. *Then* $(\mathbf{v}(x,t)\cdot\nabla)\mathbf{v}(x,t)\in L_2(\Omega\times(0,2\pi))$ *and*

$$\int\limits_0^t \|E_{\boldsymbol\beta}^{(k)}(\mathbf{v}(\cdot,\tau)\cdot\nabla)\mathbf{v}(\cdot,\tau);L_2(\Omega)\|^2 d\tau \le \varepsilon\int\limits_0^t \|E_{\boldsymbol\beta}^{(k)}\mathbf{v}(\cdot,\tau);W_2^2(\Omega)\|^2 d\tau$$

$$+c_\varepsilon B_1\int\limits_0^t \|E_{\boldsymbol\beta}^{(k)}\nabla\mathbf{v}(\cdot,\tau);L_2(\Omega)\|^4 d\tau,\ \ \forall t\in[0,2\pi], \tag{4.24}$$

where B_1 *is defined in (4.15) and the constant c is independent of* $t\in[0,2\pi]$ *and* k.

Lemma 4.3. *Let* $\mathbf{v}(x,t)\in W_2^{2,1}(\Omega\times(0,2\pi))$. *Then* $(\mathbf{V}(x,t)\cdot\nabla)\mathbf{v}(x,t)\in L_2(\Omega\times(0,2\pi)), (\mathbf{v}(x,t)\cdot\nabla)\mathbf{V}(x,t)\in L_2(\Omega\times(0,2\pi))$ *and*

$$\int\limits_0^t \|E_{\boldsymbol\beta}^{(k)}(\mathbf{V}(\cdot,\tau)\cdot\nabla)\mathbf{v}(\cdot,\tau);L_2(\Omega)\|^2 d\tau$$

$$+\int\limits_0^t \|E_{\boldsymbol\beta}^{(k)}(\mathbf{v}(\cdot,\tau)\cdot\nabla)\mathbf{V}(\cdot,\tau);L_2(\Omega)\|^2 d\tau$$

$$\le c\varepsilon\int\limits_0^t \|E_{\boldsymbol\beta}^{(k)}\mathbf{v}(\cdot,\tau);W_2^2(\Omega)\|^2 d\tau$$

$$+c_\varepsilon A_1\int\limits_0^t Y(\tau)\|E_{\boldsymbol\beta}^{(k)}\nabla\mathbf{v}(\cdot,\tau);L_2(\Omega)\|^2 d\tau,\ \ \forall t\in[0,2\pi], \tag{4.25}$$

where A_1 is defined by (4.10),

$$Y(t) = \sum_{j=1}^{J} \|\mathbf{U}^{(j)}(\cdot,t); W_2^2(\sigma_j)\|^2 + \|\mathbf{W}(\cdot,t); W_2^2(\Omega_{(3)})\|^2,$$

and the constant c is independent of $t \in [0,2\pi]$ and k.

Lemmas 4.2 and 4.3 are proved in [66].

4.4 Weighted estimates of the solution

Lemma 4.4. *Let $\widetilde{\mathbf{f}}(x,t) \in L_2(\Omega \times (0,2\pi)), t \in [0,2\pi]$, and let $\mathbf{v}(x,t) \in W_2^{2,1}(\Omega \times (0,2\pi))$ be a solution of (4.8). If the number γ_* in the inequality (1.3_3) for the weight-function $E_{\boldsymbol{\beta}}(x)$ is sufficiently small and the condition (4.14) is satisfied, then there holds the estimate*

$$\int_0^t (\|E_{\boldsymbol{\beta}}^{(k)}\mathbf{v}_\tau(\cdot,\tau); L_2(\Omega)\|^2 + \|E_{\boldsymbol{\beta}}^{(k)}\mathbf{v}(\cdot,\tau); W_2^2(\Omega)\|^2)d\tau$$

$$+ \int_0^t \|E_{\boldsymbol{\beta}}^{(k)}\nabla\widetilde{p}(\cdot,\tau); L_2(\Omega)\|^2 d\tau \le cB_2^{(k)}, \ \ \forall t \in [0,2\pi], \qquad (4.26)$$

where

$$B_2^{(k)} = (A_5^{(k)} + A_1 + A_1^2)(1 + e^{(B_1+A_1)^2}(A_1+B_1)^2), \qquad (4.27)$$

$$A_5^{(k)} = \int_0^{2\pi} \|E_{\boldsymbol{\beta}}^{(k)}\widehat{\mathbf{f}}(\cdot,t); L_2(\Omega)\|^2 dt.$$

The constant c in (4.26) does not depend on $t \in [0,2\pi]$ and k.

Proof. Consider the solution $\mathbf{v}(x,t)$ of problem (4.8) as a solution of the linear problem (4.23). Then there holds the estimate (see (3.17))

$$\int_0^t (\|E_{\boldsymbol{\beta}}^{(k)}\mathbf{v}_\tau(\cdot,\tau); L_2(\Omega)\|^2 + \|E_{\boldsymbol{\beta}}^{(k)}\mathbf{v}(\cdot,\tau); W_2^2(\Omega)\|^2)d\tau$$

$$+\int_0^t \|E_{\boldsymbol{\beta}}^{(k)}\nabla\widetilde{p}(\cdot,\tau);L_2(\Omega)\|^2 d\tau$$

$$\leq c\int_0^t \|E_{\boldsymbol{\beta}}^{(k)}\mathbf{g}(\cdot,\tau;\mathbf{v}(\cdot,\tau));L_2(\Omega)\|^2 d\tau \tag{4.28}$$

with the constant c independent of k. According to Lemma 1.1, $E_{\boldsymbol{\beta}}\mathbf{v}(\cdot,t)\in W_2^1(\Omega)$ and

$$\|E_{\boldsymbol{\beta}}^{(k)}\mathbf{v}(\cdot,t);W_2^1(\Omega)\|^2$$

$$\leq c\int_0^t (\|E_{\boldsymbol{\beta}}^{(k)}\mathbf{v}_\tau(\cdot,\tau);L_2(\Omega)\|^2+\|E_{\boldsymbol{\beta}}^{(k)}\mathbf{v}(\cdot,\tau);W_2^2(\Omega)\|^2)d\tau. \tag{4.29}$$

The constant in the (4.29) is independent of t and k. Relations (4.28), (4.29) together with inequalities (4.24), (4.25) yield

$$\int_\Omega E_{\boldsymbol{\beta}}^{(k)}(x)|\nabla\mathbf{v}(x,t)|^2dx+\int_0^t (\|E_{\boldsymbol{\beta}}^{(k)}\mathbf{v}_\tau(\cdot,\tau);L_2(\Omega)\|^2+\|E_{\boldsymbol{\beta}}^{(k)}\mathbf{v}(\cdot,\tau);W_2^2(\Omega)\|^2)d\tau$$

$$+\int_0^t \|E_{\boldsymbol{\beta}}^{(k)}\nabla\widetilde{p}(\cdot,\tau);L_2(\Omega)\|^2 d\tau$$

$$\leq c\int_0^t\int_\Omega E_{\boldsymbol{\beta}}^{(k)}(x)|\widetilde{\mathbf{f}}(x,\tau)|^2dxd\tau+c_*\varepsilon\int_0^t \|E_{\boldsymbol{\beta}}^{(k)}\mathbf{v}(\cdot,\tau);W_2^2(\Omega)\|^2d\tau$$

$$+c_\varepsilon B_1\int_0^t \|\nabla\mathbf{v}(\cdot,\tau);L_2(\Omega)\|\|E_{\boldsymbol{\beta}}^{(k)}\nabla\mathbf{v}(\cdot,\tau);L_2(\Omega)\|^2d\tau$$

$$+c_\varepsilon A_1\int_0^t Y(\tau)\|E_{\boldsymbol{\beta}}^{(k)}\nabla\mathbf{v}(\cdot,\tau);L_2(\Omega)\|^2d\tau.$$

Fixing $\varepsilon=1/2c_*$, from the last inequality we find

$$\int_\Omega E_{\boldsymbol{\beta}}^{(k)}(x)|\nabla\mathbf{v}(x,t)|^2dx+\int_0^t (\|E_{\boldsymbol{\beta}}^{(k)}\mathbf{v}(\cdot,\tau);L_2(\Omega)\|^2+\|E_{\boldsymbol{\beta}}^{(k)}\mathbf{v}(\cdot,\tau);W_2^2(\Omega)\|^2)d\tau$$

$$+\int_0^t \|E_{\boldsymbol{\beta}}^{(k)}\nabla\widetilde{p}(\cdot,\tau);L_2(\Omega)\|^2 d\tau \leq c\int_0^t\int_\Omega E_{\boldsymbol{\beta}}^{(k)}(x)|\widetilde{\mathbf{f}}(x,\tau)|^2 dxd\tau$$

$$+c(A_1+B_1)\int_0^t (\|\nabla\mathbf{v}(\cdot,\tau);L_2(\Omega)\|^2+Y(\tau))\|E_{\boldsymbol{\beta}}^{(k)}\nabla\mathbf{v}(\cdot,\tau);L_2(\Omega)\|^2 d\tau. \quad (4.30)$$

Omitting the second and the third terms on the left-hand side of (4.30), we also have

$$\int_\Omega E_{\boldsymbol{\beta}}^{(k)}(x)|\nabla\mathbf{v}(x,t)|^2 dx \leq c\int_0^t\int_\Omega E_{\boldsymbol{\beta}}^{(k)}(x)|\widetilde{\mathbf{f}}(x,\tau)|^2 dxd\tau$$

$$+c(A_1+B_1)\int_0^t (\|\nabla\mathbf{v}(\cdot,\tau);L_2(\Omega)\|^2+Y(\tau))\|E_{\boldsymbol{\beta}}^{(k)}\nabla\mathbf{v}(\cdot,\tau);L_2(\Omega)\|^2 d\tau. \quad (4.31)$$

Denote

$$\|\nabla\mathbf{v}(\cdot,t);L_2(\Omega)\|^2+Y(t)=Z(t), \int_0^t \|E_{\boldsymbol{\beta}}^{(k)}\nabla\mathbf{v}(\cdot,\tau);L_2(\Omega)\|^2 Z(\tau)d\tau = X(t).$$

Multiplying both sides of relation (4.31) by $Z(t)$, we receive the inequality

$$\frac{d}{dt}X(t) \leq c_1(A_1+B_1)Z(t)X(t)+c_2 Z(t)\Phi(t),$$

where

$$\Phi(t)=\int_0^t\int_\Omega E_{\boldsymbol{\beta}}^{(k)}(x)|\widetilde{\mathbf{f}}(x,\tau)|^2 dxd\tau.$$

Therefore,

$$\frac{d}{dt}\Big(X(t)e^{-c_1(A_1+B_1)\int_0^t Z(\tau)d\tau}\Big) \leq c_2 e^{-c_1(A_1+B_1)\int_0^t Z(\tau)d\tau} Z(t)\Phi(t). \qquad (4.32)$$

Integrate (4.32) over t:

$$X(t)\leq c_2 e^{c_1(A_1+B_1)\int_0^t Z(\tau)d\tau}\int_0^t e^{-c_1(A_1+B_1)\int_0^\tau Z(s)ds} Z(\tau)\Phi(\tau)d\tau$$

$$\leq c_2 e^{c_1(A_1+B_1)\int\limits_0^t Z(\tau)d\tau} \int\limits_0^t Z(\tau)\Phi(\tau)d\tau. \tag{4.33}$$

By the definition of $Z(t)$ and using inequalities (4.3), (4.9), (4.11), (4.15) we get

$$\int\limits_0^t Z(\tau)d\tau = \int\limits_0^t \|\nabla \mathbf{v}(\cdot,\tau); L_2(\Omega)\|^2 + \sum_{j=1}^J \int\limits_0^t \|\mathbf{U}^{(j)}(\cdot,\tau); W_2^2(\sigma_j)\|^2 d\tau$$

$$+ \int\limits_0^t \|\mathbf{W}(\cdot,\tau); W_2^2(\Omega_{(3)})\|^2 d\tau \leq c(A_1+B_1), \tag{4.34}$$

$$\int\limits_0^t Z(\tau)\Phi(\tau)d\tau \leq \int\limits_0^t \|E_{\boldsymbol{\beta}}^{(k)}\widetilde{\mathbf{f}}(\cdot,\tau); L_2(\Omega)\|^2 d\tau \int\limits_0^t Z(\tau)d\tau$$

$$\leq c(A_1+B_1)(A_5^{(k)} + A_1 + A_1^2), \tag{4.35}$$

where $\int\limits_0^{2\pi} \|E_{\boldsymbol{\beta}}^{(k)}\widehat{\mathbf{f}}(\cdot,\tau); L_2(\Omega)\|^2 d\tau = A_3^{(k)}$. From (4.33)–(4.35) follows the estimate

$$X(t) \leq ce^{c(A_1+B_1)^2}(A_1+B_1)(A_5^{(k)} + A_1 + A_1^2)$$

and, therefore, (4.30) furnishes

$$\int\limits_0^t \Big(\|E_{\boldsymbol{\beta}}^{(k)}\mathbf{v}_\tau(\cdot,\tau); L_2(\Omega)\|^2 + \|E_{\boldsymbol{\beta}}^{(k)}\mathbf{v}(\cdot,\tau); W_2^2(\Omega)\|^2\Big)d\tau$$

$$+ \int\limits_0^t \|E_{\boldsymbol{\beta}}^{(k)}\nabla\widetilde{p}(\cdot,\tau); L_2(\Omega)\|^2 d\tau$$

$$\leq c(A_5^{(k)} + A_1 + A_1^2)(1 + e^{(B_1+A_1)^2}(A_1+B_1)^2) \equiv B_2^{(k)}.$$

The lemma is proved.

Remark 4.1. *Taking $\beta_j = 0, j = 1, ..., J$, from (4.26) we get the inequality*

$$\int\limits_0^t (\|\mathbf{v}_\tau(\cdot,\tau); L_2(\Omega)\|^2 + \|\mathbf{v}(\cdot,\tau); W_2^2(\Omega)\|^2)d\tau$$

$$+\int_0^t \|\nabla\widetilde{p}(\cdot,\tau);L_2(\Omega)\|^2 d\tau \le cB_2, \ \ \forall t\in[0,2\pi], \tag{4.36}$$

where $B_2 = B_2^{(0)}$.

Remark 4.2. *Considerations of this section are similar to those of the paper [66].*

4.5 Solvability of problem (4.8)

Let $\Omega^{(l)}$ be a bounded domain with the boundary $\partial\Omega(l) \in C^2$ such that $\Omega_{(l+1)} \subset \Omega^{(l)} \subset \Omega_{(l+2)}$. Consider in $\Omega^{(l)}$ the following problem

$$\begin{cases} \mathbf{v}_t^{(l)}(x,t) - \nu\Delta\mathbf{v}^{(l)}(x,t) + (\mathbf{v}^{(l)}(x,t)\cdot\nabla)\mathbf{v}^{(l)}(x,t) \\ \qquad +(\mathbf{V}(x,t)\cdot\nabla)\mathbf{v}^{(l)}(x,t) + (\mathbf{v}^{(l)}(x,t)\cdot\nabla)\mathbf{V}(x,t) \\ \qquad\qquad +\nabla\widetilde{p}^{(l)}(x,t) = \widetilde{\mathbf{f}}(x,t), \\ \mathrm{div}\mathbf{v}^{(l)}(x,t) = 0, \\ \mathbf{v}^{(l)}(x,t)|_{\partial\Omega} = 0, \\ \mathbf{v}^{(l)}(x,0) = \mathbf{v}^{(l)}(x,2\pi). \end{cases} \tag{4.37}$$

Assuming that $\widetilde{\mathbf{f}}(x,t)$ is fixed, we can reduce problem (4.8) to an operator equation in the space $W_2^{2,1}(\Omega^{(l)}\times(0,2\pi))$:

$$\mathbf{v}^{(l)}(x,t) = \mathcal{S}^{-1}\mathcal{N}\mathbf{v}^{(l)}(x,t) = \mathcal{B}\mathbf{v}^{(l)}(x,t), \tag{4.38}$$

where $\mathcal{S}^{-1}: \mathbf{g}(x,t)\in L_2(\Omega^{(l)}\times(0,2\pi)) \to \mathbf{v}^{(l)}(x,t) \in W_2^{2,1}(\Omega^{(l)}\times(0,2\pi))$ is the bounded inverse operator of the linear time periodic Stokes problem

$$\begin{cases} \mathbf{v}_t^{(l)}(x,t) - \nu\Delta\mathbf{v}^{(l)}(x,t) + \nabla\widetilde{p}^{(l)}(x,t) = \mathbf{g}(x,t), \\ \mathrm{div}\mathbf{v}^{(l)}(x,t) = 0, \\ \mathbf{v}^{(l)}(x,t)|_{\partial\Omega} = 0, \\ \mathbf{v}^{(l)}(x,0) = \mathbf{v}^{(l)}(x,2\pi), \end{cases} \tag{4.39}$$

and

$$\begin{aligned} \mathcal{N}\mathbf{v}^{(l)}(x,t) = \widetilde{\mathbf{f}}(x,t) - (\mathbf{v}^{(l)}(x,t)\cdot\nabla)\mathbf{v}^{(l)}(x,t) - \\ -(\mathbf{V}(x,t)\cdot\nabla)\mathbf{v}^{(l)}(x,t) - (\mathbf{v}^{(l)}(x,t)\cdot\nabla)\mathbf{V}(x,t). \end{aligned} \tag{4.40}$$

Lemma 4.5. *For any bounded domain $\Omega^{(l)}$ the operator $\mathcal{N} : W_2^{2,1}(\Omega^{(l)}\times(0,2\pi)) \to L_2(\Omega^{(l)} \times (0,2\pi))$ is compact.*

The proof of this lemma could be found in [66].

Theorem 4.1. *For any $\widetilde{\mathbf{f}} \in L_2(\Omega^{(l)} \times (0,2\pi))$, problem (4.37) admits at least one solution $(\mathbf{v}^{(l)}(x,t), p^{(l)}(x,t))$ satisfying the estimate*

$$\|\mathbf{v}^{(l)}; W_2^{2,1}(\Omega^{(l)} \times (0,2\pi))\|^2 + \|\nabla p; L_2(\Omega^{(l)} \times (0,2\pi))\|^2 \leq cB_2. \tag{4.41}$$

The number B_2 is defined in (4.36) and the constant c in (4.41) is independent of l.

Proof. The solvability of problem (4.37) is equivalent to the solvability of operator equation (4.38). Since the operator $\mathcal{S}^{-1}$ is bounded and the operator $\mathcal{N}$ is compact, we conclude that $\mathcal{B}$ is compact and, according to Leray–Schauder theorem, (4.38) has at least one solution, if the norms $\|\mathbf{v}^{(l,\lambda)}; W_2^{2,1}(\Omega^{(l)} \times (0,2\pi))\|$ of all possible solutions to the equation

$$\mathbf{v}^{(l,\lambda)}(x,t) = \lambda\mathcal{B}\mathbf{v}^{(l,\lambda)}(x,t), \quad \lambda \in [0,1], \tag{4.42}$$

are bounded by the same constant independent of λ. Now repeating all considerations of Section 4.4, we find that for any solutions $\mathbf{v}^{(l,\lambda)}(x,t) \in W_2^{2,1}(\Omega^{(l)} \times$

$(0, 2\pi))$ of the problem

$$\begin{cases} \mathbf{v}_t^{(l,\lambda)}(x,t) - \nu\Delta\mathbf{v}^{(l,\lambda)}(x,t) + \lambda(\mathbf{v}^{(l,\lambda)}(x,t)\cdot\nabla)\mathbf{v}^{(l,\lambda)}(x,t) + \\ \lambda(\mathbf{V}(x,t)\cdot\nabla)\mathbf{v}^{(l,\lambda)}(x,t) + \lambda(\mathbf{v}^{(l,\lambda)}(x,t)\cdot\nabla)\mathbf{V}(x,t) + \nabla\widetilde{p}(x,t) = \lambda\widetilde{\mathbf{f}}(x,t), \\ \mathrm{div}\mathbf{v}^{(l,\lambda)}(x,t) = 0, \\ \mathbf{v}^{(l,\lambda)}(x,t)|_{\partial\Omega} = 0, \\ \mathbf{v}^{(l,\lambda)}(x,0) = \mathbf{v}^{(l,\lambda)}(x,2\pi), \end{cases}$$

which is equivalent to operator equation (4.42), holds the estimate

$$\|\mathbf{v}^{(l,\lambda)}; W_2^{2,1}(\Omega^{(l)} \times (0,2\pi))\|^2 + \|\nabla p^{(l,\lambda)}; L_2(\Omega^{(l)} \times (0,2\pi))\|^2 \le cB_2,$$

(for the definition of B_2 see remark (2.1)). It is easy also to verify that the constant in this estimate does not depend on λ. Thus, by Leray-Schauder theorem, operator equation (4.38) admits at least one solution $(\mathbf{v}^{(l)}(x,t), p^{(l)}(x,t))$ and there holds estimate (4.41). The theorem is proved.

Theorem 4.2. *Let* $\partial\Omega \in C^2, F_j(t) \in W_2^1(0,2\pi)$*, and let the external force* $\mathbf{f}(x,t)$ *are represent in the form (4.2) with*

$$f_2^{(j)}(x,t) \in L_2(\sigma_j \times (0,2\pi)), \;\; \widehat{\mathbf{f}}(x,t) \in \mathcal{L}_{2,\boldsymbol{\beta}}(\Omega \times (0,2\pi)), \quad j = 1,...,J.$$

Let the number γ_* *in* (1.3_3) *is sufficiently small and let there the condition (4.14) holds. Then problem (4.8) admits at least one solution* $(\mathbf{v}(x,t) \in \mathcal{W}_{2,\boldsymbol{\beta}}^{2,1}(\Omega \times (0,2\pi)), \nabla\widetilde{p}(x,t) \in \mathcal{L}_{2,\boldsymbol{\beta}}(\Omega \times (0,2\pi)))$*. There holds the estimate*

$$\|\mathbf{v}; \mathcal{W}_{2,\boldsymbol{\beta}}^{2,1}(\Omega \times (0,2\pi))\|^2 + \|\nabla\widetilde{p}; \mathcal{L}_{2,\boldsymbol{\beta}}(\Omega \times (0,2\pi))\|^2 \le cB_3, \tag{4.43}$$

where $B_3 = \lim_{k\to\infty} B_2^{(k)}$.

Proof. Let us consider the solutions $\mathbf{v}(x,t) \in W_2^{2,1}(\Omega^{(l)} \times (0,2\pi))$ of problem (4.37) in bounded domains $\Omega^{(l)}$. By Lemma 1.1, for almost $t \in [0,2\pi]$ holds the inclusions $\mathbf{v}^{(l)}(\cdot,t)\in W_2^1(\Omega^{(l)})$ and

$$\sup_{t\in[0,2\pi]}\|\mathbf{v}^{(l)}(\cdot,t);W_2^1(\Omega^{(l)})\|^2\le c\|\mathbf{v}^{(l)};W_2^{2,1}(\Omega^{(l)}\times(0,2\pi))\|^2. \qquad (4.44)$$

The constant in the last inequality is independent of l. The functions $\mathbf{v}^{(l)}(x,t)$ can be considered as elements of space $\mathring{W}_2^{1,1}(\Omega^{(l)}\times(0,2\pi))$. Extending them by zero to the whole domain Ω, we get a sequence $\{\mathbf{v}^{(l)}(x,t)\}\subset W_2^{1,1}(\Omega\times(0,2\pi))$ such that $\{\mathbf{v}^{(l)}(\cdot,t)\}\subset\mathring{W}_2^{1,1}(\Omega)$ for all $t\in[0,2\pi]$. In virtue of (4.44), (4.41),

$$\sup_{t\in[0,2\pi]}\|\mathbf{v}^{(l)}(\cdot,t);W_2^1(\Omega)\|^2+\|\mathbf{v}^{(l)}(\cdot,t);W_2^{1,1}(\Omega\times(0,2\pi))\|^2\le cB_2. \quad (4.45)$$

The constant c in (4.45) is independent of l. Therefore, there exists a subsequence $\{\mathbf{v}^{(l_m)}(x,t)\}$ which converges weakly in $W_2^{1,1}(\Omega\times(0,2\pi))$ to some $\mathbf{v}(x,t)\in W_2^{1,1}(\Omega\times(0,2\pi))$ and for $\mathbf{v}(x,t)$ remains valid estimate (4.45). Each $\mathbf{v}^{(l_m)}(x,t)$ satisfies the integral identity

$$\int_0^t\int_\Omega \mathbf{v}_\tau^{(l_m)}(x,\tau)\cdot\boldsymbol{\eta}(x,\tau)dxd\tau+\nu\int_0^t\int_\Omega\nabla\mathbf{v}_\tau^{(l_m)}(x,\tau)\cdot\nabla\boldsymbol{\eta}(x,\tau)dxd\tau$$

$$+\int_0^t\int_\Omega((\mathbf{V}(x,\tau)+\mathbf{v}^{(l_m)}(x,\tau))\cdot\nabla)\mathbf{v}^{(l_m)}(x,\tau)\cdot\boldsymbol{\eta}(x,\tau)dxd\tau$$

$$+\int_0^t\int_\Omega(\mathbf{v}^{(l_m)}(x,\tau)\cdot\nabla)\mathbf{V}(x,\tau)\cdot\boldsymbol{\eta}(x,\tau)dxd\tau$$

$$=\int_0^t\int_\Omega\widetilde{\mathbf{f}}^{(l_m)}(x,\tau)\cdot\boldsymbol{\eta}(x,\tau)dxd\tau,\ \ \forall t\in[0,2\pi], \qquad (4.46)$$

for any divergence free $\boldsymbol{\eta}(\cdot,t)\in W_2^{1,0}(\Omega)$ with $\mathrm{supp}_x\boldsymbol{\eta}(x,t)\subset\Omega^{(l_m)}$. Let us fix $\boldsymbol{\eta}(x,t)$ with compact support and pass in (4.46) to a limit as $l_m\to\infty$. This gives

$$\int_0^t\int_\Omega\mathbf{v}_\tau(x,\tau)\cdot\boldsymbol{\eta}(x,\tau)dxd\tau+\nu\int_0^t\int_\Omega\nabla\mathbf{v}(x,\tau)\cdot\nabla\boldsymbol{\eta}(x,\tau)dxd\tau$$

$$+\int\limits_0^t\int\limits_\Omega((\mathbf{V}(x,\tau)+\mathbf{v}(x,\tau))\cdot\nabla)\mathbf{v}(x,\tau)\cdot\boldsymbol{\eta}(x,\tau)dxd\tau$$

$$+\int\limits_0^t\int\limits_\Omega(\mathbf{v}(x,\tau)\cdot\nabla)\mathbf{V}(x,\tau)\cdot\boldsymbol{\eta}(x,\tau)dxd\tau$$

$$=\int\limits_0^t\int\limits_\Omega\widetilde{\mathbf{f}}(x,\tau)\cdot\boldsymbol{\eta}(x,\tau)dxd\tau,\ \ \forall t\in[0,2\pi]. \tag{4.47}$$

Let us show that (4.47) remains valid for any divergence free $\boldsymbol{\eta}(x,t)\in\mathring{W}_2^{1,0}(\Omega\times(0,2\pi))$. We have

$$\Big|\int\limits_0^t\int\limits_\Omega(\mathbf{v}(x,\tau)\cdot\nabla)\mathbf{v}(x,\tau)\cdot\boldsymbol{\eta}(x,\tau)dxd\tau\Big|$$

$$\leq\int\limits_0^t\|\mathbf{v}(\cdot,\tau);L_4(\Omega)\|\|\nabla\mathbf{v}(\cdot,\tau);L_2(\Omega)\|\|\boldsymbol{\eta}(\cdot,\tau);L_4(\Omega)\|d\tau$$

$$\leq c\int\limits_0^t\|\nabla\mathbf{v}(\cdot,\tau);L_2(\Omega)\|^2\|\nabla\boldsymbol{\eta}(\cdot,\tau);L_2(\Omega)\|d\tau$$

$$\leq c\sup_{t\in[0,2\pi]}\|\nabla\mathbf{v}(\cdot,\tau);L_2(\Omega)\|\int\limits_0^t(\|\nabla\mathbf{v}(\cdot,\tau);L_2(\Omega)\|^2+\|\nabla\boldsymbol{\eta}(\cdot,\tau);L_2(\Omega)\|^2)d\tau.$$

Further,

$$\int\limits_\Omega|\mathbf{V}(x,t)|^2|\nabla\mathbf{v}(x,t)|^2dx\leq\sum_{j=1}^J\sum_{s=0}^\infty\int\limits_{\omega_{js}}|\mathbf{V}(x,t)|^2|\nabla\mathbf{v}(x,t)|^2dx$$

$$+\int\limits_{\Omega_{(3)}}|\mathbf{V}(x,t)|^2|\nabla\mathbf{v}(x,t)|^2dx\leq\sum_{j=1}^J\sum_{s=0}^\infty\sup_{x\in\overline{\omega}_{js}}|\mathbf{V}(x,t)|^2\int\limits_{\omega_{js}}|\nabla\mathbf{v}(x,t)|^2dx$$

$$+\sup_{x\in\overline{\Omega}_{(3)}}|\mathbf{V}(x,t)|^2\int\limits_{\Omega_{(3)}}|\nabla\mathbf{v}(x,t)|^2dx$$

$$\leq \sum_{j=1}^{J}\sum_{s=0}^{\infty}\|\mathbf{V}(\cdot,t);W_2^2(\omega_{js})\|^2\int\limits_{w_{js}}|\nabla\mathbf{v}(x,t)|^2dx$$

$$+c\|\mathbf{V}(\cdot,t);W_2^2(\Omega_{(3)})\|^2\int\limits_{\Omega_{(3)}}|\nabla\mathbf{v}(x,t)|^2dx\leq Y(t)\int\limits_{\Omega}|\nabla\mathbf{v}(x,t)|^2dx \quad (4.48)$$

and

$$\int\limits_{\Omega}|\mathbf{v}(x,t)|^2|\nabla\mathbf{V}(x,t)|^2dx\leq\sum_{j=1}^{J}\sum_{s=0}^{\infty}\|\mathbf{v}(\cdot,t);L_4(\omega_{js})\|^2\|\nabla\mathbf{V}(\cdot,t);L_4(w_{js})\|^2$$

$$+\|\mathbf{v}(\cdot,t);L_4(\Omega_{(3)})\|^2\|\nabla\mathbf{V}(\cdot,t);L_4(\Omega_{(3)})\|^2$$

$$\leq \sum_{j=1}^{J}\sum_{s=0}^{\infty}\|\mathbf{V}(\cdot,t);W_2^2(\omega_{js})\|^2\int\limits_{\omega_{js}}|\nabla\mathbf{v}(x,t)|^2dx$$

$$+c\|\mathbf{V}(\cdot,t);W_2^2(\Omega_{(3)})\|^2\int\limits_{\Omega_{(3)}}|\nabla\mathbf{v}(x,t)|^2dx\leq cY(t)\int\limits_{\Omega}|\nabla\mathbf{v}(x,t)|^2dx, \quad (4.49)$$

where $Y(t)$ is defined in Lemma 4.3. From (4.48) and (4.49) follow the estimates

$$\Big|\int\limits_0^t\int\limits_{\Omega}(\mathbf{V}(x,\tau)\cdot\nabla)\mathbf{v}(x,\tau)dxd\tau\Big|+\Big|\int\limits_0^t\int\limits_{\Omega}(\mathbf{v}(x,\tau)\cdot\nabla)\mathbf{V}(x,\tau)dxd\tau\Big|$$

$$\leq\frac{1}{2}\int\limits_0^t\int\limits_{\Omega}\Big(|\mathbf{V}(x,\tau)|^2|\nabla\mathbf{v}(x,\tau)|^2+|\mathbf{v}(x,\tau)|^2|\nabla\mathbf{V}(x,\tau)|^2\Big)dxd\tau$$

$$+\int\limits_0^t\int\limits_{\Omega}|\boldsymbol{\eta}(x,\tau)|^2dxd\tau\leq c\int\limits_0^t Y(t)\int\limits_{\Omega}\|\nabla\mathbf{v}(x,\tau)|^2+\int\limits_0^t\int\limits_{\Omega}|\boldsymbol{\eta}(x,\tau)|^2dxd\tau$$

$$\leq c\sup_{t\in[0,2\pi]}\|\nabla\mathbf{v}(\cdot,\tau);L_2(\Omega)\|^2\int\limits_0^t Y(\tau)d\tau+\int\limits_0^t\int\limits_{\Omega}|\boldsymbol{\eta}(x,\tau)|^2dxd\tau$$

$$\leq cA_1\sup_{t\in[0,2\pi]}\|\nabla\mathbf{v}(\cdot,\tau);L_2(\Omega)\|^2+\int\limits_0^t\int\limits_{\Omega}|\boldsymbol{\eta}(x,\tau)|^2dxd\tau.$$

Thus, for any $\boldsymbol{\eta}(x,t) \in \mathring{W}_2^{1,0}(\Omega \times (0,2\pi))$ and $\mathbf{v}(x,t)$ satisfying (4.45), all integrals in identity (4.47) are finite and, since smooth $\boldsymbol{\eta}(x,t)$ with compact supports are dense in $\mathring{W}_2^{1,0}(\Omega \times (0,2\pi))$, we conclude that (4.45) is valid for all $\boldsymbol{\eta}(x,t) \in \mathring{W}_2^{1,0}(\Omega \times (0,2\pi))$. From this we get that for almost all $t \in [0,2\pi]$ holds the integral inequality

$$\nu \int\limits_{\Omega} \nabla \mathbf{v}(x,\tau) \cdot \nabla \boldsymbol{\eta}(x) dx + \int\limits_{\Omega} (\mathbf{v}(x,\tau)) \cdot \nabla)\mathbf{v}(x,\tau) \cdot \boldsymbol{\eta}(x) dx$$

$$= \int\limits_{\Omega} \mathbf{H}(x,t) \cdot \boldsymbol{\eta}(x) dx, \ \ \forall \boldsymbol{\eta} \in W_2^1(\Omega),$$

where $\mathbf{H}(x,t) = \widetilde{\mathbf{f}}(x,t) - \mathbf{v}_t(x,t) - (\mathbf{V}(x,t)) \cdot \nabla)\mathbf{v}(x,t) - (\mathbf{v}(x,t)) \cdot \nabla)\mathbf{V}(x,t)$. Therefore, $\mathbf{v}(x,t) \in \mathring{W}_2^1(\Omega)$ can be considered as a weak solution of steady Navier-Stokes problem with the right-hand side $\mathbf{H}(x,t)$ and zero fluxes. Then for $\mathbf{v}(x,t)$ hold the estimate (see [27], estimate (36) in Chapter V)

$$\begin{aligned}
\|\mathbf{v}(\cdot,t); W_2^2(\omega_{js})\|^2 &\leq \big(\|\mathbf{v}(\cdot,t); W_2^1(\widehat{\omega}_{js})\|^8 + \|\mathbf{H}(\cdot,t); L_2(\widehat{\omega}_{js})\|^2\big) \\
&\leq c\big(\|\mathbf{v}(\cdot,t); W_2^1(\Omega)\|^6 \|\mathbf{v}(\cdot,t); W_2^1(\widehat{\omega}_{js})\|^2 + \|\widetilde{\mathbf{f}}(\cdot,t); L_2(\widehat{\omega}_{js})\|^2 \\
&\quad + \|\mathbf{v}_t(\cdot,t); L_2(\widehat{\omega}_{js})\|^2 + \|(\mathbf{V}(\cdot,t) \cdot \nabla)\mathbf{v}(\cdot,t); L_2(\widehat{\omega}_{js})\|^2 \\
&\quad + \|(\mathbf{v}(\cdot,t) \cdot \nabla)\mathbf{V}(\cdot,t); L_2(\widehat{\omega}_{js})\|^2\big).
\end{aligned}$$

Summing these relations over s and using (4.48) and (4.49), we get

$$\begin{aligned}
&\|\mathbf{v}(\cdot,t); W_2^2(\Omega_j)\|^2 \leq c\Bigg(\|\mathbf{v}(\cdot,t); W_2^1(\Omega)\|^6 \|\mathbf{v}(\cdot,t); W_2^1(\Omega_j)\|^2 \\
&+ \|\widetilde{\mathbf{f}}(\cdot,t); L_2(\Omega_j)\|^2 + \|\mathbf{v}_t(\cdot,t); L_2(\Omega_j)\|^2 + \|(\mathbf{V}(\cdot,t) \cdot \nabla)\mathbf{v}(\cdot,t); L_2(\Omega_j)\|^2 \\
&+ \|(\mathbf{v}(\cdot,t) \cdot \nabla)\mathbf{V}(\cdot,t); L_2(\Omega_j)\|^2\Bigg) \leq c\Bigg(\|\mathbf{v}(\cdot,t); W_2^1(\Omega)\|^6 \|\mathbf{v}(\cdot,t); W_2^1(\Omega)\|^2 \\
&+ \|\widetilde{\mathbf{f}}(\cdot,t); L_2(\Omega)\|^2 + \|\mathbf{v}_t(\cdot,t); L_2(\Omega)\|^2 + Y(t)\|\nabla \mathbf{v}(\cdot,t); L_2(\Omega)\|^2\Bigg).
\end{aligned}$$

Integration of the last inequality with respect to t yields in virtue of (4.45)

$$\int\limits_0^{2\pi} \|\mathbf{v}(\cdot,t);W_2^2(\Omega_j)\|^2dt \leq \Big(c \sup_{t\in[0,2\pi]} \|\mathbf{v}(\cdot,t);W_2^1(\Omega)\|^6 \int\limits_0^{2\pi} \|\mathbf{v}(\cdot,t);W_2^1(\Omega)\|^2dt$$

$$+ \int\limits_0^{2\pi} \|\widetilde{\mathbf{f}}(\cdot,t);L_2(\Omega)\|^2dt + \int\limits_0^{2\pi} \|\mathbf{v}_t(\cdot,t);L_2(\Omega)\|^2dt$$

$$+c \sup_{t\in[0,2\pi]} \|\mathbf{v}(\cdot,t);W_2^1(\Omega)\|^2 \int\limits_0^{2\pi} Y(t)dt\Big) \leq c(A_1^2+B_1^2)$$

$$+ \int\limits_0^{2\pi} \|\widetilde{\mathbf{f}}(\cdot,t);L_2(\Omega)\|^2dt.$$

Analogously, we get that

$$\int\limits_0^{2\pi} \|\mathbf{v}(\cdot,t);W_2^2(\Omega_{(3)})\|^2dt \leq c(A_1^2+B_1^2) + \int\limits_0^{2\pi} \|\widetilde{\mathbf{f}}(\cdot,t);L_2(\Omega)\|^2dt.$$

Thus, $\mathbf{v}(x,t) \in W_2^{2,1}(\Omega\times(0,2\pi))$ and from (4.48) we conclude that equations (4.8) are satisfied almost everywhere in $\Omega\times(0,2\pi)$. By Lemma 3.4, $\mathbf{v}(x,t)$ obeys estimate (4.26), i. e,

$$\int\limits_0^{2\pi} \Big(\|E_{\boldsymbol{\beta}}^{(k)}\mathbf{v}_\tau(\cdot,\tau);L_2(\Omega)\|^2 + \|E_{\boldsymbol{\beta}}^{(k)}\mathbf{v}(\cdot,\tau);W_2^2(\Omega)\|^2\Big)d\tau$$

$$+ \int\limits_0^{2\pi} \|E_{\boldsymbol{\beta}}^{(k)}\nabla\widetilde{p}(\cdot,\tau);L_2(\Omega)\|^2 \leq cB_2^{(k)}.$$

The constant c is independent of k. Since $\widehat{\mathbf{f}}(x,t) \in \mathcal{L}_{2,\boldsymbol{\beta}}(\Omega\times(0,2\pi))$,

$$\int\limits_0^{2\pi} \|E_{\boldsymbol{\beta}}^{(k)}\widehat{\mathbf{f}}(\cdot,t);L_2(\Omega)\|^2dt \leq \|\widehat{\mathbf{f}};\mathcal{L}_{2,\boldsymbol{\beta}}(\Omega\times(0,2\pi))\|^2 = A_3$$

and

$$\int_0^{2\pi} \Big(\|E_{\boldsymbol\beta}^{(k)} \mathbf{v}_\tau(\cdot,\tau); L_2(\Omega)\|^2 + \|E_{\boldsymbol\beta}^{(k)} \mathbf{v}(\cdot,\tau); W_2^2(\Omega)\|^2 \Big) d\tau$$

$$+ \int_0^{2\pi} \|E_{\boldsymbol\beta}^{(k)} \nabla \widetilde{p}(\cdot,\tau); L_2(\Omega)\|^2 \leq cB_3, \qquad (4.50)$$

with $B_3 = \lim_{k\to\infty} B_2^{(k)}$. Now we can let $k \to \infty$ in (4.44) and, thus, we obtain (4.43). The theorem is proved.

5

Three-dimensional time periodic Navier-Stokes problem

5.1 Formulation of the problem

In this chapter we shall study (4.1) problem in three-dimensional domain $\Omega \in \mathbb{R}^3$. Note that now $\sigma_j \in \mathbb{R}^2$ is a bounded domain.

Assume that external force $\mathbf{f}(x,t)$ is represented in the form

$$\mathbf{f}(x,t) = \sum_{j=1}^{J} \zeta(x_3^{(j)})\mathbf{f}^{(j)}(x^{(j)\prime},t) + \widehat{\mathbf{f}}(x,t), \tag{5.1}$$

where

$$\mathbf{f}^{(j)}(x^{(j)\prime},t) = \Big(f_1^{(j)}(x^{(j)\prime},t), f_2^{(j)}(x^{(j)\prime},t), f_3^{(j)}(x^{(j)\prime},t)\Big),$$

$\zeta(\tau)$ is a smooth cut-off function with $\zeta(\tau) = 0$, for $\tau \leq 1$ and $\zeta(\tau) = 1$, for $\tau \geq 2$, $\mathbf{f}^{(j)}(x,t) \in W_2^{2,1}(\sigma_j \times (0,2\pi))$; $\widehat{\mathbf{f}}(x,t) \in \mathcal{W}^{2,1}_{2,\boldsymbol{\beta}}(\Omega \times (0,2\pi))$. Moreover, let $F_j(t) \in W_2^1(0,2\pi)$ and also $F_j(0) = F_j(2\pi), j = 1,...,J$. Then in each cylinder $\prod_j = \sigma_j \times (0,2\pi)$ there exits a generalized Poiseuille solution $(\mathbf{U}^{(j)}(x^{(j)\prime},t), P^{(j)}(x^{(j)},t))$ having the form (3.34) and satisfying estimate (3.36). Let us define $\mathbf{V}(x,t)$ by formula (3.39) while $\mathbf{U}(x,t)$ and $P(x,t)$ are given by (3.37). Consider Navier-Stokes problem (4.1). Looking for the solution

$(\mathbf{u}(x,t), p(x,t))$ of (4.1) in the form

$$\mathbf{u}(x,t) = \mathbf{v}(x,t) + \mathbf{V}(x,t), \ \ p(x,t) = \widetilde{p}(x,t) + P(x,t), \tag{5.2}$$

we obtain for $(\mathbf{v}(x,t), \widetilde{p}(x,t))$ three-dimensional problem

$$\begin{cases} \mathbf{v}_t(x,t) - \nu\Delta\mathbf{v}(x,t) + (\mathbf{v}(x,t)\cdot\nabla)\mathbf{v}(x,t) + (\mathbf{V}(x,t)\cdot\nabla)\mathbf{v}(x,t) \\ \qquad + (\mathbf{v}(x,t)\cdot\nabla)\mathbf{V}(x,t) + \nabla\widetilde{p}(x,t) = \widetilde{\mathbf{f}}(x,t), \\ \mathrm{div}\mathbf{v} = 0, \\ \mathbf{v}(x,t)|_{\partial\Omega} = 0, \\ \mathbf{v}(x,0) = \mathbf{v}(x,2\pi), \\ \int\limits_{\sigma_j} \mathbf{v}(x,t)\cdot\mathbf{n}(x)ds = 0, \ \ j = 1,...,J, \end{cases} \tag{5.3}$$

with $\widetilde{\mathbf{f}}(x,t) = \widehat{\mathbf{f}}(x,t) + \mathbf{f}_{(1)}(x,t) + \mathbf{f}_{(2)}(x,t)$,

$$\mathbf{f}_{(1)}(x,t) = (\mathbf{f}^{(1)'}(x,t), f_3^{(1)}(x,t)),$$

$$\mathbf{f}'_{(1)}(x,t) = \sum_{j=1}^{J}\Bigg(\nu\zeta''(x_3^{(j)})\mathbf{U}^{(j)'}(x^{(j)'},t)$$
$$-\zeta(x_3^{(j)})\zeta'(x_3^{(j)})U_3^{(j)}(x^{(j)'},t)\mathbf{U}^{(j)'}(x^{(j)'},t)$$
$$-\zeta'(x_3^{(j)})(\zeta'(x_3^{(j)})-1)(\mathbf{U}^{(j)'}(x^{(j)'},t)\cdot\nabla')\mathbf{U}^{(j)'}(x^{(j)'},t)\Bigg),$$

$$f_{3(1)}(x,t) = \sum_{j=1}^{J}\Bigg(\nu\zeta''(x_3^{(j)})U_3^{(j)'}(x^{(j)'},t)$$
$$-\zeta(x_3^{(j)})\zeta'(x_3^{(j)})|U_3^{(j)}(x^{(j)'},t)|^2$$
$$-\zeta'(x_3^{(j)})(\zeta'(x_3^{(j)})-1)(\mathbf{U}^{(j)'}(x^{(j)'},t)\cdot\nabla')U_3^{(j)}(x^{(j)'},t)-$$

$$-\zeta'(x_3^{(j)})x_3^{(j)}q^{(j)}(t)\Big),$$

$$\mathbf{f}_{(2)} = -\mathbf{W}_t(x,t) + \nu\Delta\mathbf{W}(x,t) - (\mathbf{W}(x,t)\cdot\nabla)\mathbf{W}(x,t)$$
$$-(\mathbf{U}(x,t)\cdot\nabla)\mathbf{W}(x,t) - (\mathbf{W}(x,t)\cdot\nabla)\mathbf{U}(x,t).$$

Using Sobolev embedding inequalities, estimate (3.36) for $\mathbf{U}^{(j)}(x^{(j)\prime},t)$ and estimate (3.38) for $\mathbf{W}(x,t)$, we obtain the inequalities

$$\int_0^t\int_\Omega |\mathbf{f}_{(1)}(x,\tau)|^2dxd\tau \le c\sum_{j=1}^J\int_0^t\int_{\sigma_j}(|\mathbf{U}^{(j)}(x^{(j)\prime},\tau)|^2 + |\mathbf{U}^{(j)}(x^{(j)\prime},\tau)|^4$$
$$+|\mathbf{U}^{(j)}(x^{(j)\prime},\tau)|^2|\nabla'\mathbf{U}^{(j)}(x^{(j)\prime},\tau)|^2 + |q^{(j)}(\tau)|^2)dxd\tau$$
$$\le c\sum_{j=1}^J\int_0^t\int_{\sigma_j}(|\mathbf{U}^{(j)}(x^{(j)\prime},\tau)|^2 + |q^{(j)}(\tau)|^2dx^{(j)\prime}d\tau$$
$$+c\sum_{j=1}^J\int_0^t\sup_{x^{(j)\prime}\in\overline{\sigma}_j}(|\mathbf{U}^{(j)}(x^{(j)\prime},\tau)|^2)\|\mathbf{U}^{(j)}(\cdot,\tau);W_2^1(\sigma_j)\|^2d\tau$$
$$\le (E_0+E_1) + \sum_{j=1}^J\sup_{t\in[0,2\pi]}(\|\mathbf{U}^{(j)}(\cdot,\tau);W_2^1(\sigma_j)\|^2)\int_0^t\|\mathbf{U}^{(j)}(\cdot,\tau);W_2^2(\sigma_j)\|^2d\tau$$
$$\le c(E_0+E_1)(1+E_0+E_1) := cE_2,$$

$$\int_0^t\int_\Omega |\mathbf{f}_{(2)}(x,\tau)|^2dxd\tau \le c\int_0^t\int_{\Omega_{(3)}}\Big(|\mathbf{W}_t(x,\tau)|^2 + |\Delta\mathbf{W}(x,\tau)|^2$$
$$+\Big(|\mathbf{W}(x,\tau)|^2 + |\mathbf{U}(x,\tau)|^2\Big)|\nabla\mathbf{W}(x,\tau)|^2 + |\mathbf{W}(x,\tau)|^2|\nabla\mathbf{U}(x,\tau)|^2\Big)dxd\tau$$
$$\le cE_1 + c\int_0^t\Big\{\sup_{x\in\overline{\Omega}_{(3)}}\Big(|\mathbf{W}(x,\tau)|^2 + |\mathbf{U}(x,\tau)|^2\Big)\int_{\Omega_{(3)}}(|\nabla\mathbf{W}|^2 + |\nabla\mathbf{U}|^2)dx\Big\}d\tau$$
$$\le cE_1 + c\sup_{t\in[0,2\pi]}\Big[\|\mathbf{W}(\cdot,t);W_2^1(\Omega_{(3)})\|^2 + \sum_{j=1}^J\|\mathbf{U}^{(j)}(\cdot,t);W_2^1(\sigma_j)\|^2\Big]$$

$$\times \int_0^t \left(\|\mathbf{W}(\cdot,\tau); W_2^2(\Omega_{(3)})\|^2 + \sum_{j=1}^J \|\mathbf{U}^{(j)}(\cdot,t); W_2^1(\sigma_j)\|^2 \right) d\tau \le cE_2. \quad (5.4)$$

We denote $E_0 = \sum_{j=1}^J E_0^{(j)}$, where $E_0^{(j)} = \|\mathbf{U}^{(j)\prime}; W_2^{2,1}(\sigma \times (0,2\pi))\|^2$,

$$E_1 = \sum_{j=1}^J (\|F_j; W_2^1(0,2\pi)\|^2 + \|f_3^{(j)}; W_2^1(\sigma_j \times (0,2\pi))\|^2).$$

5.2 Estimates of nonlinear terms

Lemma 5.1. *Let* $\mathbf{v}(x,t) \in \mathcal{W}^{2,1}_{2,\boldsymbol{\beta}}, \beta_j \ge 0, j = 1,...,J$. *Then* $(\mathbf{v}(x,t)\cdot\nabla)\mathbf{v}(x,t) \in \mathcal{L}_{2,\boldsymbol{\beta}}(\Omega \times (0,2\pi))$ *and*

$$\int_0^t \|(\mathbf{v}(\cdot,\tau)\cdot\nabla)\mathbf{v}(\cdot,\tau); \mathcal{L}_{2,\boldsymbol{\beta}}(\Omega)\|^2 d\tau$$

$$\le c\|\mathbf{v}; \mathcal{W}^{2,1}_{2,\boldsymbol{\beta}}(\Omega \times (0,t))\|^4, \ \ \forall t \in [0,2\pi], \quad (5.5)$$

where the constant c is independent of $t \in [0,2\pi]$.

Lemma 5.2. *Let* $\mathbf{u}(x,t), \mathbf{v}(x,t) \in \mathcal{W}^{2,1}_{2,\boldsymbol{\beta}}(\Omega \times (0,2\pi)), \quad \beta_j \ge 0, \quad j = 1,...,J.$ *Then* $(\mathbf{u}(x,t)\cdot\nabla)\mathbf{v}(x,t) \in \mathcal{L}_{2,\boldsymbol{\beta}}(\Omega \times (0,2\pi))$, $(\mathbf{v}(x,t)\cdot\nabla)\mathbf{u}(x,t) \in \mathcal{L}_{2,\boldsymbol{\beta}}(\Omega \times (0,2\pi))$ *there holds the estimates*

$$\int_0^t \|(\mathbf{u}(\cdot,\tau)\cdot\nabla)\mathbf{v}(\cdot,\tau); \mathcal{L}_{2,\boldsymbol{\beta}}(\Omega)\|^2 d\tau \le c_\varepsilon \int_0^t \|\mathbf{u}; \mathcal{W}^{2,1}_{2,\boldsymbol{\beta}}(\Omega \times (0,\tau))\|^2 d\tau$$

$$+\varepsilon c\|\mathbf{v}; \mathcal{W}^{2,1}_{2,\boldsymbol{\beta}}(\Omega \times (0,t))\|^4 \|\mathbf{u}; \mathcal{W}^{2,1}_{2,\boldsymbol{\beta}}(\Omega \times (0,t))\|^2, \quad (5.6)$$

and

$$\int_0^t \|(\mathbf{u}(\cdot,\tau)\cdot\nabla)\mathbf{v}(\cdot,\tau); \mathcal{L}_{2,\boldsymbol{\beta}}(\Omega)\|^2 d\tau \le c_\varepsilon \int_0^t \|\mathbf{v}; \mathcal{W}^{2,1}_{2,\boldsymbol{\beta}}(\Omega \times (0,\tau))\|^2 d\tau$$

$$+\varepsilon c\|\mathbf{v}; \mathcal{W}^{2,1}_{2,\boldsymbol{\beta}}(\Omega \times (0,t))\|^4 \|\mathbf{u}; \mathcal{W}^{2,1}_{2,\boldsymbol{\beta}}(\Omega \times (0,t))\|^2, \quad (5.7)$$

Constants in (5.6), (5.7) are independent of $t \in [0,2\pi]$.

Lemma 5.3. *Let* $\mathbf{v}(x,t) \in \mathcal{W}^{2,1}_{2,\boldsymbol{\beta}}(\Omega \times (0,2\pi))$. *Then* $(\mathbf{V}(x,t)\cdot\nabla)\mathbf{v}(x,t) \in \mathcal{L}_{2,\boldsymbol{\beta}}(\Omega \times (0,2\pi))$, $(\mathbf{v}(x,t)\cdot\nabla)\mathbf{V}(x,t) \in \mathcal{L}_{2,\boldsymbol{\beta}}(\Omega \times (0,2\pi))$ *and*

$$\int\limits_0^t \|(\mathbf{V}(\cdot,\tau)\cdot\nabla)\mathbf{v}(\cdot,\tau);\mathcal{L}_{2,\boldsymbol{\beta}}(\Omega)\|^2 d\tau + \int\limits_0^t \|(\mathbf{v}(\cdot,\tau)\cdot\nabla)\mathbf{v}(\cdot,\tau);\mathcal{L}_{2,\boldsymbol{\beta}}(\Omega)\|^2 d\tau$$

$$\leq c(E_0+E_1)\|\mathbf{v};\mathcal{W}^{2,1}_{2,\boldsymbol{\beta}}(\Omega\times(0,t))\|^2, \quad \forall t\in[0,2\pi], \tag{5.8}$$

where the constant c *is independent of* $t \in [0,2\pi]$.

Lemma 5.4. *Let* $\mathbf{v}(x,t) \in \mathcal{W}^{2,1}_{2,\boldsymbol{\beta}}(\Omega \times (0,2\pi))$, $t \in [0,2\pi]$. *Then there holds the estimate*

$$\int\limits_0^t \|(\mathbf{V}(\cdot,\tau)\cdot\nabla)\mathbf{v}(\cdot,\tau);\mathcal{L}_{2,\boldsymbol{\beta}}(\Omega)\|^2 d\tau + \int\limits_0^t \|(\mathbf{v}(\cdot,\tau)\cdot\nabla)\mathbf{v}(\cdot,\tau);\mathcal{L}_{2,\boldsymbol{\beta}}(\Omega)\|^2 d\tau$$

$$\leq \varepsilon(E_0^2+E_1^2)\|\mathbf{v};\mathcal{W}^{2,1}_{2,\boldsymbol{\beta}}(\Omega\times(0,t))\|^2 + c_\varepsilon \int\limits_0^t \|\mathbf{v};\mathcal{W}^{2,1}_{2,\boldsymbol{\beta}}(\Omega\times(0,\tau))\|^2 d\tau, \tag{5.9}$$

where the constant c *is independent of* $t \in [0,2\pi]$.

The proof of lemmas 5.2–5.4 could be find in [66].

5.3 Solvability of (5.3) problem

Now we prove the main result of the chapter.

Theorem 5.1. *Let* $\partial\Omega \in C^2$, $F_j(t) \in W^1_2(0,2\pi)$, *and let the external force* $\mathbf{f}(x,t)$ *is represented in the form (5.1) with*

$$\widehat{\mathbf{f}}(x,t) \in \mathcal{L}_{2,\boldsymbol{\beta}}(\Omega\times(0,2\pi)), \;\; \mathbf{f}^{(j)}(x',t) \in L_2(\sigma_j\times(0,2\pi)), \;\; \beta_j \geq 0, \;\; j=1,...,J.$$

Moreover, assume that $F_j(0) = F_j(2\pi)$ *and that the number* γ_* *in inequality* (1.3_3) *for the weight function* $E_{\boldsymbol{\beta}}(x)$ *is sufficiently small. If*

$$c_2(E_0+E_1) < 1,$$

$$4c_1c_2(E_2+E_3) < \Big(1 - c_2(E_0+E_1)\Big)^2, \tag{5.10}$$

where E_0, E_1, E_2 are defined in section 5.1,

$$E_3 = \|\widehat{\mathbf{f}}; \mathcal{L}_{2,\boldsymbol{\beta}}(\Omega \times (0, 2\pi))\|^2,$$

and c_1, c_2 are absolute constants defined below, then the (5.3) problem admits at least one solution $\mathbf{v}(x,t) \in \mathcal{W}^{2,1}_{2,\boldsymbol{\beta}}(\Omega \times (0,2\pi))$, $\nabla\widetilde{p}(x,t) \in \mathcal{L}_{2,\boldsymbol{\beta}}(\Omega \times (0,2\pi))$. There holds the estimate

$$\|\mathbf{v}; \mathcal{W}^{2,1}_{2,\boldsymbol{\beta}}(\Omega \times (0, 2\pi))\|^2 + \|\nabla\widetilde{p} \in \mathcal{L}_{2,\boldsymbol{\beta}}(\Omega \times (0, 2))\|^2 \le c r_0, \tag{5.11}$$

where

$$r_0 = \frac{2\alpha_0}{1 - \alpha_1 + \sqrt{(1-\alpha_1)^2 - 4\alpha_0\alpha_2}}, \tag{5.12}$$

$$\alpha_0 = c_1(E_0 + E_1 + E_2 + E_3), \quad \alpha_1 = c_2(E_0 + E_1), \quad \alpha_2 = c_2.$$

Proof. The existence of the solution to problem (5.3) we prove using method of successive approximations. Let us consider problem (5.3) as linear time periodic Stokes problem

$$\begin{cases} \mathbf{v}_t(x,t) - \nu\Delta\mathbf{v}(x,t) + \nabla\widetilde{p}(x,t) = \mathbf{g}(x,t;\mathbf{v}(x,t)), \\ \mathrm{div}\mathbf{v}(x,t) = 0, \\ \mathbf{v}(x,t)|_{\partial\Omega} = 0, \ \ \mathbf{v}(x,0) = \mathbf{v}(x,2\pi), \\ \int\limits_{\sigma_j} \mathbf{v}(x,t)\cdot\mathbf{n}(x)ds = 0, \quad j = 1, ..., J, \end{cases} \tag{5.13}$$

with

$$\mathbf{g}(x,t;\mathbf{v}(x,t)) = \widetilde{\mathbf{f}}(x,t) - (\mathbf{v}(x,t)\cdot\nabla)\mathbf{v}(x,t) - (\mathbf{V}(x,t)\cdot\nabla)\mathbf{v}(x,t).$$
$$-(\mathbf{v}(x,t)\cdot\nabla)\mathbf{V}(x,t).$$

Let us put $\mathbf{v}^{(0)}(x,t) = 0$, $\widetilde{p}^0(x,t) = 0$, and define the successive approximations recurrently as solutions of linear problems

$$\begin{cases} \mathbf{v}_t^{(l+1)}(x,t) - \nu\Delta\mathbf{v}^{(l+1)}(x,t) + \nabla\widetilde{p}^{(l+1)}(x,t) = \mathbf{g}(x,t;\mathbf{v}^{(l)}(x,t)), \\ \operatorname{div}\mathbf{v}^{(l+1)}(x,t) = 0, \\ \mathbf{v}^{(l+1)}(x,t)|_{\partial\Omega} = 0, \ \ \mathbf{v}^{(l+1)}(x,0) = \mathbf{v}^{(l+1)}(x,2\pi), \\ \int\limits_{\sigma_j} \mathbf{v}^{(l+1)}(x,t)\cdot\mathbf{n}(x)ds = 0, \quad j = 1,...,J. \end{cases} \tag{5.14}$$

In virtue of (5.5), (5.6), (5.7), (5.8), (5.9) the right-hand side $\mathbf{g}(x,t;\mathbf{v}^{(l)}(x,t))$ admits the estimate

$$\|\mathbf{g}(x,t;\mathbf{v}^{(l)}(x,t));\mathcal{L}_{2,\boldsymbol{\beta}}(\Omega\times(0,2\pi))\|^2 \le c\|\widetilde{\mathbf{f}};\mathcal{L}_{2,\boldsymbol{\beta}}(\Omega\times(0,2\pi))\|^2$$
$$+c\Big(\|\mathbf{v}^{(l)};\mathcal{W}^{2,1}_{2,\boldsymbol{\beta}}(\Omega\times(0,2\pi))\|^4 + (E_0+E_1)\|\mathbf{v}^{(l)};\mathcal{W}^{2,1}_{2,\boldsymbol{\beta}}(\Omega\times(0,2\pi))\|^2\Big)$$
$$\le c\Bigg(\|\widehat{\mathbf{f}};\mathcal{L}_{2,\boldsymbol{\beta}}(\Omega\times(0,2\pi))\|^2 + \|\mathbf{f}_{(1)};\mathcal{L}_{2,\boldsymbol{\beta}}(\Omega\times(0,2\pi))\|^2$$
$$+\|\mathbf{f}_{(2)};\mathcal{L}_{2,\boldsymbol{\beta}}(\Omega\times(0,2\pi))\|^2\Bigg) + c\|\mathbf{v}^{(l)};\mathcal{W}^{2,1}_{2,\boldsymbol{\beta}}(\Omega\times(0,2\pi))\|^4$$
$$+c(E_0+E_1)\|\mathbf{v}^{(l)};\mathcal{W}^{2,1}_{2,\boldsymbol{\beta}}(\Omega\times(0,2\pi))\|^2 \le c(E_0+E_1+E_2+E_3)$$
$$+c\Big(\|\mathbf{v}^{(l)};\mathcal{W}^{2,1}_{2,\boldsymbol{\beta}}(\Omega\times(0,2\pi))\|^4 + (E_0+E_1)\|\mathbf{v}^{(l)};\mathcal{W}^{2,1}_{2,\boldsymbol{\beta}}(\Omega\times(0,2\pi))\|^2\Big)$$

Therefore, all approximations $(\mathbf{v}^{(l+1)}(x,t),\widetilde{p}^{(l+1)}(x,t))$ are well defined and satisfy the estimates

$$\|\mathbf{v}^{(l+1)}(x,t);\mathcal{W}^{2,1}_{2,\boldsymbol{\beta}}(\Omega\times(0,2\pi))\|^2 + \|\nabla\widetilde{p}^{(l+1)};\mathcal{L}_{2,\boldsymbol{\beta}}(\Omega\times(0,2\pi))\|^2$$
$$\le c\|\mathbf{g}(\cdot,\mathbf{v}^{(l)});\mathcal{L}_{2,\boldsymbol{\beta}}(\Omega\times(0,2\pi))\|^2$$
$$\le c_1(E_0+E_1+E_2+E_3) + c_2\Bigg(\|\mathbf{v}^{(l)};\mathcal{W}^{2,1}_{2,\boldsymbol{\beta}}(\Omega\times(0,2\pi))\|^4 +$$

$$(E_0 + E_1)\|\mathbf{v}^{(l)}; \mathcal{W}^{2,1}_{2,\boldsymbol{\beta}}(\Omega \times (0, 2\pi))\|^2 \Bigg)$$

$$= \alpha_0 + \alpha_1\|\mathbf{v}^{(l)}; \mathcal{W}^{2,1}_{2,\boldsymbol{\beta}}(\Omega \times (0, 2\pi))\|^2 + \alpha_2\|\mathbf{v}^{(l)}; \mathcal{W}^{2,1}_{2,\boldsymbol{\beta}}(\Omega \times (0, 2\pi))\|^4. \quad (5.15)$$

If

$$\alpha_1 < 1, \;\; 4\alpha_0\alpha_2 < (1 - \alpha_1)^2, \quad (5.16)$$

the quadratic equation $\alpha_2 y^2 + (\alpha_1 - 1)y + \alpha_0 = 0$ has positive roots minimal r_0 of which is given by formula (5.12). From (5.15), (5.16) it follows that, if $\|\mathbf{v}^{(l)}; \mathcal{W}^{2,1}_{2,\boldsymbol{\beta}}(\Omega \times (0, 2\pi))\|^2 \le r_0$, then also

$$\|\mathbf{v}^{(l+1)}(x,t); \mathcal{W}^{2,1}_{2,\boldsymbol{\beta}}(\Omega\times(0,2\pi))\|^2 + \|\nabla\widetilde{p}^{(l+1)}; \mathcal{L}_{2,\boldsymbol{\beta}}(\Omega\times(0,2\pi))\|^2 \le r_0. \quad (5.17)$$

Since obviously $\|\mathbf{v}^{(0)}; \mathcal{W}^{2,1}_{2,\boldsymbol{\beta}}(\Omega \times (0, 2\pi))\|^2 \le r_0$, we conclude that (5.17) is valid for $\forall l \ge 0$. Let us show now that the sequence $\{\mathbf{v}^{(l+1)}(x,t), \widetilde{p}^{(l+1)}(x,t)\}$ converges to the solution $(\mathbf{v}(x,t), \widetilde{p}(x,t))$ of problem (5.3). The differences

$$\mathbf{w}^{(l)}(x,t) = \mathbf{v}^{(l+1)}(x,t) - \mathbf{v}^{(l)}(x,t), \;\; q^{(l)}(x,t) = \widetilde{p}^{(l+1)}(x,t) - \widetilde{p}^{(l)}(x,t)$$

are the solutions of the following linear problems

$$\begin{cases} \mathbf{w}^{(l)}_t(x,t) - \nu\Delta\mathbf{w}^{(l)}(x,t) + \nabla q^{(l)}(x,t) = \mathbf{g}(x,t;\mathbf{v}^{(l+1)}) - \mathbf{g}(x,t;\mathbf{v}^{(l)}) \\ = -\mathbf{V}(x,t)\cdot\nabla)\mathbf{w}^{(l)}(x,t) - (\mathbf{w}^{(l)}(x,t)\cdot\nabla)\mathbf{V}(x,t) \\ -(\mathbf{v}(x,t)\cdot\nabla)\mathbf{V}(x,t) - (\mathbf{w}^{(l)}(x,t)\cdot\nabla)\mathbf{v}^{(l)}(x,t) - (\mathbf{v}^{(l)}(x,t)\cdot\nabla)\mathbf{w}^{(l)}(x,t), \\ \mathrm{div}\mathbf{w}(x,t) = 0, \\ \mathbf{w}(x,t)|_{\partial\Omega} = 0, \\ \mathbf{w}(x,0) = \mathbf{w}(x,2\pi), \\ \int\limits_{\sigma_j} \mathbf{w}(x,t)\cdot\mathbf{n}(x)ds = 0, \;\; j = 1,...,J. \end{cases}$$

Let $X^{(l+1)}(t) = \|\mathbf{w}^{(l+1)}; \mathcal{W}^{2,1}_{2,\boldsymbol{\beta}}(\Omega \times (0,t))\|^2 + \|\nabla q^{(l+1)}; \mathcal{L}_{2,\boldsymbol{\beta}}(\Omega \times (0,t))\|^2$. Using Lemmas 5.2, 5.4 and estimate (5.17), we derive the inequality

$$X^{(l+1)}(t) = c\|\mathbf{g}(\cdot;\mathbf{v}^{(l+1)}) - \mathbf{g}(\cdot;\mathbf{v}^{(l)}); \mathcal{L}_{2,\boldsymbol{\beta}}(\Omega \times (0,t))\|^2$$

$$\leq c\Bigg(\|(\mathbf{V}\cdot\nabla)\mathbf{w}^{(l)}; \mathcal{L}_{2,\boldsymbol{\beta}}(\Omega \times (0,t))\|^2 + \|(\mathbf{w}^{(l)}\cdot\nabla)\mathbf{V}; \mathcal{L}_{2,\boldsymbol{\beta}}(\Omega \times (0,t))\|^2$$

$$+\|(\mathbf{w}^{(l)}\cdot\nabla)\mathbf{v}^{(l)}; \mathcal{L}_{2,\boldsymbol{\beta}}(\Omega \times (0,t))\|^2 + \|(\mathbf{v}^{(l)}\cdot\nabla)\mathbf{w}^{(l)}; \mathcal{L}_{2,\boldsymbol{\beta}}(\Omega \times (0,t))\|^2\Bigg)$$

$$\leq \varepsilon c\Bigg(E_0^2 + E_1^2 + \|\mathbf{v}^{(l)}; \mathcal{W}^{2,1}_{2,\boldsymbol{\beta}}(\Omega \times (0,2\pi))\|^4\Bigg)\|\mathbf{w}^{(l)}; \mathcal{W}^{2,1}_{2,\boldsymbol{\beta}}(\Omega \times (0,t))\|^2$$

$$+c_\varepsilon\int_0^t \|\mathbf{w}^{(l)}; \mathcal{W}^{2,1}_{2,\boldsymbol{\beta}}(\Omega\times(0,\tau))\|^2 d\tau \leq \varepsilon c_*(E_0^2+E_1^2+r_0^2)\|\mathbf{w}^{(l)}; \mathcal{W}^{2,1}_{2,\boldsymbol{\beta}}(\Omega\times(0,t))\|^2$$

$$+c_\varepsilon\int_0^t \|\mathbf{w}^{(l)}; \mathcal{W}^{2,1}_{2,\boldsymbol{\beta}}(\Omega \times (0,t))\|^2 d\tau. \tag{5.18}$$

Let us fix $\varepsilon = \dfrac{1}{2c_*(E_0^2 + E_1^2 + r_0^2)}$ and sum relations (5.18) by l from 1 to M. This yields

$$\sum_{m=2}^{M+1} X^{(l)}(t) \leq \frac{1}{2}\sum_{m=1}^{M} X^{(l)}(t) + c\int_0^t \sum_{m=1}^{M} X^{(l)}(t)d\tau. \tag{5.19}$$

Setting $Y^{(M)}(t) = \sum\limits_{m=1}^{M} X^{(l)}(t)$, from (5.19) we get

$$Y^{(M)}(t) \leq 2X^{(1)}(t) + c\int_0^t Y^{(M)}(\tau)d\tau$$

and, by Gronwall inequality,

$$Y^{(M)}(t) \leq 2e^{ct}X^{(1)}(t).$$

Therefore, the series $\sum_{l=1}^{\infty} X^{(1)}(t)$ converges for any finite t and, consequently, $\{\mathbf{v}^{(l+1)}(x,t), \nabla\widetilde{p}^{(l+1)}(x,t)\}$ converges in the norm of the space $\mathcal{W}^{2,1}_{2,\boldsymbol{\beta}}(\Omega\times(0,2\pi))\times \mathcal{L}_{2,\boldsymbol{\beta}}(\Omega\times(0,2\pi))$ to the solution $(\mathbf{v}(x,t), \widetilde{p}(x,t))$ of problem (5.3). Obviously, for $(\mathbf{v}(x,t), \widetilde{p}(x,t))$ remains valid inequality (5.19).
The theorem is proved.

General conclusions

In the dissertation, the following results are established.

1. The existence and uniqueness of a non-stationary Poiseuille solution to Navier-Stokes system in a straight cylinder is proved.
2. The existence and uniqueness of a solution to the time periodic Stokes problem in domains with cylindrical outlets to infinity is proved in weighted Sobolev spaces.
3. The existence of a solution for the two-dimensional time periodic Navier-Stokes problem in domains with cylindrical outlets to infinity is proved in weighted Sobolev spaces. The results were obtained for "small" fluxes. However, the external forces could be "large".
4. The existence of a solution for the three-dimensional time periodic Navier-Stokes problem in domains with cylindrical outlets to infinity is proved in weighted Sobolev spaces. The results were obtained for all "small" data, including both fluxes and external forces.
5. It is shown that the obtained solutions in every outlet to infinity asymptotically tend as $|x| \to \infty$ to the corresponding time periodic Poiseuille solution.
6. From the obtained results, we conclude that in the case of "rapidly" vanishing external forces, the time periodic solution of Navier–Stokes equations

in outlets to infinity are almost coincident for a large $|x|$ with a corresponding time periodic Poiseuille solution. In particular, if there is no external force, then the time periodic solution of the Navier–Stokes problem exponentially tends towards the time periodic Poiseuille flow. Thus, in studying viscous fluid flow problems in complicated systems of pipes one can get the information for large distances solving much easier linear parabolic problems describing the time periodic Poiseuille flows in pipes.

References

[1] ABERGEL, F.; BONA, J. L. A. Mathematical theory for viscous free-surface flows over a perturbed plane. *Arch. Rational Mech. Anal.*, 1992, 118, p. 71–93.

[2] ADAMS, R. A. *Sobolev spaces*. New York, San Francisco, London: Academic press, 1975.

[3] AMICK, C. J. Steady solutions of the Navier-Stokes equations in unbounded channels and pipes. *Ann. Scuola Norm. Sup. Pisa Cl. Sci.*, 1977, 4(4), p. 473–513.

[4] AMICK, C. J. Properties of steady Navier-Stokes equations for certain unbounded channels and pipes. *Nonlinear analysis, Theory, Meth., Appl.*, 1978, 2, p. 689–720.

[5] AMICK, C. J.; FRAENKEL, L. E. Steady solutions of the Navier-Stokes equations representing plane flow in channels of various types. *Acta Math.*, 1980, 144, p. 81–152.

[6] AMBRAZEVIČIUS, A., DOMARKAS, A. *Matematinės fizikos lygtys 2*. Vilnius: Aldorija, 1999.

[7] BABIN, A. V. Asymptotic expansion at infinity of a strongly perturbed Poiseuille flow. *Advances in Soviet Mathematics*, 1992, 10, p. 1–83.

[8] BEIRAO DA VEIGA, H. On time–periodic solutions of the Navier–Stokes equations in an unbounded cylindrical domains. Leray's problem for periodic flows. *Arch. Rat. Mech. Anal.*, 2005, 178(3), p. 301–325.

[9] BORCHERS, W.; PILECKAS, K. Existence, uniqueness and asymptotics of steady jets. *Arc. Rational Mech. Anal.*, 1992, 120, p. 1–49.

[10] BORCHERS, W.; GALDI, G. P.; PILECKAS, K. On the uniqueness of Leray-Hopf solutions for the flow through an aperture. *Arch. Rational Mech. Anal.*, 1993, 122, p. 19–33.

[11] COSCIA, V.; PATRIA, M. C. Existence, uniqueness and asymptotic decay of steady flow of an incompressible fluid in a half space. *Stability Appl. Anal. Cont. Media*, 1992, 2, p. 101–127.

[12] FRAENKEL, L. E. Laminar flow in symmetrical channels with slightly curved walls. I. On the Jefferry-Hamel solutions for flow between plane walls. *Proc. R. Soc. London, Ser. A.*, 1962, 267, p. 119–138.

[13] FRAENKEL, L. E. Laminar flow in symmetrical channels with slightly curved walls. II. An asymptotic series for the stream function. *Proc. R. Soc. London, Ser. A.*, 1963, 272, p. 406–428.

[14] FRAENKEL, L. E. On the theory of laminar flow in channels of certain class. *Proc. Camb. Phil. Soc.*, 1973, 73, p. 361–390.

[15] FRAENKEL, L. E.; EAGLES, P. M. On the theory of laminar flow in channels of certain class. *Math. Proc. Camb. Phil. Soc.*, 1975, 77, p. 199–224.

[16] GALDI, G. P. *An introduction to the Mathematical Theory of Navier-Stokes equation*. Vol. I and II, Springer Tracts in Natural Philosophy, **39**, 1994.

[17] GALDI, G. P.; PADULA, M.; SOLONNIKOV, V. A. Existence, uniqueness and asymptotic behavior of solutions of steady-state Navier-Stokes equations in a plane aperture domain. *Indiana Univ. Math. J.*, 1996, 45(4), p. 961–995.

[18] GALDI, G. P.; PILECKAS, K.; SILVESTRE, A. On the unsteady Poiseuille flow in a pipe. *Zeitschrift fur angew. Mathematik und Physik*, 2007, 58(6), p. 994–1007.

[19] GALDI, G. P.; ROBERTSON, A. M. The relation between flow rate and axial pressure gradient for time-periodic Poiseuille flow in a pipe. *J. Math. Fluid Mech.*, 2004, 7(2), p. 215–223.

[20] HEYWOOD, J. G. On uniqueness questions in the theory of viscous flow. *Acta. Math.*, 1976, 136, p. 61–102.

[21] HU, X. Existence of time-periodic solutions to incompressible Navier-Stokes equations in the whole space. *Electr. J. Diff. Eq.*, 2005, 104, p. 1–6.

[22] KANIEL, S.; SHINBROT, M. A reproductive property of the Navier-Stokes equations. *Arch. Rational Mech. Anal.*, 1967, 24, p. 265–288.

[23] KAPITANSKII, L. V. Stationary solutions of the Navier-Stokes equations in periodic tubes. *J. Sov. Math.*, 1983, 28, p. 689–695.

[24] KAPITANSKII, L. V. On spaces of solenoidal vector fields and boundary value problems for the Navier-Stokes equations in domains with noncompact boundaries. *Proc. Math. Inst. Steklov*, 1984, 159(2), p. 3–34.

[25] KAPITANSKII, L. V.; PILECKAS, K. Certain problems of vector analysis. *J. Sov. Math.*, 1986, 32(5), p. 469–483.

[26] KOZONO, H.; NAKAO, M. Periodic solutions of the Navier-Stokes equations in unbounded domains. *Tohoku Math. J.*, 1996, 48, p. 33–50.

[27] LADYZHENSKAYA, O. A. *The mathematical theory of viscous incompressible flow*, New York, London, Paris: Gordon and Breach, 1969.

[28] LADYZHENSKAYA, O. A.; SOLONNIKOV, V. A. On determination of solutions to boundary value problems for Stokes and Navier-Stokes stationary equations having unbounded Dirichlet integral. *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov.*(LOMI), 1977, 96, p. 117–160.

[29] LADYZHENSKAJA, O. A.; SOLONNIKOV, V. A. On some problems of vector analysis and generalized formulations of boundary value problems for the Navier-Stokes equations. *J. Sov. Math.*, 1978, 10(2), p. 257–285.

[30] LADYZHENSKAJA, O. A.; SOLONNIKOV, V. A. On the stability of boundary value problems for the Navier-Stokes equations in regions with noncompact boundaries. *Vestnik Leningrad Univ. Math.*, 1982, 10, p. 271–280.

[31] LADYZHENSKAJA, O. A.; SOLONNIKOV, V. A. On initial-boundary value problem for the linearized Navier-Stokes system in domains with noncompact boundaries. *Proc. Math. Inst. Steklov*, 1984, 159(2), p. 35–40.

[32] LADYZHENSKAJA, O. A.; SOLONNIKOV, V. A.; URALTSEVA, N. N. *Linear and quasilinear equations of parabolic type*, Moscow: Nauka, 1967 (in Russian).

[33] LANDAU, L. D.; LIFSHITS, E. M. *Continuum Mechanics*, Moscow: Gostekhizdat, 1954 (in Russian).

[34] MIKHLIN, S. G.; *Integral equations*, Moscow: Gostekhizdat, 1947 (In

Russian).

[35] NAZAROV, S. A. On the two-dimensional aperture problem for Navier-Stokes equations. *C. R. Acad. Sci. Paris, Ser. I. Math.*, 1996, 3, p. 699–703.

[36] NAZAROV, S. A.; PILECKAS, K. On the behavior of solutions of the Stokes and Navier-Stokes systems in domains in with periodically varying section. *Proc. Math. Inst. Steklov*, 1984, 159(2), p. 141–154.

[37] NAZAROV, S. A.; PILECKAS, K. On noncompact free boundary problems for the plane stationary Navier-Stokes equations. *J. für reine und angewandte Math.*, 1993, 438, p. 103–141.

[38] NAZAROV, S. A.; PILECKAS, K. The asymptotics of solutions to steady Stokes and Navier-Stokes equations in domains with noncompact boundaries. *Rend. Sem. Math. Univ. Padova*, 1998, 99, p. 33–76.

[39] NAZAROV, S. A.; PILECKAS, K. The asymptotic properties of the solutions to the Stokes problem in domains that are layer-like at infinity. *J. Math. Fluid Mech*, 1999, 1(2), p. 131–167.

[40] NAZAROV, S. A.; PILECKAS, K. On the solvability of the Stokes and Navier-Stokes problems in domains that are layer-like at infinity. *J. Math. Fluid Mech.*, 1999, 1(1), p. 78–116.

[41] NAZAROV, S. A.; PILECKAS, K. Asymptotic conditions at infinity for the Stokes and Navier-Stokes problems in domains with cylindrical outlets to infinity. *Quaderni di Matematica, Advances in Fluid Dynamics*, 1999, 6, p. 32–132.

[42] NAZAROV, S. A.; PILECKAS, K. On the Fredholm property to the Stokes operator in a layer-like domain. *Z.Anal. Anwendungen*, 2001, 20(1), p. 155–182.

[43] NAZAROV, S. A.; SEQUEIRA, A.; VIDEMAN, J. H. Steady flows of Jeffrey-Hamel type from the half-plane into an infinite channel. 1. Linearization on an antisymmetric solution. *J. Math. Pures Appl.*, 2001, 80(10), p. 1069–1098.

[44] ORLOVSKY, D. G. Weak and strong solutions of the inverse problems for differential equations in Banach space. *Differentsialnye Uravneniya*, 1991, 27(5), p. 807–874.

[45] ORLOVSKY, D. G. On solvability of a problem for a parabolic equation in the Hölder class. *Mat. Zametki*, 1991, 50(3), p. 107–112.

[46] PRILEPKO, A. I.; ORLOVSKY, D. G.; VASIN, I. A. *Methods for solving inverse problems in mathematical physics*, New York, Basel: Marsel Dekker, 1999.

[47] PILECKAS, K. Three-dimensional solenoidal vectors. *J. Sov. Math.*, 1983, 21, p. 821–823.

[48] PILECKAS, K. On spaces of solenoidal vectors. *Proc. Math. Ins. Steclov*, 1984, 159(2), p. 141–154.

[49] PILECKAS, K. Existence of solutions for the Navier-Stokes equations having an infinite dissipation of energy, in a class of domains with noncompact boundaries. *J. Sov. Math.*, 1984, 25(1), p. 932–947.

[50] PILECKAS, K. On the problem of motion of heavy viscous incompressible fluid with noncompact free boundary. *Lithuanian Math. J.*, 1988, 28(2), p. 157–184.

[51] PILECKAS, K. The example of non-uniqueness of the solutions to a noncompact free boundary problem for the Navier-Stokes system. *Differential Equations and their Application*, 1988, 42, p. 59–65.

[52] PILECKAS, K. On plane motion of a viscous incompressible cappilary liquid with a noncompact free boundary. *Arch. Rational Mech.*, 1989, 41, p. 329–342.

[53] PILECKAS, K. Weighted L^q-theory and point-wise estimates for steady Stokes and Navier-Stokes equations in domains with exits to infinity. *Proceedings of third International Conference on Navier-Stokes equations and related nonlinear problems*, Plenum press (Edited by A. Sequeira), 1995, p. 33–53.

[54] PILECKAS, K. Weighted L^q-solvability of steady Stokes system in domains with noncompact boundaries. *Math. Models and Methods in Applied Sciences*, 1996, 6(1), p. 97–136.

[55] PILECKAS, K. Classical solvability and uniform estimates for the steady Stokes system in domains with noncompact boundaries. *Math. Models and Methods in Applied Sciences*, 1996, 6(2), p. 149–167.

[56] PILECKAS, K. Strong solutions of the steady nonlinear Navier-Stokes system in domains with exits to infinity. *Rend. Sem. Math. Univ. Padova*, 1997, 97, p. 117–160.

[57] PILECKAS, K. Recent advances in the theory of Stokes and Navier-Stokes equations in domains with noncompact boundaries. *Nonlinear Partial Differ-*

ential Equations and Their Applications. Pitmann Notes in Math. College de France Seminar, 1996, 304, p. 30–85.

[58] PILECKAS, K. On asymptotic behavior of solutions of a stationary system of Navier-Stokes equations in a layer-like domain. *Mat. Sbornik*, 2002, 193(12), p. 69–104.

[59] PILECKAS, K. On the behavior of the nonstationary Poiseuille solution as $t \to \infty$. *Siberian Math. J.*, 2005, 46(4), p. 890–900.

[60] PILECKAS, K. On the nonstationary linearized Navier–Stokes problem in domains with cylindrical outlets to infinity. *Math. Annalen*, 2005, 332, p. 395–419.

[61] PILECKAS, K. Existence of solutions with the prescribed flux of the Navier–Stokes system in an infinite pipe. *J. Math. Fluid. Mech.*, 2006, 8(4), p. 542–563.

[62] PILECKAS, K. On nonstationary two–dimensional Leray's problem for Poiseuille flow. *Advances in Mathematical Sciences and Applications*, 2006, 16(1), p. 141–174.

[63] PILECKAS, K. On the nonstationary two–dimensional Navier–Stokes problem in domains with cylindrical outlets to infinity. *Hyperbolic Problems and Regularity Questions, Trends in Mathematics*, Birkhauser Verlag, 2006, p. 159–171.

[64] PILECKAS, K. Global solvability in $W_2^{2,1}$–weighted spaces of the two–dimensional Navier–Stokes problem in domains with strip–like outlets to infinity. *J. Math. Fluid. Mech.*, 2007, 7(1), p. 235–264.

[65] PILECKAS, K. Solvability in weighted spaces of the three–dimensional Navier–Stokes problem in domains with cylindrical outlets to infinity. *Topological Methods in Nonlinear Analysis*, 2007, 29(1), p. 453–472.

[66] PILECKAS, K. *Navier–Stokes system in domains with cylindrical outlets to infinity. Leray's problem*. Handbook of Mathematical Fluid Dynamics, 4(8), 445–647, Elsevier, 2007.

[67] PILECKAS, K. On the three-dimensional time dependent Navier-Stokes problem in domains with cylindrical outlets to infinity. *J. Math. Fluid. Mech.*, 2007, 10(1), p. 145–160.

[68] PILECKAS, K. SOCOLOWSKY, J. Analysis of two linearized problems modeling viscous two-layer flow. *Math. Nachrichten*, 2002, 245, p. 129–166.

[69] PILECKAS, K. SOLONNIKOV, V. A. On stationary Stokes and Navier-Stokes systems in an open infinite channel. *Lithuanian Math. J.*, 1989, 29(2), p. 235–256.

[70] PILECKAS, K.; SOCOLOWSKY, J. Viscous two-fluid flows in perturbed unbounded domains. *Math. Nachrichten*, 2005, 278(5), p. 511–623.

[71] PILECKAS, K. ZAJECZKOWSKI, W. Global solvability for a large flux of a three-dimensional time dependent Navier-Stokes problem in a straight cylinder. *Math. Meth. App. Sci.*, 2008, 4(1), p. 102–132.

[72] PILECKAS, K. ZALESKIS, L. On steady three-dimensional noncompact free boundary problem for the Navier-Stokes equations. *Zapiski POMI*, 2003, 306, p. 134–164.

[73] RIVKIND, L.; SOLONNIKOV, V. A. Jeffery-Hamel asymptotics for steady Navier-Stokes flow in domains with sector-like outlets to infinity. *J. Math. Fluid Mech.*, 2000, 2(4), p. 325–352.

[74] SERRIN, J. A note on the existence of periodic solutions of the Navier-Stokes equations. *Arch. Rational Mech. Anal.*, 1959, 3, p. 120–122.

[75] SOLONNIKOV, V. A. On boundary value problems for linear parabolic systems of general differential equations. *Trudy Mat. Inst. Nauk SSSR*, 1965, 83, p. 1–162.

[76] SOLONNIKOV, V. A. On the solvability of boundary and initial-boundary value problems for the Navier-Stokes system in domains with noncompact boundaries. *Pasific J. Math.*, 1981, 93(2), p. 443–458.

[77] SOLONNIKOV, V. A. Stokes and Navier-Stokes equations in domains with noncompact boundaries. *Nonlinear Partial Differential Equations and Their Applications. Pitmann Notes in Math. College de France Seminar*, 1983, 3, p. 210–349.

[78] SOLONNIKOV, V. A. On solutions of stationary Navier-Stokes equations with an infinite Dirichlet integral. *J. Sov. Math.*, 1985, 28(5), p. 792–799.

[79] SOLONNIKOV, V. A. Solvability of the problem of effluence of a viscous incompressible fluid into an open basin. *Proc. Math. Inst. Steklov*, 1989, 179(2), p. 193–225.

[80] SOLONNIKOV, V. A. Boundary and initial-boundary value problems for the Navier-Stokes equations in domains with noncompact boundaries. *Math. Topics in Fluid Mechanics, Pitman Research Notes in Mathematics Series*, 1991, 274 (eds. Rodriques, J. F.; Sequeira, A.), p. 117–162.

[81] SOLONNIKOV, V. A. On problems of hydrodynamics of a viscous incompressible fluid in domains with noncompact boundaries. *Algebra i Analiz*, 1992, 4(6), p. 28–53.

[82] SOLONNIKOV, V. A.; PILECKAS, K. Certain spaces of solenoidal vectors and the solvability of the boundary value problem for the Navier-stokes system of equations in domains with noncompact boundaries. *J. Sov. Math.*, 1986, 34(6), p. 2101–2111.

[83] SOLONNIKOV, V. A. Stokes and Navier–Stokes equations in domains with noncompact boundaries // Nonlinear partial differential equations and their applications. *Pitmann Notes in Math., College de France Seminar*, 1983, 3, p. 210–349.

[84] TAKESHITA, A. On the reproductive property of two–dimensional Navier-Stokes equations. *J. Fac. Sci. Univ. Tokyo Sec. IA*, 1970, 16, p. 297–311.

[85] TEMAM, R. *Navier-Stokes equations.Theory and numerical analysis*, Amsterdam: North-Holland Publishing Co., 1984.

[86] VEJVODA, O.; HERRMANN, L.; LOVICAR, V.; SOVA, M.; STRASKRABA, I.; STEDRY, M. *Partial differential equations: time periodic solutions*, The Hague: Martinus Nijhoff Publishers, 1981.

List of V. Keblikas published works on the topic of the dissertation

[A1] KEBLIKAS, V., PILECKAS, K., On the existence of non-stationary Poiseuille solution, *Siberian Math. J.*, 2005, 46(3), 514–526.

[A2] KEBLIKAS, V., On the time-periodic problem for the Stokes system in domains with cylindrical outlets to infinity, *Lithuanian Math. J.*, 2007, 47(2), 147–163.

Vaidas KEBLIKAS

TIME PERIODIC PROBLEMS FOR NAVIER–STOKES EQUATIONS
IN DOMAINS WITH CYLINDRICAL OUTLETS TO INFINITY

Doctoral dissertation
Physical Sciences, Mathematics (01P)

2008 09 02. 7 sp. l. Tiražas 20 egz.
Vilnius Gedimino technikos universiteto
leidykla „Technika“, Saulėtekio al. 11, 10223 Vilnius
http://leidykla.vgtu.lt
Spausdino UAB „Baltijos kopija“
Kareivių g. 13B, 09109 Vilnius
http://www.kopija.lt